Lect. 9. Accuracy and reliability of target location

Two types of localization errors: normal and anomalous ones.

WGSIN-model: localization accuracy and signal shape.

Localization accuracy in non-white noise.

Fundamental threshold in target location reliability.

Target localization reliability in the presence of foreign non-overlapping objects

Case study: Correlational accumulation for image denoising, deblurring and “super-resolution”

Image averaging: \[ \hat{a}_k = \frac{\sum_{m=1}^{M} b_k^{(m)}}{\sum_{m=1}^{M} \frac{1}{\sigma_n^{(m)}}} \]

Image mutual alignment: \[ \frac{AV_{\Omega_x}(\gamma_r^{(m)}, \alpha_r^{(l)})}{AV_{\Omega_x}(\alpha_r^{(l)})^2} = \exp \left( \frac{i2\pi u^{(m,j)}_r}{N} \right) \]

where \( u^{(m,j)}_r \) is mutual misalignment.

Image alignment and averaging \[ \hat{a}_r^{(m)} = \frac{\sum_{l=1}^{M} \exp \left[ i2\pi \frac{u^{(m,j)}_r}{N} \right] \beta_r^{(l)} / \sigma_n^{(l)}}{\sum_{l=1}^{M} 1 / \sigma_n^{(l)}} \]

Expected performance of the correlational accumulation

Misalignment errors: normal and anomalous. Normal misalignment errors limit additive noise variance reduction factor to \( M \cdot \left( 1 - P_{an,err} \right) \) and cause restored signal blur with PSF determined by variance of normal errors. Anomalous misalignment errors cause a spurious signal resulted from accumulation of noise realizations that exhibited high correlation with the signal template.

Additional reading:
TWO TYPES OF THE LOCALIZATION ERRORS

Correlator’s output: the case of “normal” error

Correlator’s output: the case of anomalous error
VARIANCE OF THE LOCALIZATION ERRORS

For one-dimensional signals:

\[ \text{Corr}_{sh}(x) = \text{corr}[(s(x - x_0) + n(x))h(x)]; \ h(x) - \text{impulse response of the filter} \]

\[ \frac{d\text{Corr}}{dx} \bigg|_{x = \hat{x}_0} + \frac{d\text{Corr}_{sh}}{dx} \bigg|_{x = \hat{x}_0} + \frac{d\text{Corr}_{nh}}{dx} \bigg|_{x = \hat{x}_0} = 0; \quad \hat{x}_0 = x_0 + \varepsilon; \]

\[ \frac{d\text{Corr}_{sh}(x)}{dx} \bigg|_{x = \hat{x}_0} = 0; \]

\[ \varepsilon \frac{d^2\text{Corr}_{sh}}{dx^2} \bigg|_{x = \hat{x}_0} + \varepsilon \frac{d^2\text{Corr}_{nh}}{dx^2} \bigg|_{x = \hat{x}_0} = 0; \]

\[ \left( \varepsilon^2 \right) \left( \frac{d^2\text{Corr}_{sh}}{dx^2} \bigg|_{x = \hat{x}_0} \right)^2 = \left( \frac{d\text{Corr}_{nh}}{dx} \bigg|_{x = \hat{x}_0} \right)^2. \]

\[ \sigma_x^2 = \left( \varepsilon^2 \right) = \frac{N_0}{4\pi^2 \bar{f}_x^2 E_a} \int_{-\infty}^{\infty} f^2 |\alpha(f)|^2 df \int_{-\infty}^{\infty} |H(f)|^2 df \]

\[ \int_{-\infty}^{\infty} f^2 \alpha(f)H(f) df \]

\[ \frac{N_0}{4\pi^2 \bar{f}_x^2 E_a} = \frac{N_0}{4\pi^2 \bar{f}_{aa}^2 E_{aa}}, \]

when \( H(f) = \alpha^*(f), \)

\[ \alpha(f) = \int_{-\infty}^{\infty} s(x) \exp(i2\pi f x) dx; \quad H(f) = \int_{-\infty}^{\infty} h(x) \exp(i2\pi f x) dx; \quad E_a = \int_{-\infty}^{\infty} |\alpha(f)|^2 df \]

