

Lect. 9. Accuracy and reliability of target location

Two types of localization errors: normal and anomalous ones.

WGSIN-model: localization accuracy and signal shape.

Localization accuracy in non-white noise.

Fundamental threshold in target location reliability.

Target localization reliability in the presence of foreign non-overlapping objects

Case study: Correlational accumulation for image denoising, deblurring and “super-resolution”

$$\text{Image averaging: } \hat{a}_k = \frac{\sum_{m=1}^M b_k^{(m)} / (\sigma_n^{(m)})^2}{\sum_{m=1}^M 1 / (\sigma_n^{(m)})^2}$$

$$\text{Image mutual alignment: } \frac{AV_{\Omega_A}(\alpha_r^{(m)} \alpha_r^{(l)})}{AV_{\Omega_A}(|\alpha_r^{(l)}|^2)} = \exp\left(i2\pi \frac{u^{(m,l)} r}{N}\right), \text{ where } u^{(m,l)} \text{ is mutual misalignment}$$

$$\text{Image alignment and averaging } \hat{\alpha}_r^{(m)} = \frac{\sum_{l=1}^M \exp\left[i2\pi \frac{u^{(m,l)} r}{N}\right] \beta_r^{(l)} / \sigma_n^{(l)}}{\sum_{l=1}^M 1 / \sigma_n^{(l)}}$$

Expected performance of the correlational accumulation

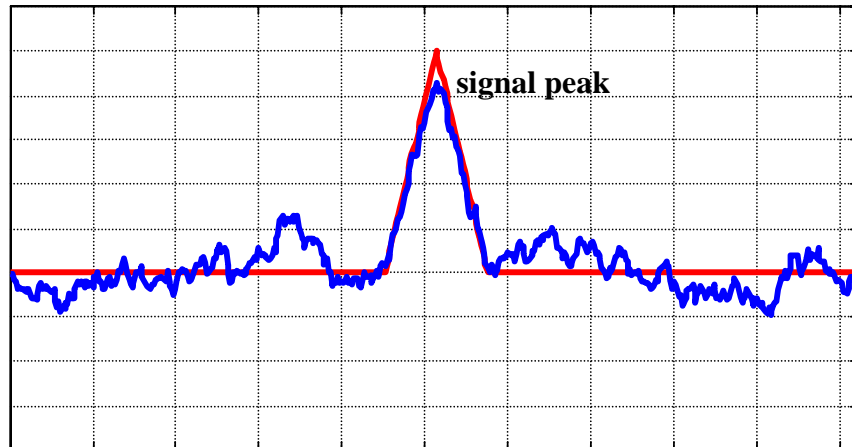
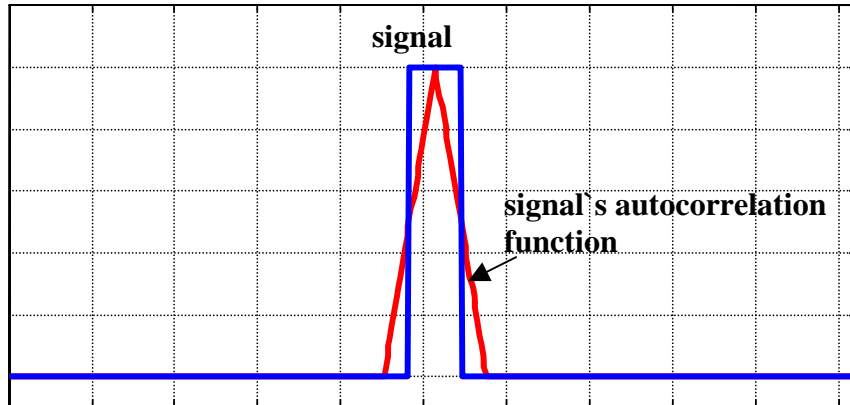
Misalignment errors: normal and anomalous. Normal misalignment errors limit additive noise variance reduction factor to $M \cdot (1 - P_{an.err})$ and cause restored signal blur with PSF determined by variance of normal errors. Anomalous misalignment errors cause a spurious signal resulted from accumulation of noise realizations that exhibited high correlation with the signal template.

