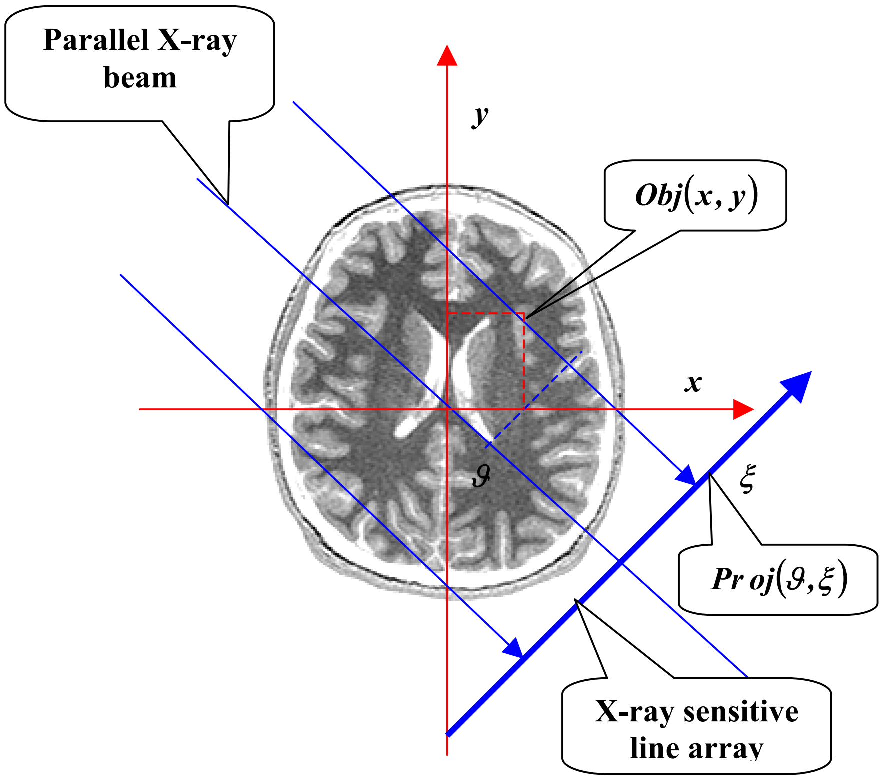
L. Yaroslavsky,

FROM PHOTOGRAPHY TO *.GRAPHIES: UNCONVENTIONAL IMAGING TECHNIQUES

A short course at Tampere University of Technology,
Tampere, Finland, Sept. 3 – Sept. 14, 2001

Lecture 4.
PRINCIPLES OF RECONSTRUCTIVE TOMOGRAPHY

Lecture 4 PRINCIPLES OF RECONSTRUCTIVE TOMOGRAPHY



Schematic diagram of parallel beam projection tomography

In computer tomography, a set of object's projections taken a different observation angles is measured and used for subsequent reconstruction of the object.
The relationship between object $\boldsymbol{Obj(x, y)}$ and its projection

$$\Pr oj(\vartheta,\xi) = \int_X \int_Y Obj(x,y)\delta(\xi - x\cos\vartheta - y\sin\vartheta)dxdy$$

is an integral one and is called ***Radon Transform***. Since projections are taken over a finite set of the observation angles, the process of projecting can be regarded as a specific type of 2-D signal discretization.

Projection theorem:

$$\boldsymbol{Pr\,Sp(\vartheta, f) = \int_{-\infty}^{\infty} Pr\,oj(\vartheta,\xi)exp(i2\pi f\xi)d\xi =}$$

$$\boldsymbol{\int_{-\infty}^{\infty}\left\{\int_X \int_Y Obj(x,y)\delta(\xi - x\cos\vartheta - y\sin\vartheta)dxdy\right\}exp(i2\pi f\xi)d\xi =}$$

