m satisfies [7], [8]

$$\text{Heller:}\quad d_{\text{free}} \le \min_{i\ge 1} \left\lfloor \frac{(m+i)c}{2(1-2^{-bi})} \right\rfloor \tag{1}$$

$$\text{Griesmer:}\quad \sum_{j=0}^{bi-1} \left\lceil \frac{d_{\text{free}}}{2^j} \right\rceil \le (m+i)c, \qquad i=1,2,\cdots. \tag{2}$$

For systematic encoding matrices we have the corresponding bounds [3]

$$\text{Heller:}\quad d_{\text{free}} \le \min_{i\ge 1} \left\lfloor \frac{(m(1-R)+i)c}{2(1-2^{-bi})} \right\rfloor \tag{3}$$

$$\text{Griesmer:}\quad \sum_{j=0}^{bi-1} \left\lceil \frac{d_{\text{free}}}{2^j} \right\rceil \le (m(1-R)+i)c, \qquad i=1,2,\cdots. \tag{4}$$

In Table III we list rate $1/3$ systematic polynomial ODP encoders for memories $1 \le m \le 30$. The corresponding nonsystematic encoders for memories $1 \le m \le 19$ are listed in Table IV. In Fig. 1 the free distances are compared with Heller's and Griesmer's upper bounds. For comparison we also show the optimum distance profile. (The distance profile is always the same for systematic and nonsystematic encoders [1], [3].)

Rate $1/4$ systematic polynomial ODP convolutional encoders for memories $1 \le m \le 30$ are listed in Table V and rate $1/4$ nonsystematic polynomial ODP convolutional encoders for memories $1 \le m \le 21$ are listed in Table VI. Finally, in Fig. 2 the free distances are related to Heller's and Griesmer's bounds.

The new convolutional codes combine a large free distance with an optimum distance profile and, thus might be attractive for use in various communication systems.

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The Weighted Coordinates Bound and Trellis Complexity of Block Codes and Periodic Packings

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***Abstract*—Weighted entropy profiles and a new bound, the weighted coordinates bound, on the state complexity profile of block codes are presented. These profiles and bound generalize the notion of dimension/length profile (DLP) and entropy/length profile (ELP) to block codes whose symbols are not drawn from a common alphabet set, and in particular, group codes. Likewise, the new bound may improve upon the DLP and ELP bounds for linear and nonlinear block codes over fields. However, it seems that the major contribution of the proposed bound is to the study of trellis complexity of block codes whose different coordinates are drawn from different alphabet sets. The label code of lattice and nonlattice periodic packings usually has this property. The construction of a trellis diagram for a lattice and some related bounds are generalized to periodic packings by introducing the fundamental module of the packing, and using the new bound on the state complexity profile. This generalization is limited to a given coordinate system. We show that any bounds on the trellis structure of block codes, and in particular, the bound presented in this work, are applicable to periodic packings.**

***Index Terms*—Entropy/dimension profiles, entropy/length profiles, lattices, periodic packings, trellis complexity.**

I. INTRODUCTION

Trellis diagrams suggest an efficient framework for soft-decision decoding algorithms for codes and lattices, such as the maximum-likelihood or the maximum *a posteriori* algorithms. Trellis complexity is a fundamental descriptive characteristic of both codes and lattices since it reflects the decoding complexity of these algorithms. The investigation of trellis diagrams of linear block codes has been an active research area during the last decade. Less attention has been directed to group codes and lattices in recent literature hitherto.

Under a given symbol permutation, any group code has a unique minimal biproper trellis [14]. An algorithm for computing the minimal trellis for a group code over a finite Abelian group has been presented by Vazirani *et al*. [27]. This algorithm extends the work of Kschischang and Sorokine [15] which treats linear codes over fields. The generalization of Vazirani *et al*. introduces the notions of p*-linear combinations* and p*-generator sequences*. The trellis product of the codewords of a p-generator sequence is minimal if and only if this sequence is *two-way proper*. A two-way proper p-generator sequence is a generalization of the trellis-oriented generator matrix [6], [15], for linear block codes over fields.