Additional reading:

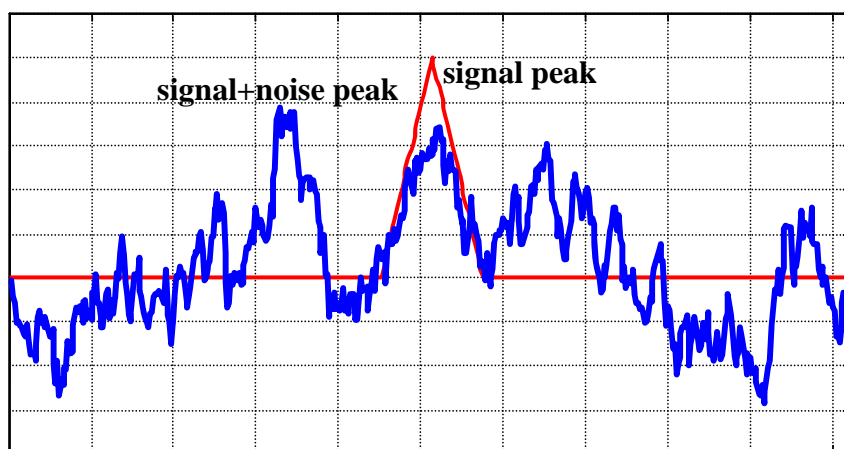
L.P.Yaroslavsky, H.J. Caulfield, Deconvolution of Multiple Images of the same object, Appl. Opt., Vol. 33, No 11, p. 2157-21

L. Yaroslavsky, M. Eden, Correlational Accumulation as a Method for Signal Restoration, Signal Processing, 39 (1994) p. 89-106

TWO TYPES OF THE LOCALIZATION ERRORS



Correlator's output: the case of "normal" error



Correlator's output: the case of anomalous error

VARIANCE OF THE LOCALIZATION ERRORS

For one-dimensional signals:

$Corr_{sn}(x) = conv[(s(x - x_0) + n(x))h(x)]$; $h(x)$ - impulse response of the filter

$$\frac{dCorr}{dx} \Big|_{x=\hat{x}_0} = \frac{dCorr_{sh}}{dx} \Big|_{x=\hat{x}_0} + \frac{dCorr_{nh}}{dx} \Big|_{x=\hat{x}_0} = 0; \quad \hat{x}_0 = x_0 + \varepsilon;$$

$$\frac{dCorr_{sh}(x)}{dx} \Big|_{x=x_0} = 0;$$

$$\varepsilon \frac{d^2 Corr_{sh}}{dx^2} \Big|_{x=x_0} + \frac{dCorr_{nh}}{dx} \Big|_{x=\hat{x}_0} = 0; \quad \langle \varepsilon^2 \rangle \left| \frac{d^2 Corr_{sh}}{dx^2} \Big|_{x=x_0} \right|^2 = \left\langle \left| \frac{dCorr_{nh}}{dx} \right|^2 \right\rangle$$

$$\sigma_\varepsilon^2 = \langle \varepsilon^2 \rangle = \frac{N_0}{4\pi^2 \bar{f}_2^2 E_a} \frac{\int_{-\infty}^{\infty} f^2 |\alpha(f)|^2 df \int_{-\infty}^{\infty} f^2 |H(f)|^2 df}{\left(\int_{-\infty}^{\infty} f^2 \alpha(f) H(f) df \right)^2} \geq \frac{N_0}{4\pi^2 \bar{f}_2^2 E_a} = \frac{N_0}{4\pi^2 E_{aa}},$$

when $H(f) = \alpha^*(f)$, where

$$\alpha(f) = \int_{-\infty}^{\infty} s(x) \exp(i2\pi fx) dx; \quad H(f) = \int_{-\infty}^{\infty} h(x) \exp(i2\pi fx) dx; \quad E_a = \int_{-\infty}^{\infty} |\alpha(f)|^2 df$$

