Lect. 9. Accuracy and reliability of target location

Two types of localization errors: normal and anomalous ones.
WGSIN-model: localization accuracy and signal shape.
Localization accuracy in non-white noise.
Fundamental threshold in target location reliability.
Target localization reliability in the presence of foreign non-overlapping objects
Case study: Correlational accumulation for image denoising, deblurring and "super-resolution"

Image averaging: $\hat{a}_k = \sum_{m=1}^{M} b_k^{(m)} / \left(\sigma_n^{(m)}\right)^2 \Big/ \sum_{m=1}^{M} 1 / \left(\sigma_n^{(m)}\right)^2$

Image mutual alignment: $\dfrac{AV_{\Omega_A}\left(\alpha_r^{(m)} \alpha_r^{(l)}\right)}{AV_{\Omega_A}\left(\left|\alpha_r^{(l)}\right|^2\right)} = \exp\left(i2\pi \dfrac{u^{(m,l)} r}{N}\right)$, where $u^{(m,l)}$ is mutual misalignment

Image alignment and averaging $\hat{\alpha}_r^{(m)} = \dfrac{\sum_{l=1}^{M} \exp\left[i2\pi \dfrac{u^{(m,l)}}{N} r\right] \beta_r^{(l)} / \sigma_n^{(l)}}{\sum_{l=1}^{M} 1 / \sigma_n^{(l)}}$

Expected performance of the correlational accumulation
Misalignment errors: normal and anomalous. Normal misalignment errors limit additive noise variance reduction factor to $M \cdot \left(1 - P_{an.err}\right)$ and cause restored signal blur with PSF determined by variance of normal errors. Anomalous misalignment errors cause a spurious signal resulted from accumulation of noise realizations that exhibited high correlation with the signal template.

Additional reading:

L.P.Yaroslavsky, H.J. Caulfield, Deconvolution of Multiple Images of the same object, Appl. Opt., Vol. 33, No 11, p. 2157-21
L. Yaroslavsky, M. Eden, Correlational Accumulation as a Method for Signal Restoration, Signal Processing, 39 (1994) p. 89-106

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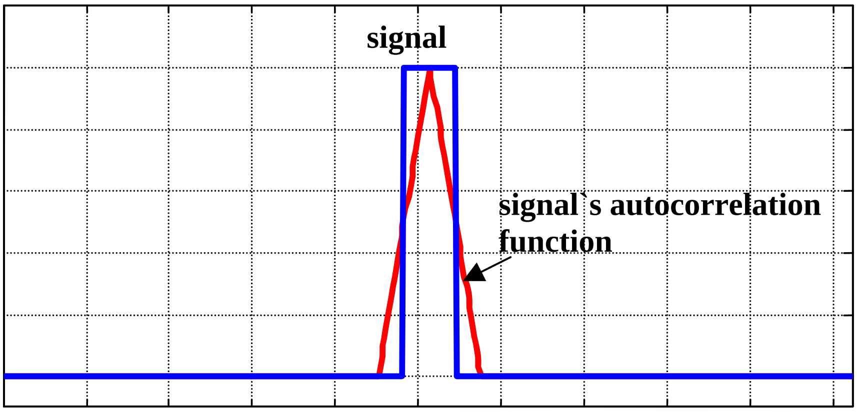
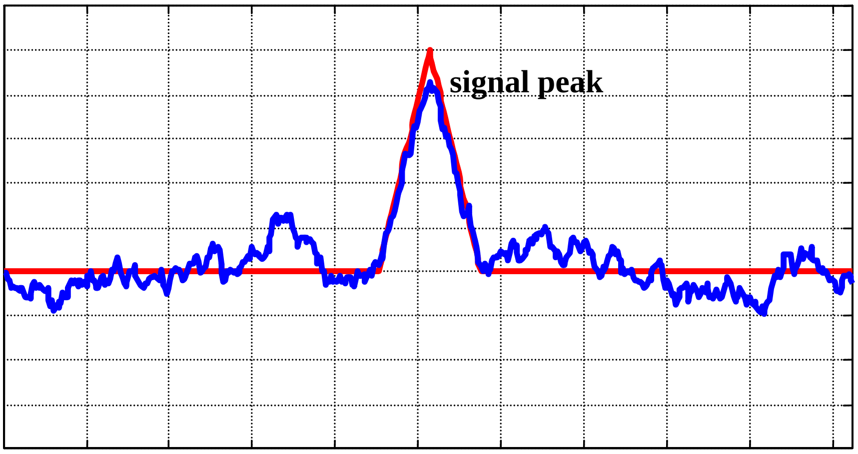
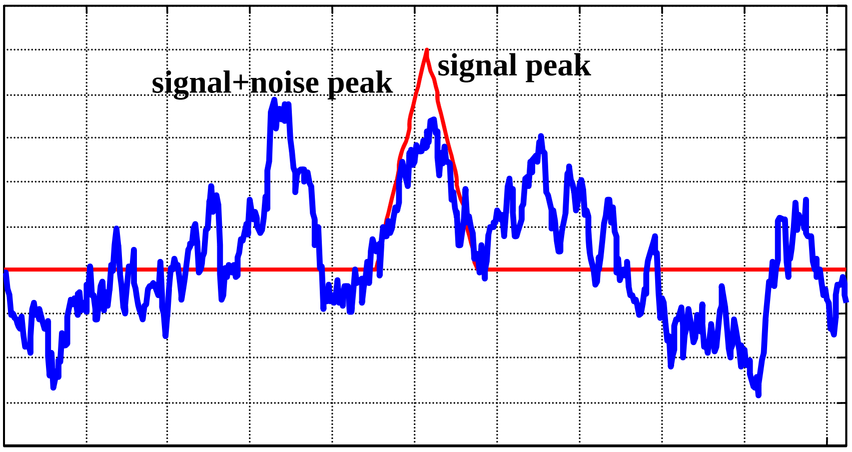
TWO TYPES OF THE LOCALIZATION ERRORS