Measures of trellis complexity of block codes over a fixed alphabet set are bounded by the *entropy/length profile* (ELP) [18] which extends the *dimension/length profile* (DLP) of linear codes [7] to nonlinear codes. Several studies have addressed the problem of finding efficient permutations that meet the DLP bound, and hence minimize measures of trellis complexity (e.g., [3], [12], [13]). There is no measure equivalent to the DLP and ELP for block codes whose symbols are taken from alphabets of different sizes, such as

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Euclidean-space codes and group codes. An important application of these codes is to the construction of minimal trellises for lattices and more generally, packings. The coordinate system for a lattice is not unique. Distinct representations of a packing in different coordinate systems may not incorporate the same number of paths. For lattices and packings this degree of freedom is superimposed to the permutation problem for group codes.

The majority of the work in the field of trellis representation of group codes and lattices is included in [6], [8], and [9]. These papers introduce the basic concepts and define the algebraic derivation of trellises for lattices. In two recent papers [22] and [23], Tarokh and Blake have proposed three measures of trellis complexity which were defined as functions of the coding gain of the lattice. The study of the behavior of these functions led to lower bounds on the trellis complexity of lattices. These lower bounds were given as an explicit power of the coding gain. The main result of the above two papers is that the number of states and branches of trellis diagrams of lattices grows exponentially with the coding gain. Tarokh and Vardy [24] have shown that not even any *rational* lattice (which hence has a finite trellis) has a coordinate system in which the average number of states is upper-bounded by a function of the coding gain or the dimension. However, several measures of trellis complexity of *integral* lattices were bounded in the afore-mentioned work by the volume of the lattice. These bounds were improved and generalized by Banihashemi and Blake [2]. Likewise, in [1], focusing on the number of distinct paths of the trellis diagram of a lattice as the complexity measure, they lower-bounded this measure by a function of the coding gain. Trellis diagrams that minimize this complexity measure were derived for some well-known lattices in the above-mentioned work. The scope of all the above-referenced works is limited to lattice packings.

The construction of trellises for nonlattice packings has not been defined heretofore. The study of trellis representation of nonlattice packings has both theoretical and practical importance. This study may shed further light into the structure of these packings. Practically, this study is closely related to the complexity of soft-decoding algorithms for these packings which are the densest packings known in some dimensions. In dimensions 10, 11, 13, and 20 there are nonlattice packings denser than any known lattice ([5], [25]). Likewise, in many other dimensions there are nonlattice packings that have the same density as lattices.

This work comprises two intimately related main sections. The first one introduces a new lower bound on the state complexity profile of block codes under any coordinate ordering. The new bound applies to any block code. This bound, which will be henceforth referred to as the *weighted coordinates bound*, generalizes the notions of ELP and ELP bounds to block codes whose components are drawn from distinct alphabets. In particular, it may improve upon the well-known DLP bound on the state complexity profile (of linear block codes), and upon the ELP bound [18] for the general case. The new bound assigns arbitrary *weights* to the coordinates of the code, and accordingly modified (weighted) ELP are evaluated. The ELP bound, and in particular the DLP bound, are special cases of the new general class of bounds. It is illustrated that the new bound may be tighter than the DLP/ELP bound, and thereby distinguish codes that do not satisfy the two-way chain condition [7], [28].

The remainder of the correspondence is devoted to the study of the trellis complexity of *periodic packings*. A periodic packing is a union of a finite number of translates of an *orthogonal lattice*. All useful constructions of nonlattice packings (and evidently also lattice packings) comply with the above definition of periodic packings.

We define a *fundamental module* of a packing, and use this definition in conjunction with the weighted coordinates bound to generalize the construction of a trellis diagram for a lattice and some related ideas to periodic packings. There is a one-to-one correspondence between the points of the fundamental module and the label code of a periodic packing. Hence the fundamental module extracts the dynamical properties of the trellis diagram for the packing. In general, the fundamental module is a Euclidean-space code, and in particular, the fundamental module of a lattice is a group code. The results of [18] and those of the present work can thus be applied to the label code of periodic packings. However, the proposed generalization is limited to a *given* coordinate system, and thus it does not provide much information about the minimal trellis complexity of the packing.