$$\bar{f}_x^2 = \frac{\int_{-\infty}^{\infty} f^2 |\alpha(f)|^2 df}{\int_{-\infty}^{\infty} |\alpha(f)|^2 df}; \quad E_{aa} = 4\pi^2 \int_{-\infty}^{\infty} f^2 |\alpha(f)|^2 df = \int_{-\infty}^{\infty} \left| \frac{da(x)}{dx} \right|^2 dx - \text{energy of signal's}$$

derivative

For 2-D signals:

$$\sigma_x^2 = \frac{1}{4\pi^2} \frac{\bar{f}_y^2}{\bar{f}_x^2 \bar{f}_y^2 - (\bar{f}_{xy})^2} \frac{N_0}{E_a}; \quad \sigma_y^2 = \frac{1}{4\pi^2} \frac{\bar{f}_x^2}{\bar{f}_x^2 \bar{f}_y^2 - (\bar{f}_{xy})^2} \frac{N_0}{E_a};$$

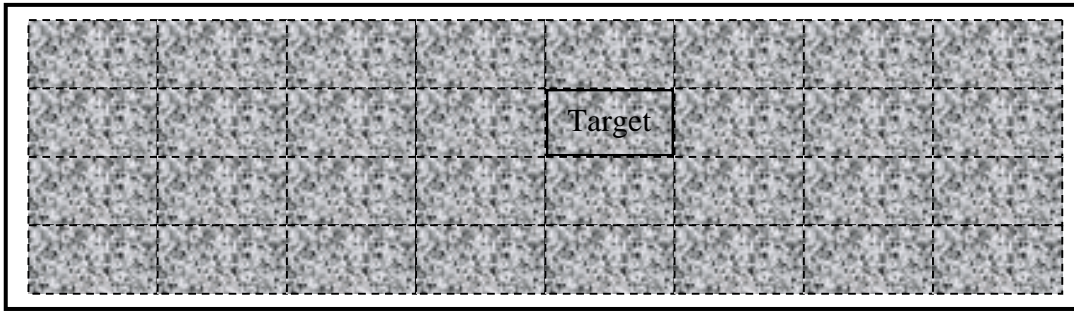
$$\sigma_{xy}^2 = \frac{1}{4\pi^2} \frac{\bar{f}_{xy}^2}{\bar{f}_x^2 \bar{f}_y^2 - (\bar{f}_{xy})^2} \frac{N_0}{E_a}.$$

where

$$E_a = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\alpha(f_x, f_y)|^2 df_x df_y; \quad \bar{f}_x^2 = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_x^2 |\alpha(f_x, f_y)|^2 df_x df_y}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\alpha(f_x, f_y)|^2 df_x df_y}$$

$$\bar{f}_y^2 = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_y^2 |\alpha(f_x, f_y)|^2 df_x df_y}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\alpha(f_x, f_y)|^2 df_x df_y}; \quad \bar{f}_{xy}^2 = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_x f_y |\alpha(f_x, f_y)|^2 df_x df_y}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\alpha(f_x, f_y)|^2 df_x df_y}$$