\[ \bar{f}_x^2 = \frac{\int \int_{-\infty}^{\infty} f^2 |\alpha(f)|^2 df \int \int_{-\infty}^{\infty} |\alpha(f)|^2 df}{\int \int_{-\infty}^{\infty} |\alpha(f)|^2 df} \]

\[ \int_{-\infty}^{\infty} f^2 \alpha(f)H(f) df \]

\[ E_a = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\alpha(f_x, f_y)|^2 df_x df_y \]

\[ \bar{f}_x^2 = \frac{\int \int_{-\infty}^{\infty} f_x^2 |\alpha(f_x, f_y)|^2 df_x df_y}{\int \int_{-\infty}^{\infty} |\alpha(f_x, f_y)|^2 df_x df_y} \]

For 2-D signals:

\[ \sigma_x^2 = \frac{1}{4\pi^2} \frac{\bar{f}_y^2}{\bar{f}_x^2 \bar{f}_y - \bar{f}_{xy}^2} N_0; \quad \sigma_y^2 = \frac{1}{4\pi^2} \frac{\bar{f}_x^2}{\bar{f}_x^2 \bar{f}_y - \bar{f}_{xy}^2} N_0; \]

\[ \sigma_{xy}^2 = \frac{1}{4\pi^2} \frac{\bar{f}_{xy}^2}{\bar{f}_x^2 \bar{f}_y - \bar{f}_{xy}^2} N_0, \]

where

\[ E_a = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\alpha(f_x, f_y)|^2 df_x df_y \]

\[ \bar{f}_x^2 = \frac{\int \int_{-\infty}^{\infty} f_x^2 |\alpha(f_x, f_y)|^2 df_x df_y}{\int \int_{-\infty}^{\infty} |\alpha(f_x, f_y)|^2 df_x df_y} \]

\[ \bar{f}_y^2 = \frac{\int \int_{-\infty}^{\infty} f_y^2 |\alpha(f_x, f_y)|^2 df_x df_y}{\int \int_{-\infty}^{\infty} |\alpha(f_x, f_y)|^2 df_x df_y} \]

\[ \bar{f}_{xy}^2 = \frac{\int \int_{-\infty}^{\infty} f_x f_y |\alpha(f_x, f_y)|^2 df_x df_y}{\int \int_{-\infty}^{\infty} |\alpha(f_x, f_y)|^2 df_x df_y} \]
Probability of anomalous (false detection) errors:

\[ P_{ac} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{n^2}{2}\right) \left[ 1 - \Phi\left(\frac{E_a}{\sqrt{N_0}} + n\right)\right]^{Q-1} dn \]

where \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{n^2}{2}\right) dn \), \( Q = O(\text{Search area}/\text{Target size}) \).

Experimental distribution density of the localization error for a Gaussian-shaped impulse (\( \sigma_a = 10 \) in scale 1:256) and \( \sqrt{E_a / N_0} = 1.5 \) (100000 realizations)

Standard deviation of the localization error as a function of noise-to-signal error for Gaussian-shaped impulse with (10000 realizations)
The localization reliability threshold:
By L'Hospital's rule we have:

\[
\lim_{Q \to \infty} \left( \ln \left[ \Phi \left( \frac{E_a}{\sqrt{N_0}} + n \right) \right] \right)^{-1} = \lim_{Q \to \infty} \left[ \frac{d}{dQ} \ln \Phi \left( E_a \sqrt{\ln Q} \right) / \frac{d}{dQ} \frac{1}{Q} \right] = \\
\lim_{Q \to \infty} \frac{-E}{\Phi \left( E_a \sqrt{\ln Q} \right)} Q^{\left( -e^2 / 2 \right)} = \begin{cases} -\infty, & \text{if } E^2 / 2 > 1 \\ 0, & \text{if } E^2 / 2 \leq 1 \end{cases}
\]

where \( E = \sqrt{E_a/N_0 \ln Q} \)

Therefore,

\[
\lim_{Q \to \infty} P_{ae} = \begin{cases} 1, & \text{if } E_a / 2N_0 \leq \ln Q \\ 0, & \text{if } E_a / 2N_0 > \ln Q \end{cases}
\]