This correspondence is organized as follows. The next section introduces the weighted entropy/length profiles and the weighted coordinates bound. In Section III, we present the generalization of the construction of a trellis representation for periodic packings. In Section IV, we extent the ELP concept to periodic packings, and present the *entropy/dimension profiles*. We use the weighted profiles in conjunction with the results of [18] to present bounds on trellis complexity of periodic packings. We illustrate the devised bounds by analyzing the trellis complexity of Constructions B and C of packings which are based on block codes. Finally, concluding remarks and a discussion of the computational complexity of the bound and its practical value are given in Section V.

II. The Weighted Coordinates Bound on the State Complexity Profile of Block Codes

In this section, we present a new lower bound on the state complexity profile of block codes under any coordinate ordering. The new technique can be used also to tighten other related bounds. The proofs of the claims in this section are very similar to the corresponding (unweighted) ELP-based theorems of [18], and thus they are omitted.

A length-n *sequence space* W is defined by an *index set* I, $I \triangleq \{1, 2, \cdots, n\}$ and by a set of *symbol alphabets* $\{A_i : i \in I\}$. W is the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$. That is, W is the set of all sequences $\boldsymbol{a} = (a_i, i \in I)$ with $a_i \in A_i$. A *block code* is a subset of W. When the symbol alphabets are groups, then W is called a *group sequence space*. We define the componentwise group operation in W

$$\boldsymbol{a} * \boldsymbol{b} \triangleq (a_i * b_i, i \in I)$$

where $a_i * b_i$ denotes the product of $\{a_i, b_i\} \subseteq A_i$ under the binary group operation in A_i. Evidently, W is a direct product group under the above operation. A *group code* is a subgroup of a group sequence space W.

A trellis $T = (V, A, E)$ for a block code C of length n is an edge-labeled directed graph in which the *vertex* (*state*) set V is the union of $n+1$ disjoint subsets, $V = \bigcup_{i=0}^{n} V_i$, where V_i is the set of vertices at level i, the *edge* set $E = \bigcup_{i=1}^{n} E_i$, consists of ordered triples

$$E_i = \{(u, \alpha, v) : u \in V_{i-1}, v \in V_i, \alpha \in A_i\}$$

where A_i is the alphabet set at index i, and $A = A_1 \times A_2 \times \cdots \times A_n$ is a finite alphabet set. Any path $(\boldsymbol{v}, \boldsymbol{\alpha}) \in V \times A$ from V_0 to V_n defines a state sequence, and the corresponding n-tuple sequence of edge labels $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is associated with a codeword of C. The state complexity of the trellis diagram at level i, $s_i(C)$, is the logarithm of the vertex count at this level, i.e., $s_i(C) \triangleq \log |V_i(C)|$. The sequence $\boldsymbol{s}(C) = \{s_i(C), 0 \le i \le n\}$ is the *state complexity profile* of C. All the logarithms in this study are assumed to be to the same arbitrary base. For a given coordinate ordering we denote $s_{\max}(C) \triangleq \max_i\{s_i(C)\}$. The minimum $s_{\max}(C)$ over all coordinate orderings is called the *state complexity* $s(C)$.

The block codes discussed in this section need not be linear. Furthermore, their different coordinates need not take on values from the same symbol set. We denote a block code by the pair (n, M), where n is the length of the code and M is its cardinality. The derived bounds refer to any trellis representation of the code. The diagram for nonlinear codes, which need not necessarily admit a minimal biproper trellis, need not be proper or one-to-one.

Let I be the index set $\{1, 2, \cdots, n\}$, and let J be a subset of I, $J \subseteq I$. In particular, we denote by i^- the subset consisting of the first i consecutive indices $[1, 2, \cdots, i]$, and similarly we denote $i^+ \triangleq [i+1, i+2, \cdots, n]$, with the convention $0^- = n^+ = \{\phi\}$ are the null set. The complementary set of J in I will be denoted by $I - J$. We denote by $P_J(C)$ the set of the projections of the codewords of C onto the indices of J. We make the (n, M) code C into an ensemble whose sample space is the entire set of codewords.