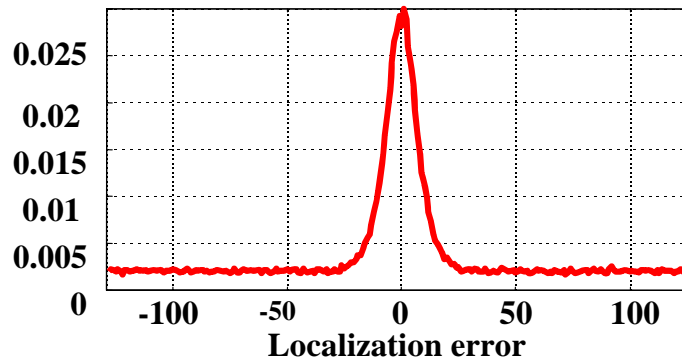
LOCALIZATION RELIABILITY



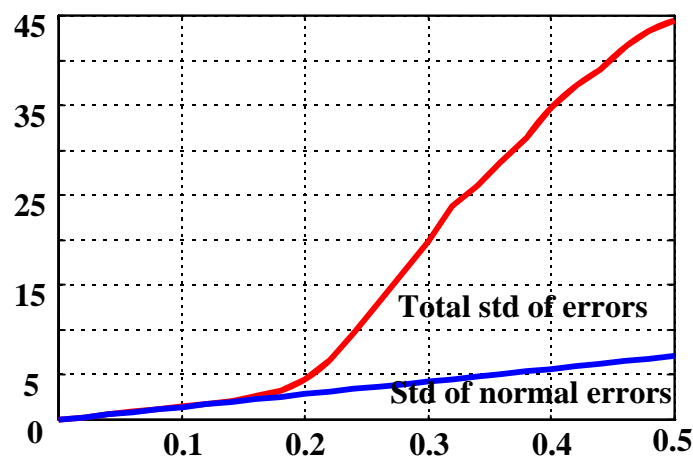
Probability of anomalous (false detection) errors:

$$P_{ae} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{n^2}{2}\right) \left\{ 1 - \left[\Phi\left(\sqrt{\frac{E_a}{N_0}} + n\right) \right]^{Q-1} \right\} dn$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{n^2}{2}\right) dn$, $Q = O(\text{Search area/Target size})$.



Experimental distribution density of the localization error for a Gaussian-shaped impulse ($\sigma_a = 10$ in scale 1:256) and $\sqrt{E_a / N_0} = 1.5$ (100000 realizations)



Standard deviation of the localization error as a function of noise-to-signal error for Gaussian-shaped impulse with (10000 realizations)

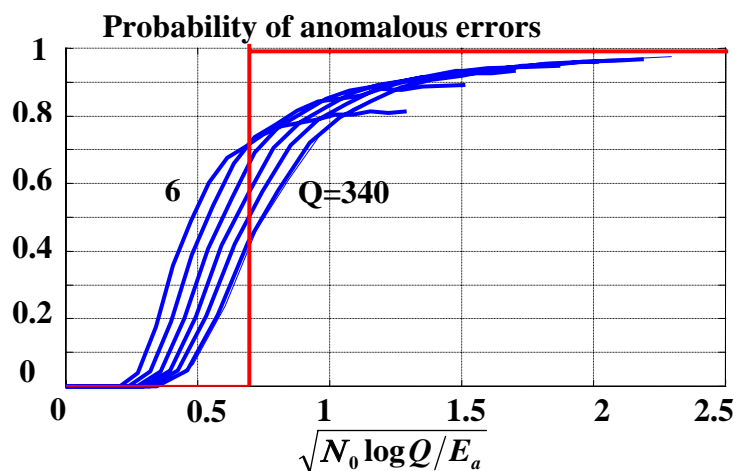
The localization reliability threshold:

By L'Hospital's rule we have:

$$\lim_{Q \rightarrow \infty} \left(\ln \left[\Phi \left(\sqrt{\frac{E_a}{N_0}} + n \right) \right] \right)^{Q-1} = \lim_{Q \rightarrow \infty} \left[\frac{d}{dQ} \ln \Phi(E \sqrt{\ln Q}) \right] / \left[\frac{d}{dQ} \frac{1}{Q} \right] =$$

$$\lim_{Q \rightarrow \infty} \frac{-E}{\Phi(E \sqrt{\ln Q})} Q^{(1-E^2/2)} = \begin{cases} -\infty, & \text{if } E^2/2 > 1 \\ 0, & \text{if } E^2/2 \leq 1 \end{cases} \text{ where } E = \sqrt{E_a / N_0 \ln Q}$$

Therefore, $\lim_{Q \rightarrow \infty} P_{ae} = \begin{cases} 1, & \text{if } E_a / 2N_0 \leq \ln Q \\ 0, & \text{if } E_a / 2N_0 > \ln Q \end{cases}$.