Correlator’s output: the case of “normal” error



Correlator’s output: the case of anomalous error

VARIANCE OF THE LOCALIZATION ERRORS

For one-dimensional signals:

$Corr_{sn}(x) = conv[(s(x - x_0) + n(x))h(x)]$; $h(x)$ - impulse response of the filter

$$\frac{dCorr}{dx}|_{x=\hat{x}_0} = \frac{dCorr_{sh}}{dx}|_{x=\hat{x}_0} + \frac{dCorr_{nh}}{dx}|_{x=\hat{x}_0} = 0; \qquad \hat{x}_0 = x_0 + \varepsilon;$$

$$\frac{dCorr_{sh}(x)}{dx}|_{x=x_0} = 0;$$

$$\varepsilon\frac{d^2Corr_{sh}}{dx^2}|_{x=x_0} + \frac{dCorr_{nh}}{dx}|_{x=\hat{x}_0} = 0; \langle\varepsilon^2\rangle\left|\frac{d^2Corr_{sh}}{dx^2}|_{x=x_0}\right|^2 = \left\langle\left|\frac{dCorr_{nh}}{dx}\right|^2\right\rangle$$

$$\sigma_\varepsilon^2 = \langle\varepsilon^2\rangle = \frac{N_0}{4\pi^2\bar{f}_2^2E_a}\frac{\int_{-\infty}^{\infty} f^2|\alpha(f)|^2df\int_{-\infty}^{\infty} f^2|H(f)|^2df}{\left(\int_{-\infty}^{\infty} f^2\alpha(f)H(f)df\right)^2} \geq \frac{N_0}{4\pi^2\bar{f}_2^2E_a} = \frac{N_0}{4\pi^2E_{aa}},$$

when $H(f) = \alpha^*(f)$, where

$$\alpha(f) = \int_{-\infty}^{\infty} s(x)\exp(i2\pi fx)dx; \; H(f) = \int_{-\infty}^{\infty} h(x)\exp(i2\pi fx)dx; \; E_a = \int_{-\infty}^{\infty}|\alpha(f)|^2df$$

$$\bar{f}_x^2 = \frac{\int_{-\infty}^{\infty} f^2|\alpha(f)|^2df}{\int_{-\infty}^{\infty}|\alpha(f)|^2df}; \; E_{aa} = 4\pi^2\int_{-\infty}^{\infty} f^2|\alpha(f)|^2df = \int_{-\infty}^{\infty}\left|\frac{da(x)}{dx}\right|^2dx$$

- energy of signal's derivative

For 2-D signals:

$$\sigma_x^2 = \frac{1}{4\pi^2}\frac{\bar{f}_y^2}{\bar{f}_x^2\bar{f}_y^2 - \left(\bar{f}_{xy}^2\right)^2}\frac{N_0}{E_a}; \qquad \sigma_y^2 = \frac{1}{4\pi^2}\frac{\bar{f}_x^2}{\bar{f}_x^2\bar{f}_y^2 - \left(\bar{f}_{xy}^2\right)^2}\frac{N_0}{E_a};$$