Similarly to [18], we assign the codewords a uniform probability of $1/M$. We denote by X_J a random $|J|$-tuple variable that takes on the values of the set $P_J(C)$ with probabilities that are induced by the uniform distribution of the codewords of C. We use the basic information-theoretic measures: the *entropy* of an ensemble X, $H(X)$, and the *mutual information* of the joint XY ensemble $I(X;Y)$. In what follows, we assign each coordinate of the code an arbitrary weight w_i, $1 \le i \le n$, and we evaluate some entropy profiles with respect to the chosen weight set.

Definition 1: The *ordered weighted ELP* (*ordered WELP*) of $C(n, M)$ with respect to the weight set $\boldsymbol{w} = \{w_i, 1 \le i \le n\}$ will be defined as the sequence $\boldsymbol{g}'(C, \boldsymbol{w}) = \{g'_i(C, \boldsymbol{w}), 0 \le i \le n\}$ where

$$g'_i(C, \boldsymbol{w}) \triangleq H(X_{i^-}) + \sum_{j=1}^{i} w_j. \tag{2.1}$$

Definition 2: The *ordered conditional WELP* of $C(n, M)$ with respect to the weight set $\boldsymbol{w} = \{w_i, 1 \le i \le n\}$ will be defined as the sequence $\boldsymbol{g}(C) = \{g_i(C, \boldsymbol{w}), 0 \le i \le n\}$, where

$$g_i(C, \boldsymbol{w}) \triangleq H(X_{i^-} \mid X_{i^+}) + \sum_{j=1}^{i} w_j. \tag{2.2}$$

Definition 3: The *weighted entropy/length profile* (*WELP*) of $C(n, M)$ with respect to the weight set $\boldsymbol{w} = \{w_i, 1 \le i \le n\}$ is the sequence $\boldsymbol{h}'(C) = \{h'_i(C, \boldsymbol{w}), 0 \le i \le n\}$ where

$$h'_i(C, \boldsymbol{w}) \triangleq \min_J \left\{ H(X_J) + \sum_{j \in J} w_j : |J| = i \right\}. \tag{2.3}$$

Definition 4: The *conditional weighted entropy/length profile* (*conditional WELP*) of $C(n, M)$ with respect to the weight set $\boldsymbol{w} = \{w_i, 1 \le i \le n\}$ is the sequence $\boldsymbol{h}(C) = \{h_i(C, \boldsymbol{w}), 0 \le i \le n\}$ where

$$h_i(C, \boldsymbol{w}) \triangleq \max_J \left\{ H(X_J \mid X_{I-J}) + \sum_{j \in J} w_j : |J| = i \right\}. \tag{2.4}$$

Lemma 1: For an (n, M) code C with weighted profiles $\boldsymbol{h}(C, \boldsymbol{w})$ and $\boldsymbol{h}'(C, \boldsymbol{w})$

$$h'_i(C, \boldsymbol{w}) = \log M + \sum_{j=1}^{n} w_j - h_{n-i}(C, \boldsymbol{w}). \tag{2.5}$$

Using the above lemma along with the obvious relations

$$\boldsymbol{h}(C, \boldsymbol{w}) \ge \boldsymbol{g}(C, \boldsymbol{w})$$

and

$$\boldsymbol{h}'(C, \boldsymbol{w}) \le \boldsymbol{g}'(C, \boldsymbol{w})$$

we prove the main result of this section.

Theorem 1 (The Weighted Coordinates Bound): The state complexity profile of an (n, M) code C is bounded by

$$s_i(C) \ge \log M + \sum_{j=1}^{n} w_j - h_{n-i}(C, \boldsymbol{w}) - h_i(C, \boldsymbol{w}), \quad 1 \le i \le n-1 \tag{2.6}$$

where the above weighted profiles are evaluated with respect to an arbitrary weight set $\boldsymbol{w}$.