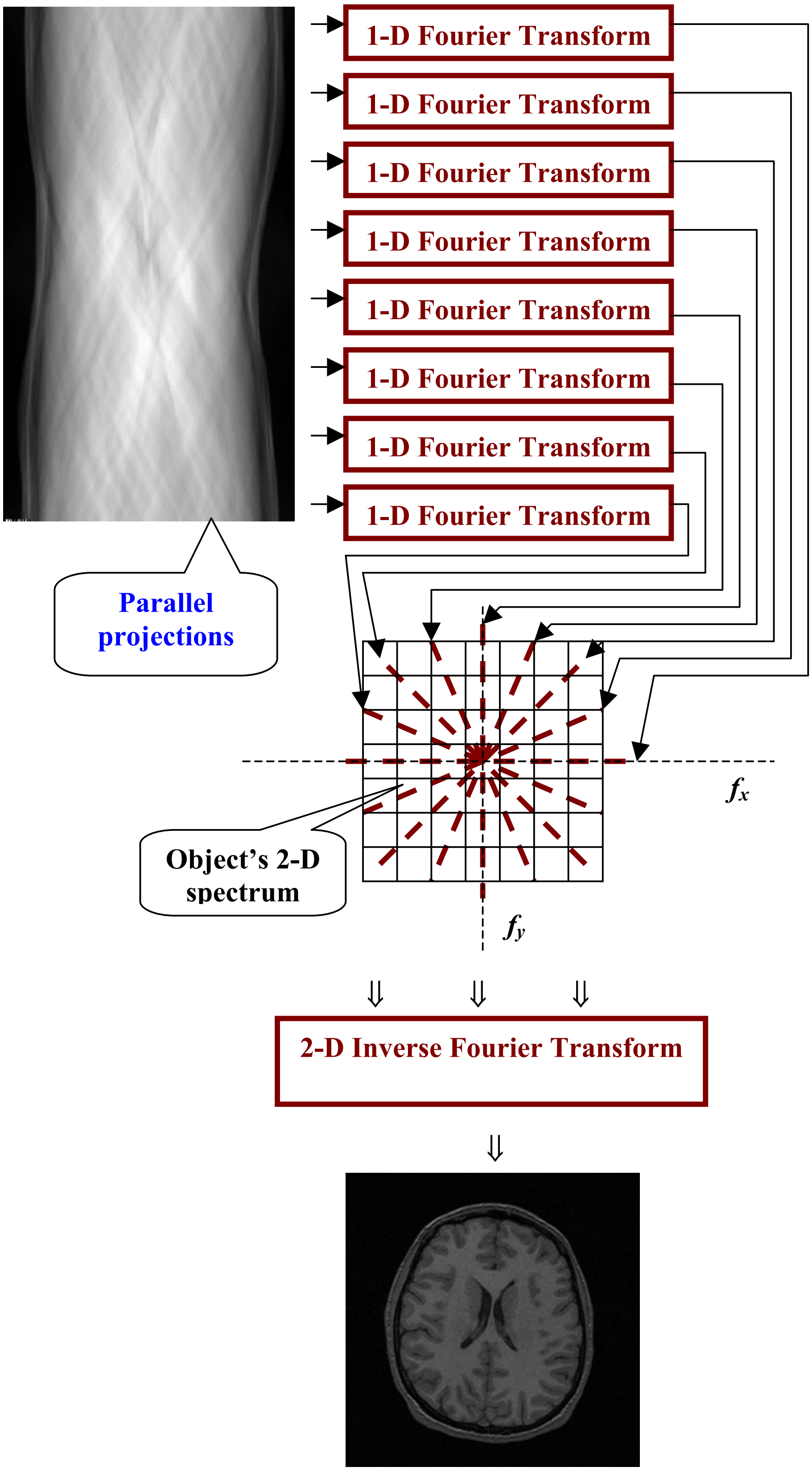
$$\boldsymbol{\int_X \int_Y Obj(x,y)exp[i2\pi(fx\cos\vartheta + fy\sin\vartheta)]dxdy =}$$