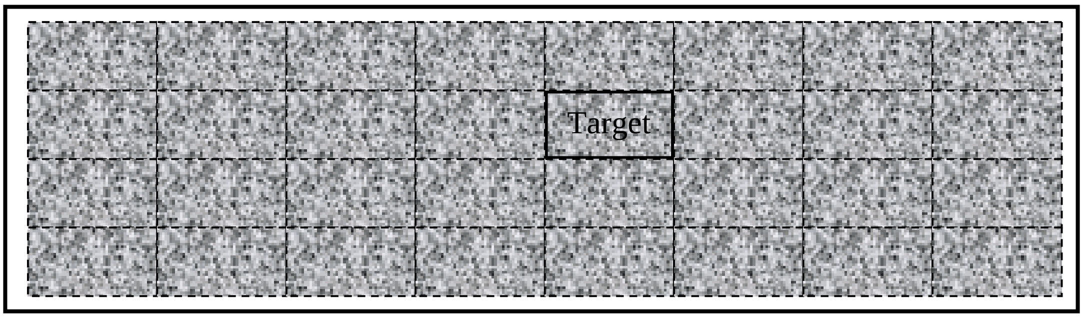
$$\sigma_{xy}^2 = \frac{1}{4\pi^2}\frac{\bar{f}_{xy}^2}{\bar{f}_x^2\bar{f}_y^2 - \left(\bar{f}_{xy}^2\right)^2}\frac{N_0}{E_a}.$$

where

$$E_a = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha(f_x, f_y)|^2df_xdf_y \qquad \bar{f}_x^2 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x^2|\alpha(f_x, f_y)|^2df_xdf_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha(f_x, f_y)|^2df_xdf_y}$$

$$\bar{f}_y^2 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_y^2|\alpha(f_x, f_y)|^2df_xdf_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha(f_x, f_y)|^2df_xdf_y}; \qquad \bar{f}_{xy}^2 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_xf_y|\alpha(f_x, f_y)|^2df_xdf_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha(f_x, f_y)|^2df_xdf_y}$$

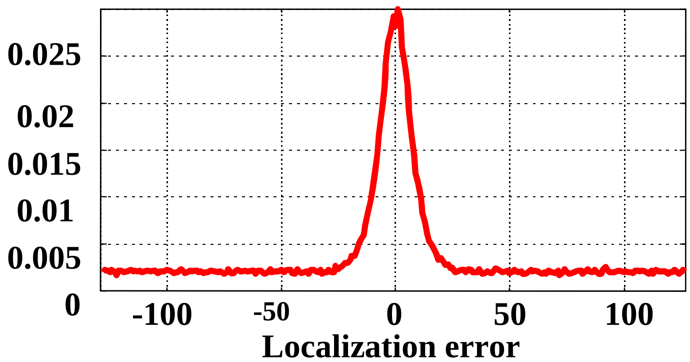
LOCALIZATION RELIABILITY



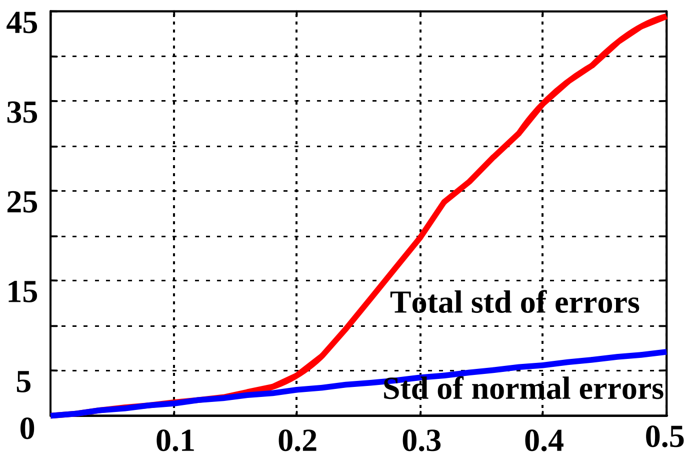
Probability of anomalous (false detection) errors:

$$P_{ae} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\frac{n^2}{2}\right)\left\{1-\left[\Phi\left(\sqrt{\frac{E_a}{N_0}}+n\right)\right]^{Q-1}\right\}dn$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} exp\left(-\frac{n^2}{2}\right)dn$, ***Q=O(Search area/Target size).***