Probability of anomalous errors as a function of normalised noise-to-signal ratio \( \sqrt{N_0 \ln Q / E_a} \) for localization of rectangle impulses of 2;5;11;21;41;81;161 samples within an interval of 1024 samples (10000 realizations). The theoretical threshold value of the normalized noise-to-signal ratio is \sqrt{2} / 2 \approx 0.707
Localization accuracy of the “optimal” filter for non-white noise with spectral density $N_w(f_x, f_y)$

$$\sigma_{x,NW}^2 = \frac{\tilde{f}_{x,NW}^2}{f_{x,NW}^2} \frac{N_0}{4\pi^2 E_{x,NW}} ; \quad \sigma_{y,NW}^2 = \frac{\tilde{f}_{y,NW}^2}{f_{y,NW}^2} - \left(\frac{\tilde{f}_{y,NW}^2}{f_{y,NW}^2}\right) \frac{N_0}{4\pi^2 E_{x,NW}}$$

$$\sigma_{y,NW}^2 = \frac{\tilde{f}_{y,NW}^2}{f_{y,NW}^2} \frac{N_0}{4\pi^2 E_{x,NW}}$$

$$E_{x,NW} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\alpha(f_x, f_y)}{H_n(f_x, f_y)} \right|^2 df_x df_y ; \quad \tilde{f}_{x,NW} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\alpha(f_x, f_y)}{H_n(f_x, f_y)} \right|^2 df_x df_y$$

Compare the potential localization accuracy in the cases of white and non-white noise for the same input noise variance. For the sake of simplicity we shall consider a one-dimensional case. In this case for non-white noise

$$\sigma_{x,NW}^2 = \frac{1}{\tilde{f}_{x,NW}^2} \frac{N_0}{4\pi^2 E_{x,NW}} .$$

Express this through the error variance $\sigma_x^2$ for the case of white noise:

$$\sigma_{x,NW}^2 = \frac{1}{\tilde{f}_{x,NW}^2} \frac{N_0}{4\pi^2 E_{x,NW}} = \sigma_x^2 \frac{E_{x,NW}}{E_{x,NW}^2} \tilde{f}_{x,NW}^2 = G \sigma_x^2 .$$

where factor $G$ shows the change of error variance owing to non-whiteness of noise. Because all functions involved in this factor are nonnegative, the following inequality takes place:

$$G = \frac{\int_{-\infty}^{\infty} \alpha(f_x)^2 df_x}{\int_{-\infty}^{\infty} \frac{\alpha(f_x)^2}{H_n(f_x)^2} df_x} \leq \frac{\int_{-\infty}^{\infty} \alpha(f_x)^2 df_x}{\int_{-\infty}^{\infty} \frac{\alpha(f_x)^2}{H_n(f_x)^2} df_x} = 1$$

Therefore, the potential localization accuracy in the presence of non-white noise is always better than that for white noise with the same variance.
Detection, by means of matched filtering, of a character with and without non-target characters in the area of search. Upper row, left image: noisy image of a printed text with standard deviation of additive noise 15 (within signal range 0-255). Upper row, right image: results of detection of character “o” (right); one can see quite a number of false detections. Bottom row, left: a noisy image with a single character “o” in its center; standard deviation of the additive noise is 175. Highlighted center of the right image shows that the character is correctly localized by the matched filter.
Correlational accumulation (averaging): Application examples

- Processing of electron micrographs

Examples of electron micrographs of virus particles and 3D reconstruction of the virus

- High resolution electrocardiography

- “Non-local methods” for image denoising

Correlational accumulation: the principle

- Selection of a template
- Alignment of sample images by locating position of maximal correlation
- Averaging aligned sample images

```
OUT=img_corr_accum demo(256,1000,0.5);
```

Correlational accumulation: SNR=0.5; Nit=1000. Template, accumulated image and 0.75-whitened template

```
Correlational accumulation: SNR=0.1; Nit=1000. Template, accumulated image and 0.75-whitened template
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Correlational accumulation; Nit=10000. Template, accumulated image and 0.75-whitened template