$$\boldsymbol{SpObj(f\cos\vartheta, f\sin\vartheta),}$$

where

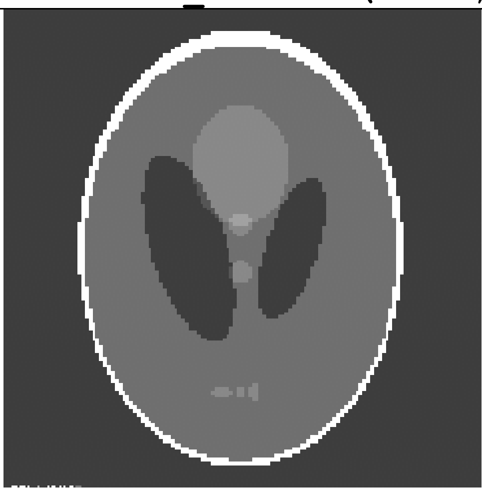
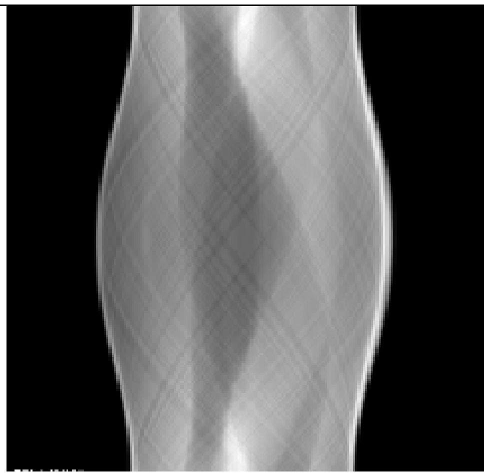
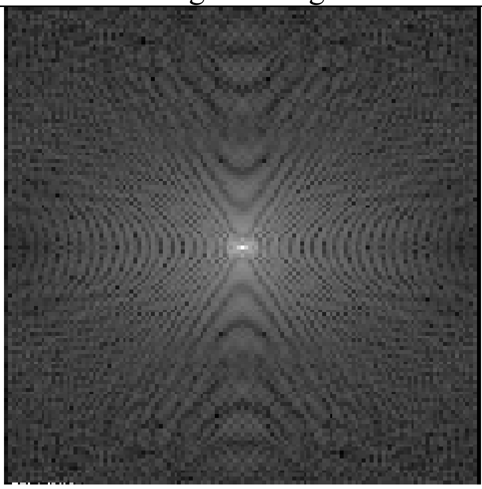
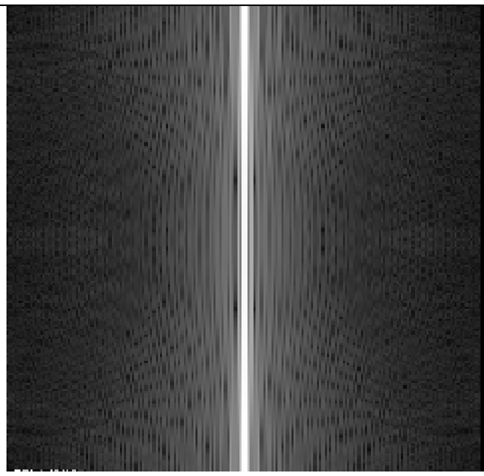
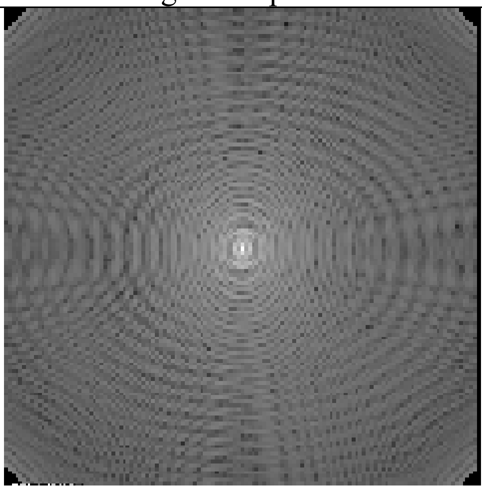
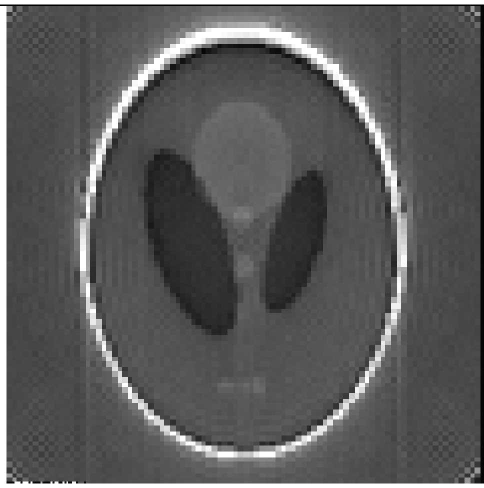
$$\boldsymbol{SpObj(f_x, f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Obj(x,y)exp[i2\pi(f_x x + f_y y)]dxdy}$$

Fourier method of tomographic reconstruction



Fourier method of tomographic reconstruction:

```
OUTIMG= inv_radonFFT(LintX,LintY)
```

 Original image	 Image projections
 Image 2-D spectrum	 1-D spectra of projections
 Reconstructed spectrum	 Reconstructed image; LintX=LintY=8

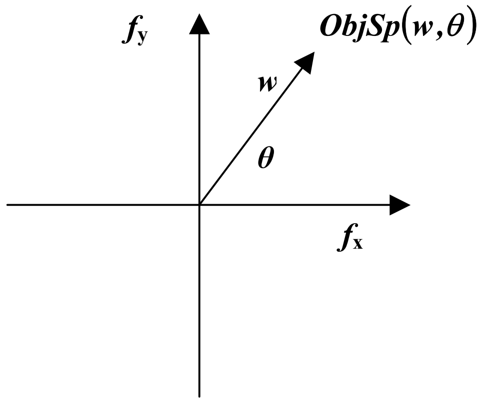
“Filtered Back Projection” method for tomographic synthesis

A 2-D function $Obj(x,y)$ can be found from its 2-D Fourier spectrum $ObjSp(f_x, f_y)$ as

$$Obj(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ObjSp(f_x, f_y)\exp[-i2\pi(f_x x+f_y y)]df_x df_y \tag{1}$$