Experimental distribution density of the localization error for a Gaussian-shaped impulse ($\sigma_a = 10$ in scale 1:256) and $\sqrt{E_a / N_0} = 1.5$ (100000 realizations)



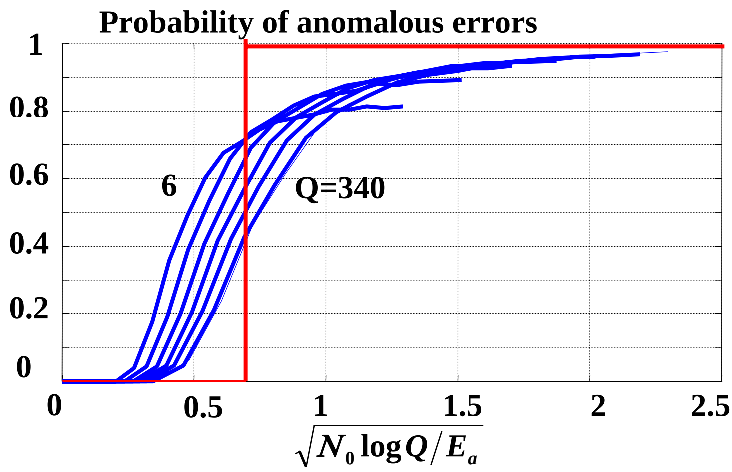
Standard deviation of the localization error as a function of noise-to-signal error for Gaussian-shaped impulse with (10000 realizations)

The localization reliability threshold:
By L'Hospital's rule we have:

$$\lim_{Q\to\infty}\left(\ln\left[\Phi\left(\sqrt{\frac{E_a}{N_0}}+n\right)\right]^{Q-1}\right)=\lim_{Q\to\infty}\left[\frac{d}{dQ}\ln\Phi\left(\mathrm{E}\sqrt{\ln Q}\right)\Big/\frac{d}{dQ}\frac{1}{Q}\right]=$$

$$\lim_{Q\to\infty}\frac{-\mathrm{E}}{\Phi\left(\mathrm{E}\sqrt{\ln Q}\right)}Q^{\left(1-\varepsilon^2/2\right)}=\begin{cases}-\infty, & if\ \ \mathrm{E}^2/2>1\\ 0, & if\ \ \mathrm{E}^2/2\le 1\end{cases}.\quad \text{where } \mathrm{E}=\sqrt{E_a/N_0\ln Q}$$

Therefore, $\lim_{Q\to\infty} P_{ae}=\begin{cases}1, if\ E_a/2N_0\le \ln Q\\ 0, if\ E_a/2N_0>\ln Q\end{cases}.$



Probability of anomalous errors as a function of normalised noise-to-signal ratio $\sqrt{N_0\, lnQ/E_a}$ for localization of rectangle impulses of 2;5;11;21;41;81;161 samples within an interval of 1024 samples (10000 realizations). The theoretical threshold value of the normalized noise-to-signal ratio is $\sqrt{2}/2\approx 0.707$

Localization accuracy of the "optimal" filter for non-white noise with spectral density $N_0\left|H_n\left(f_x, f_y\right)\right|^2$

$$\sigma^2_{x,NW} = \frac{\bar{f}^2_{y,NW}}{\bar{f}^2_{y,NW}\,\bar{f}^2_{y,NW} - \left(\bar{f}^2_{xy,NW}\right)^2}\frac{N_0}{4\pi^2 E_{a,NW}}; \qquad \sigma^2_{y,NW} = \frac{\bar{f}^2_{x,NW}}{\bar{f}^2_{y,NW}\,\bar{f}^2_{y,NW} - \left(\bar{f}^2_{xy,NW}\right)^2}\frac{N_0}{4\pi^2 E_{a,NW}}$$

$$\sigma^2_{xy,NW} = \frac{\bar{f}^2_{xy,NW}}{\bar{f}^2_{y,NW}\,\bar{f}^2_{y,NW} - \left(\bar{f}^2_{xy,NW}\right)^2}\frac{N_0}{4\pi^2 E_{a,NW}}$$

$$E_{a,NW} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|\alpha\left(f_x, f_y\right)\right|^2}{\left|H_n\left(f_x, f_y\right)\right|^2}df_x df_y; \qquad \bar{f}^2_{x,NW} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x^2 \frac{\left|\alpha\left(f_x, f_y\right)\right|^2}{\left|H_n\left(f_x, f_y\right)\right|^2}df_x df_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|\alpha\left(f_x, f_y\right)\right|^2}{\left|H_n\left(f_x, f_y\right)\right|^2}df_x df_y}$$