Probability of anomalous errors as a function of normalised noise-to-signal ratio $\sqrt{N_0 \ln Q} / E_a$ for localization of rectangle impulses of 2;5;11;21;41;81;161 samples within an interval of 1024 samples (10000 realizations). The theoretical threshold value of the normalized noise-to-signal ratio is $\sqrt{2} / 2 \approx 0.707$

Localization accuracy of the “optimal” filter for non-white noise with spectral density $N_0 |H_n(f_x, f_y)|^2$

$$\sigma_{x,NW}^2 = \frac{\bar{f}_{y,NW}^2}{\bar{f}_{y,NW}^2 \bar{f}_{y,NW}^2 - (\bar{f}_{xy,NW}^2)^2} \frac{N_0}{4\pi^2 E_{a,NW}}; \quad \sigma_{y,NW}^2 = \frac{\bar{f}_{x,NW}^2}{\bar{f}_{y,NW}^2 \bar{f}_{y,NW}^2 - (\bar{f}_{xy,NW}^2)^2} \frac{N_0}{4\pi^2 E_{a,NW}}$$

$$\sigma_{xy,NW}^2 = \frac{\bar{f}_{xy,NW}^2}{\bar{f}_{y,NW}^2 \bar{f}_{y,NW}^2 - (\bar{f}_{xy,NW}^2)^2} \frac{N_0}{4\pi^2 E_{a,NW}}$$

$$E_{a,NW} = \iint_{-\infty-\infty}^{\infty\infty} \frac{|\alpha(f_x, f_y)|^2}{|H_n(f_x, f_y)|^2} df_x df_y; \quad \bar{f}_{x,NW}^2 = \frac{\iint_{-\infty-\infty}^{\infty\infty} f_x^2 \frac{|\alpha(f_x, f_y)|^2}{|H_n(f_x, f_y)|^2} df_x df_y}{\iint_{-\infty-\infty}^{\infty\infty} \frac{|\alpha(f_x, f_y)|^2}{|H_n(f_x, f_y)|^2} df_x df_y}$$

$$\bar{f}_{y,NW}^2 = \frac{\iint_{-\infty-\infty}^{\infty\infty} f_y^2 \frac{|\alpha(f_x, f_y)|^2}{|H_n(f_x, f_y)|^2} df_x df_y}{\iint_{-\infty-\infty}^{\infty\infty} \frac{|\alpha(f_x, f_y)|^2}{|H_n(f_x, f_y)|^2} df_x df_y}; \quad \bar{f}_{xy,NW}^2 = \frac{\iint_{-\infty-\infty}^{\infty\infty} f_x f_y \frac{|\alpha(f_x, f_y)|^2}{|H_n(f_x, f_y)|^2} df_x df_y}{\iint_{-\infty-\infty}^{\infty\infty} \frac{|\alpha(f_x, f_y)|^2}{|H_n(f_x, f_y)|^2} df_x df_y}$$

Compare the potential localization accuracy in the cases of white and non-white noise for the same input noise variance. For the sake of simplicity we shall consider a one-dimensional case. In this case for non-white noise

$$\sigma_{x,NW}^2 = \frac{1}{\bar{f}_{x,NW}^2} \frac{N_0}{4\pi^2 E_{a,NW}}.$$

Express this through the error variance σ_x^2 for the case of white noise :