In polar co-ordinate system (w,θ) in frequency domain

$$f_x = w\cos\theta;\; f_y = w\sin\theta;\; df_x df_y = |w| dw d\theta;\; \theta\in[0,\pi]: \tag{2}$$



obtain:

$$\int_0^{\pi} d\theta \int_{-\infty}^{\infty} |w| ObjSp(w,\theta)\exp[-i2\pi w(x\cos\theta + y\sin\theta)]dw =$$

$$\int_0^{\pi} Fproj(x\cos\theta + y\sin\theta)d\theta, \tag{3}$$

where

$$FBproj(x\cos\theta + y\sin\theta) = \int_{-\infty}^{\infty} |w| ObjSp(w,\theta)\exp[-i2\pi w(x\cos\theta + y\sin\theta)]dw. \tag{4}$$

Eq. (3) describes reconstruction of object signal $obj(x,y)$ as a “back projection” of signal $FBproj(x\cos\theta + y\sin\theta)$ of Eq. (4) that is obtained as inverse Fourier Transform of 1-D spectrum $ObjSp(w,\theta)$ modified by a filter with frequency response $|w|$. Spectrum $ObjSp(w,\theta)$ is a section of the object’s 2-D spectrum along straight lines that goes under angle θ. According to the projection theorem, it is 1-D Fourier Transform the corresponding projection of the object. Therefore, Eq. (3)

represents reconstruction of object signal $obj(x, y)$ as a result of accumulating "back projections" of its "filtered projections" $FBproj(x\cos\theta + y\sin\theta)$.

Algorithm:

- Take 1-D Fourier transform of all projections;
- Multiply spectra by $|w|$;
- Take inverse 1-D Transform of the modified spectra.
- For each of the reconstructed filtered projections, form an image by replicating the projection along orthogonal co-ordinate; rotate obtained image by the corresponding angle.
- Sum up all obtained in this way images.

"Inverse filtering" method for tomographic synthesis

Back projection is a simple computational procedure that converts 1-D projection signal $Proj(\vartheta, \xi)$ into a 2-D signal by repeating it in a 2-D co-ordinate system (***x***,***y***) along the direction that has angle ϑ with axis ***x***: projection values at co-ordinate ξ are repeated for all ***x*** and ***y*** such that $\xi = x\cos\vartheta + y\sin\vartheta$:

$$Bproj(x, y; \vartheta) = Proj(\vartheta, \xi = x\cos\vartheta + y\sin\vartheta) \tag{5}$$

One can accumulate back projections for all range of angles

$$\overline{obj(x, y)} = \int_{\Theta} Bproj(x, y; \vartheta)d\vartheta = \int_{\Theta} Proj(x\cos\vartheta + y\sin\vartheta)d\vartheta \tag{6}$$

and then attempt to reconstruct the object from such accumulated back projections. Conversion $obj(x, y) \Rightarrow \overline{obj(x, y)}$ by projecting and back projecting is a linear transformation. Let's find its point spread function:

$$\overline{Obj(x, y)} = \int\int Obj(\bar{x}, \bar{y})PSF(x, y; \bar{x}, \bar{y}), d\bar{x}d\bar{y} = \int_{\Theta} Proj(x\cos\vartheta + y\sin\vartheta)d\vartheta =$$

$$\int_{\Theta}\int_{X}\int_{Y} Obj(\bar{x}, \bar{y})h(x\cos\vartheta + y\sin\vartheta - \bar{x}\cos\vartheta - \bar{y}\sin\vartheta)d\bar{x}d\bar{y}d\vartheta =$$

$$\int_{X}\int_{Y} Obj(\bar{x}, \bar{y})d\bar{x}d\bar{y}\int_{\Theta} h[(x - \bar{x})\cos\vartheta + (y - \bar{y})\sin\vartheta]d\vartheta . \tag{7}$$

where $h(.)$ is a point spread function of the projection sensor. Therefore,

$$PSF(x, y; \bar{x}, \bar{y}) = \int_{\Theta} h[(x - \bar{x})\cos\vartheta + (y - \bar{y})\sin\vartheta]d\vartheta , \tag{8}$$

from which it follows that that conversion $\boldsymbol{Obj}(x,y) \Rightarrow \overline{\boldsymbol{Obj}(x,y)}$ is shift invariant.

A linear transformation can be inverted by applying to the result of the transformation an inverse transformation. If the latter does not exist it can be replaced by a "pseudo-inverse" operator. For shift invariant system the inversion can be efficiently carried out in frequency domain.