$$\bar{f}^2_{y,NW} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_y^2 \frac{\left|\alpha\left(f_x, f_y\right)\right|^2}{\left|H_n\left(f_x, f_y\right)\right|^2}df_x df_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|\alpha\left(f_x, f_y\right)\right|^2}{\left|H_n\left(f_x, f_y\right)\right|^2}df_x df_y}; \qquad \bar{f}^2_{xy,NW} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x f_y \frac{\left|\alpha\left(f_x, f_y\right)\right|^2}{\left|H_n\left(f_x, f_y\right)\right|^2}df_x df_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|\alpha\left(f_x, f_y\right)\right|^2}{\left|H_n\left(f_x, f_y\right)\right|^2}df_x df_y}$$

Compare the potential localization accuracy in the cases of white and non-white noise for the same input noise variance. For the sake of simplicity we shall consider a one-dimensional case. In this case for non-white noise

$$\sigma^2_{x,NW} = \frac{1}{\bar{f}^2_{x,NW}}\frac{N_0}{4\pi^2 E_{a,NW}}.$$

Express this through the error variance σ_x^2 for the case of white noise :

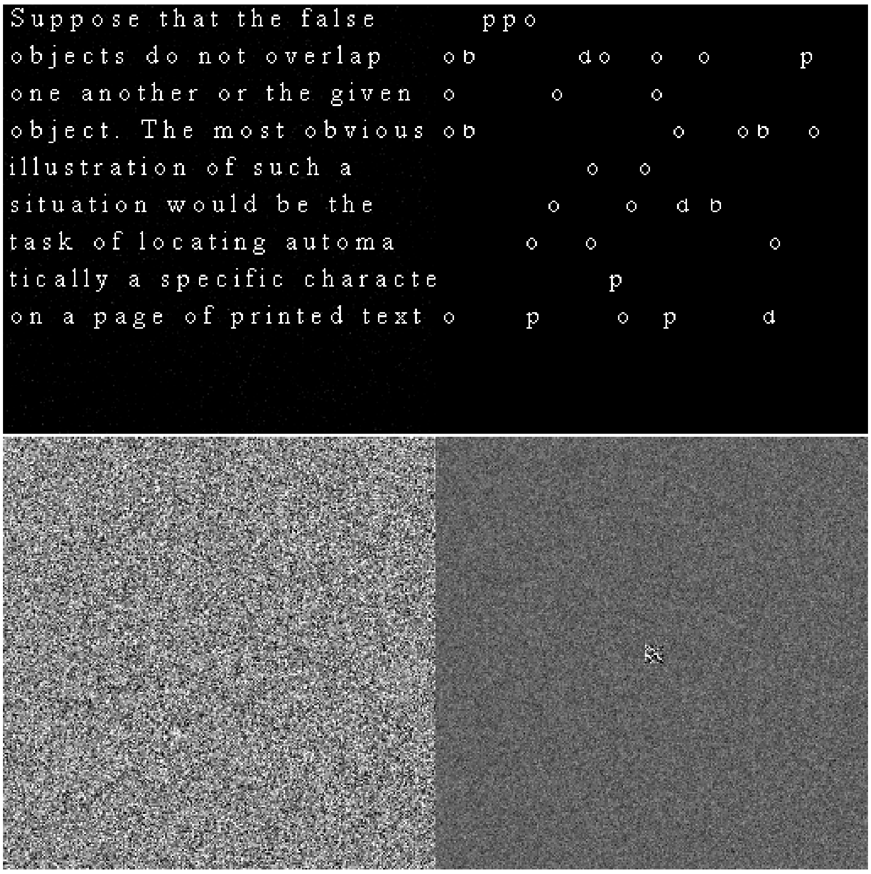
$$\sigma^2_{x,NW} = \frac{1}{\bar{f}^2_{x,NW}}\frac{N_0}{4\pi^2 E_{a,NW}} = \sigma_x^2\frac{E_a\,\bar{f}^2_x}{E_{a,NW}\,\bar{f}^2_{x,NW}} = G\sigma_x^2.$$

where factor G shows the change of error variance owing to non-whiteness of noise. Because all functions involved in this factor are nonnegative, the following inequality takes place:

$$G = \frac{\int_{-\infty}^{\infty} f_x^2\left|\alpha\left(f_x\right)\right|^2 df_x}{\int_{-\infty}^{\infty} f_x^2\frac{\left|\alpha\left(f_x\right)\right|^2}{\left|H_n\left(f_x\right)\right|^2}df_x} \le \frac{\int_{-\infty}^{\infty} f_x^2\frac{\left|\alpha\left(f_x\right)\right|^2}{\left|H_n\left(f_x\right)\right|^2}df_x \int_{-\infty}^{\infty}\left|H_n\left(f_x\right)\right|^2 df_x}{\int_{-\infty}^{\infty} f_x^2\frac{\left|\alpha\left(f_x\right)\right|^2}{\left|H_n\left(f_x\right)\right|^2}df_x} = \int_{-\infty}^{\infty}\left|H_n\left(f_x\right)\right|^2 df_x = 1$$

Therefore, the potential localization accuracy in the presence of non- white noise is always better than that for white noise with the same variance.

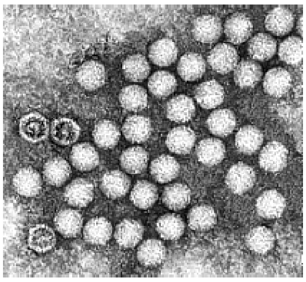
Target location in images with non-overlapping non-target objects

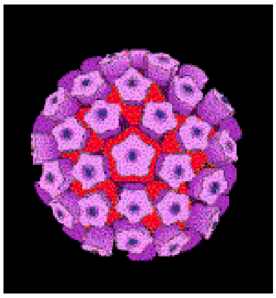


Detection, by means of matched filtering, of a character with and without non-target characters in the area of search. Upper row, left image: noisy image of a printed text with standard deviation of additive noise 15 (within signal range 0-255). Upper row, right image: results of detection of character “**o**” (right); one can see quite a number of false detections. Bottom row, left: a noisy image with a single character “**o**” in its center; standard deviation of the additive noise is 175. Highlighted center of the right image shows that the character is correctly localized by the matched filter.

Correlational accumulation (averaging): Application examples

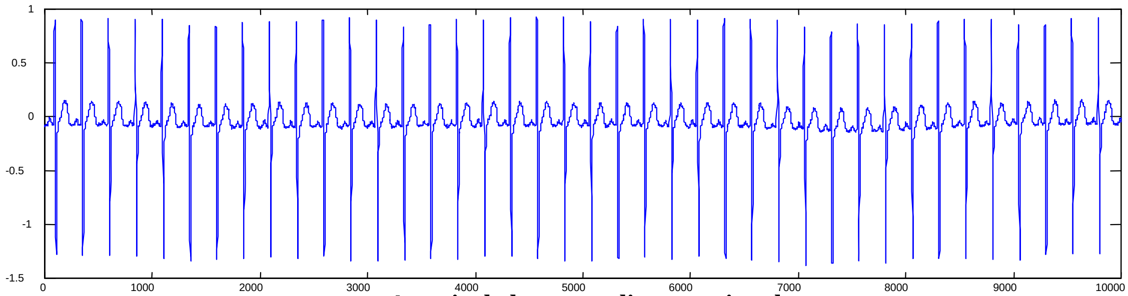
- **Processing of electron micrographs**





Examples of electron micrographs of virus particles and 3D reconstruction of the virus

- High resolution electrocardiography

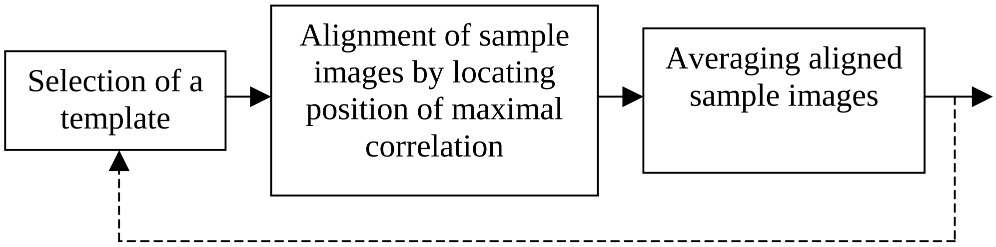


A typical electrocardiogram signal

- "Non-local methods" for image denoising

Adopted from: A.Buades B. Coll J.-M. Morel, Image and movie denoising by nonlocal means, http://www.dma.ens.fr/culturemath/maths/mathapli/imagerie-Morel/Buades-Coll-Morel-movies.pdf

Correlational accumulation: the principle



OUT=img_corr_accum_demo(256,1000,0.5);