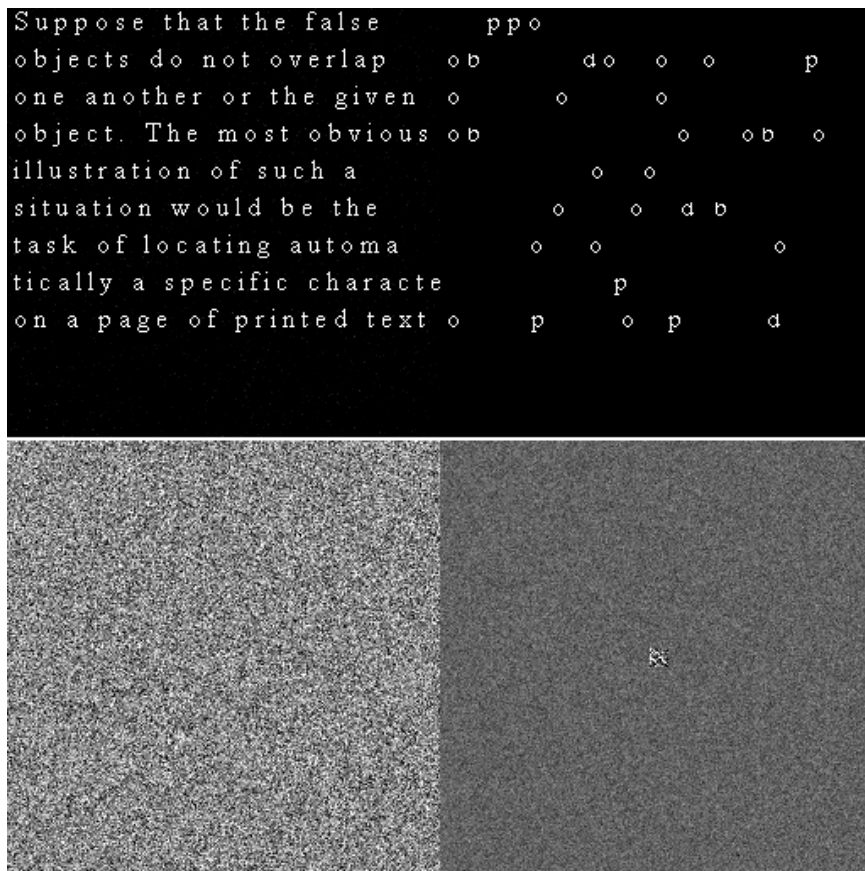
$$\sigma_{x,NW}^2 = \frac{1}{\bar{f}_{x,NW}^2} \frac{N_0}{4\pi^2 E_{a,NW}} = \sigma_x^2 \frac{E_a \bar{f}_x^2}{E_{a,NW} \bar{f}_{x,NW}^2} = G \sigma_x^2.$$

where factor G shows the change of error variance owing to non-whiteness of noise. Because all functions involved in this factor are nonnegative, the following inequality takes place:

$$G = \frac{\int_{-\infty}^{\infty} f_x^2 |\alpha(f_x)|^2 df_x}{\int_{-\infty}^{\infty} f_x^2 \frac{|\alpha(f_x)|^2}{|H_n(f_x)|^2} df_x} \leq \frac{\int_{-\infty}^{\infty} f_x^2 \frac{|\alpha(f_x)|^2}{|H_n(f_x)|^2} df_x \int_{-\infty}^{\infty} |H_n(f_x)|^2 df_x}{\int_{-\infty}^{\infty} f_x^2 \frac{|\alpha(f_x)|^2}{|H_n(f_x)|^2} df_x} = \int_{-\infty}^{\infty} |H_n(f_x)|^2 df_x = 1$$

Therefore, the potential localization accuracy in the presence of non-white noise is always better than that for white noise with the same variance.

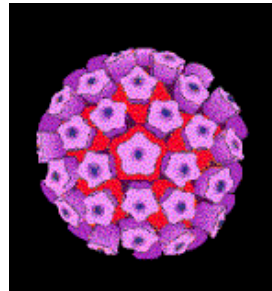
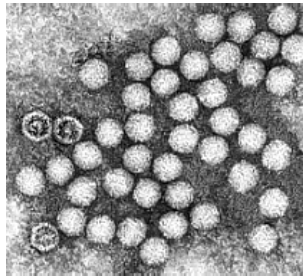
Target location in images with non-overlapping non-target objects



Detection, by means of matched filtering, of a character with and without non-target characters in the area of search. Upper row, left image: noisy image of a printed text with standard deviation of additive noise 15 (within signal range 0-255). Upper row, right image: results of detection of character “o” (right); one can see quite a number of false detections. Bottom row, left: a noisy image with a single character “o” in its center; standard deviation of the additive noise is 175. Highlighted center of the right image shows that the character is correctly localized by the matched filter.