Let $\boldsymbol{Sp\_obj}(f_x, f_y)$ is 2-D Fourier Transform of object signal $\boldsymbol{Obj}(x,y)$, $\boldsymbol{Sp\_\overline{obj}}(x,y)$ is 2-D Fourier Transform of signal $\overline{\boldsymbol{Obj}}(x,y)$, obtained by back projecting of projections of the object signal and $\boldsymbol{FrResp}(f_x, f_y)$ is frequency response of the back projecting operation. Spectra transformations made by back projection can be described as

$$\boldsymbol{Sp\_\overline{obj}}(f_x, f_y) = \boldsymbol{Sp\_obj}(f_x, f_y)\boldsymbol{FrResp}(f_x, f_y) \tag{9}$$

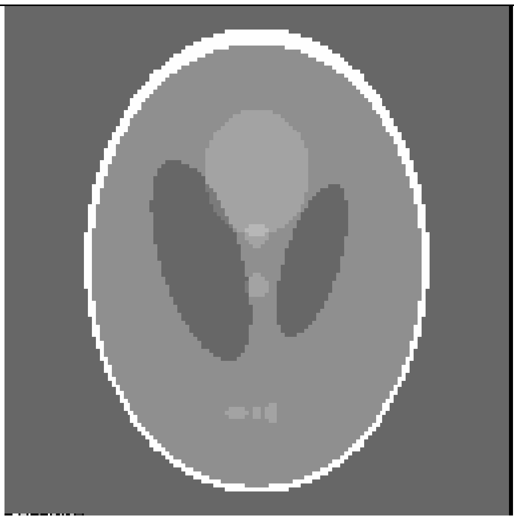
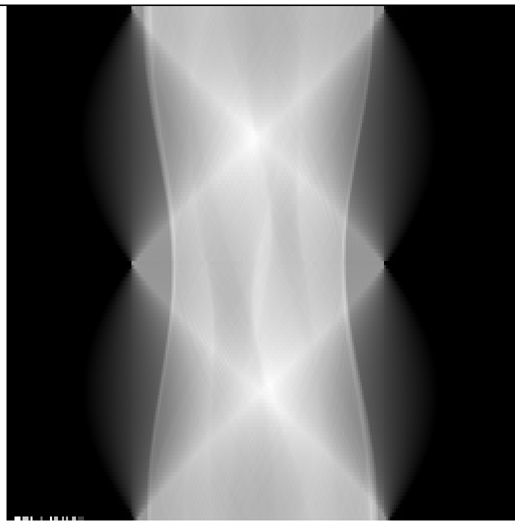
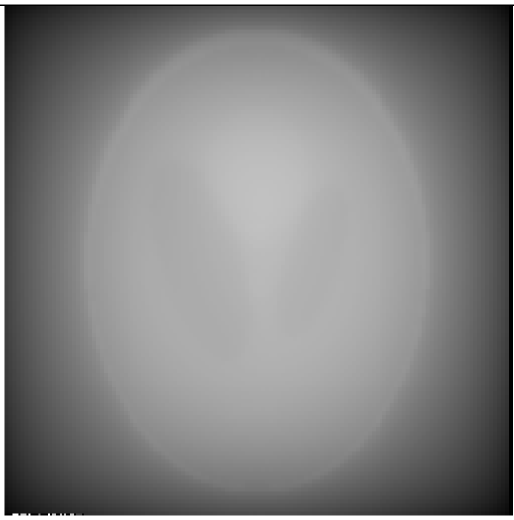
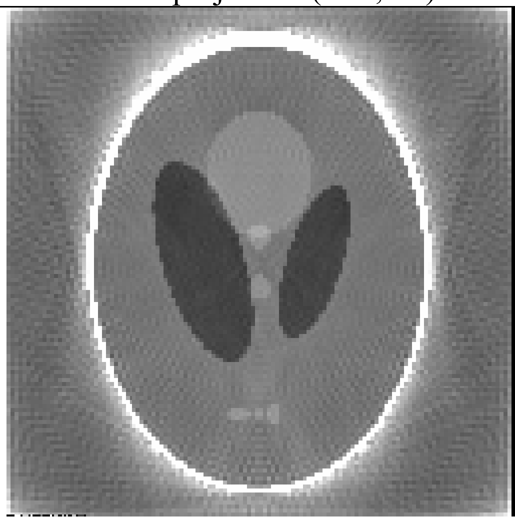
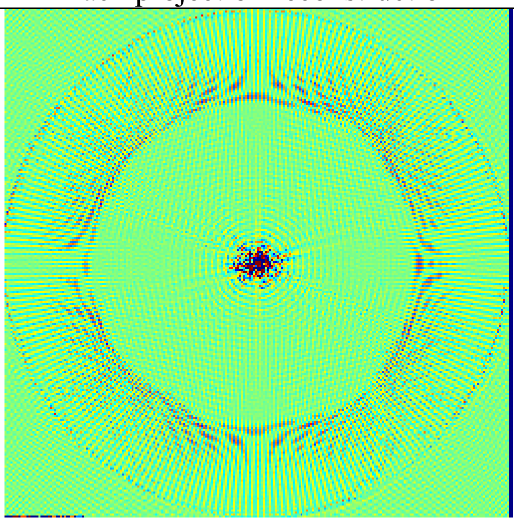
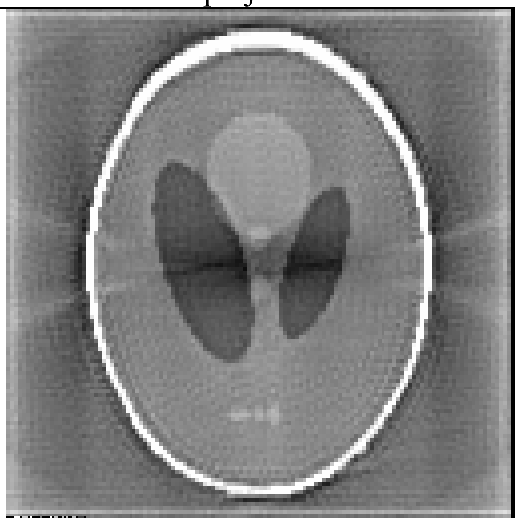
"Pseudo-inverse" filtering in frequency domain for obtaining an estimate $\overline{\boldsymbol{Sp\_obj}}(f_x, f_y)$ of the object's spectrum from the spectrum of its back projection can be implemented as filtering by a filter