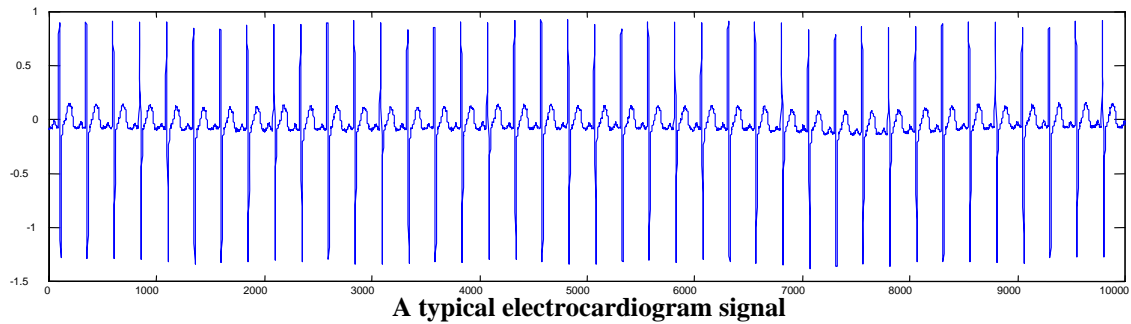
Correlational accumulation (averaging): Application examples

- Processing of electron micrographs



Examples of electron micrographs of virus particles and 3D reconstruction of the virus

- High resolution electrocardiography



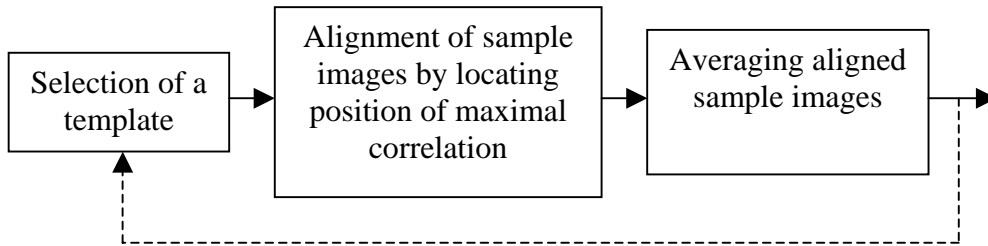
A typical electrocardiogram signal

- “Non-local methods” for image denoising



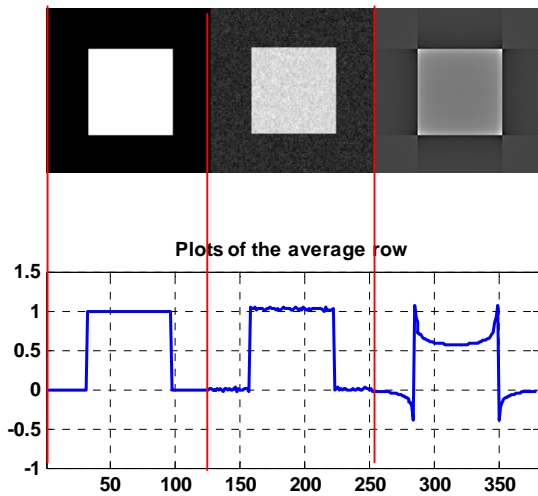
Adopted from: A.Buades B. Coll J.-M. Morel, Image and movie denoising by nonlocal means,
<http://www.dma.ens.fr/culturemath/maths/mathapli/imagerie-Morel/Buades-Coll-Morel-movies.pdf>

Correlational accumulation: the principle

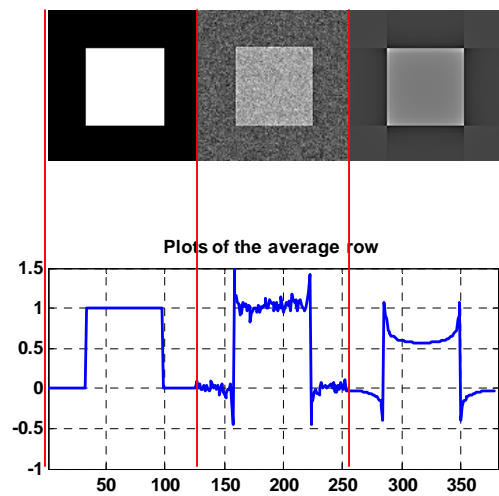


`OUT=img_corr_accum_demo(256,1000,0.5);`

Correlational accumulation: SNR=0.5; Nit=1000.
Template, accumulated image and 0.75-whitened template



Correlational accumulation: SNR=0.1; Nit=1000.
Template, accumulated image and 0.75-whitened template



Correlational accumulation; Nit=10000.
Template, accumulated image and 0.75-whitened template