$$\boldsymbol{PsInv}(f_x, f_y) = \frac{\left[\boldsymbol{FrResp}(f_x, f_y)\right]^*}{\left|\boldsymbol{FrResp}(f_x, f_y)\right| + \varepsilon^2(f_x, f_y)}, \tag{10}$$

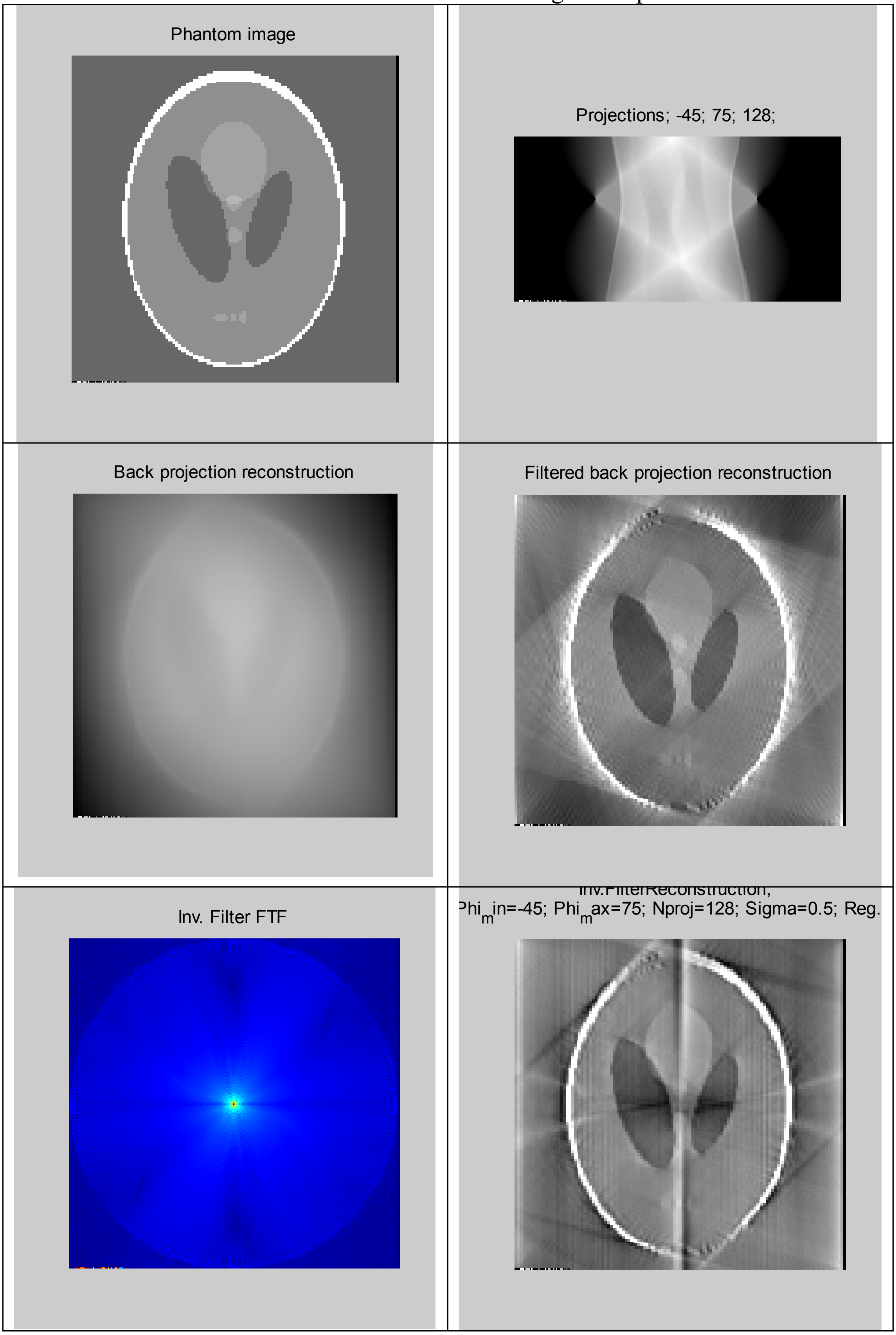
where $\varepsilon^2(f_x, f_y)$ is a regularizing function that can be interpreted as signal-to-noise ratio in frequency domain.

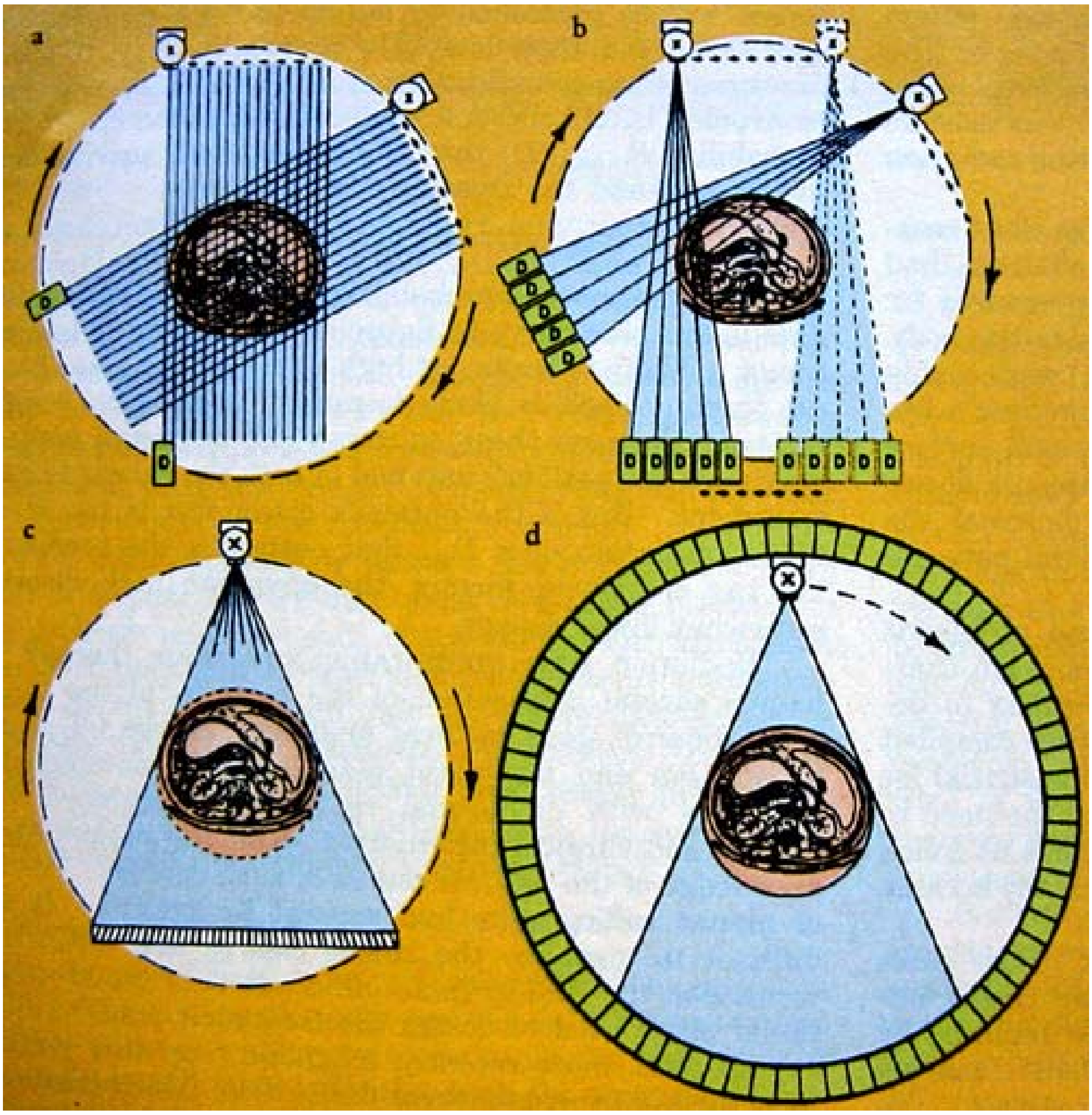
Demo:

[Projs,Phant_rec,Rest_phantom,Ph_filtered,InvFilter_PSF]=tomography_demo(128,-90,90,128,0.25);

 Test phantom	 128 projection (-90^o,90^o)
 Back projection reconstruction	 Filtered back projection reconstruction
 PsInvFilter PSF	 PsInvFilter reconstruction

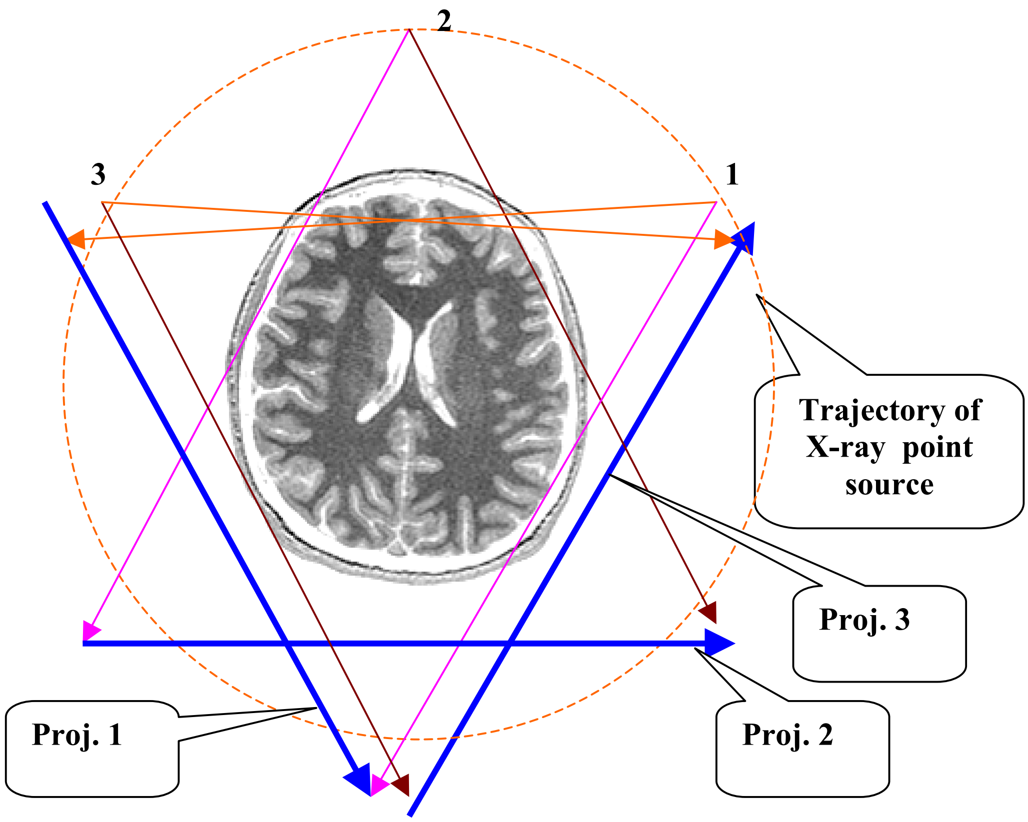
Reconstruction from incomplete set of projections:
Reconstruction limitations can be evaluated from image 2-D spectrum





Evolution of geometries of X-ray CT-scanners. a) : First generation scanner, in which the pencil x-ray beam is both translated and rotated to cover the body being imaged. b) Second generation scanner, with a diverging ffan beam and detector array that are translated and rotated. c) Third-generation scanner, with a fan-beam source that rotates around the body together with its detector bank. d) Fourth-generation detector, with a rotating fan-beam source and a stationary ring of detectors (*from: W. R. Hendee, X rays in Medicine, Physics today, Nov. 1995, p. 51*)

“Fan”- beam tomography



“Fan””-beam projection tomography schematic diagram

“Fan-beam” projections $\boldsymbol{Pr}_{fb}\left(\vartheta_{fb},\xi_{fb}\right)$ can be converted to parallel projections $\boldsymbol{Pr}_{pr}\left(\vartheta_{pr},\xi_{pr}\right)$ by means of an appropriate co-ordinate conversion $\left(\vartheta_{fb},\xi_{fb}\right)\Rightarrow\left(\vartheta_{pr},\xi_{pr}\right)$. This process requires accurate discrete signal interpolation techniques.

Test questions

1. Explain tomography as a discretization process
2. Formulate and prove projection theorem
3. Explain Fourier method of tomosynthesis and role of signal interpolation
4. Derive filtered back projection algorithm and explain associated signal processing issues
5. Explain “inverse” filtering method of tomosynthesis
6. Explain how can one evaluate potential image reconstruction distortions due to reconstruction from incomplete set of projections
7. Outline approaches to image reconstruction from fan beam projections