When the bound is applied to block codes over a fixed alphabet set then some of the resulting bounds will be looser than the ELP bound (or its DLP version for linear codes) while others may be equal to or larger than the ELP bound as illustrated in the following simple example. The lower bound on the state complexity profile may thus be defined as the maximal bound given by (2.6), where the maximization is done with respect to the weight set. For the all-zeros weight set the different profiles reduce to the corresponding entropy/length profiles. In particular, for linear block codes over a common alphabet set this choice provides the different dimension/length profiles.

Example 2.1: Consider the binary linear code C defined by the generator matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

In this example, all the logarithms are taken to the binary base. The DLP values at indices 2 and 4 are $h_2(C, \boldsymbol{0}) = 1$ and $h_4(C, \boldsymbol{0}) = 2$, and thus the ELP (DLP) bound at indices 2 and 4 is 1 state. We now assign weight $1/2$ to the first two coordinates and weight zero to the rest, namely: $\boldsymbol{w} = \{1/2, 1/2, 0, 0, 0, 0\}$. For this choice we have $h_2(C, \boldsymbol{w}) = 1$ and $h_4(C, \boldsymbol{w}) = 2$. Therefore, this choice of weights provides the bound of two states at indices 2 and 4. Clearly, this bound may be achieved at both indices in the permutation that replaces the last two positions with the third and the fourth positions.

Example 2.2: We consider a group code C which consists of the following 12 codewords:

$$\begin{matrix} 0000 & 0031 & 0220 & 0251 & 0440 & 0411 \\ 1300 & 1331 & 1520 & 1551 & 1140 & 1111. \end{matrix}$$

This code is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_2$. The WELP of the code with the all-zeros weight set is $\boldsymbol{h}(C, \boldsymbol{0}) = \{0, 0, \log 3, \log 6, \log 12\}$. This profile provides the lower bound of two states at level 2 of the trellis representation of the code under any symbol permutation. However, when we use the following weight set for the coordinates of the code $\boldsymbol{w} = \{\log 1.5, 0, 0, \log 1.5\}$ we obtain $h_2(C, \boldsymbol{w}) = \log 3$ and $h_4(C, \boldsymbol{w}) = \log 27$, yielding the lower bound of three states at level 2. This bound is achieved, for example, under the given order of the symbols.

The bound on the state complexity ([18, Theorem 6]) can be rewritten with the WELP measures replacing the ELP values.

Theorem 2 (State Complexity): Let C be an (n, M) code, and let $\{z_1, z_2, \cdots, z_N\}$ be any set of N positive integers provided that $z_1 + z_2 + \cdots + z_N = n$. The state complexity of any trellis representation for C under any coordinate ordering is bounded by

$$s(C) \ge \frac{\log M + \sum_{j=1}^{n} w_j - \sum_{j=1}^{N} h_{z_j}(C, w)}{N - 1} \tag{2.7}$$

where $h_i(C, \boldsymbol{w})$ is the ith component of the WELP of C with respect to an arbitrary weight set $\boldsymbol{w} = \{w_i, 1 \le i \le n\}$.

We now depart to a study of the trellis complexity of periodic packings. Our approach is based on the fundamental module of the packing. We use the weighted coordinates bound to address the state complexity profile of packings.

III. Trellis Construction for Periodic Packings

A packing P is a discrete set of points (vectors) in a real n-dimensional Euclidean space $\mathbb{R}^n$. If this set of points forms an additive group then this is a lattice L. The vectors that represent the center points of the packing may span m dimensions, $m < n$. In this case the packing can be expressed as a subset of $\mathbb{R}^m$. Thus without loss of generality, in this study, we consider only full-rank packings that span an n-dimensional space, and n will henceforth designate the dimension of the packing.

An n-dimensional lattice L may be defined by means of its generator matrix G. This matrix consists of n linearly independent vectors, and all the lattice points are $\mathbb{Z}$-span (integer combinations) of these vectors. The (fundamental) volume of a lattice is the volume of a fundamental n-dimensional cell, the *Voronoi cell*, associated with a single lattice point. The faces of the Voronoi cell are hyperplanes midway between two adjacent lattice points.

In the sequel, we generate high-dimensional spaces by a *direct sum* of other spaces, where the *direct sum* $A_1 \oplus A_2$ of the orthogonal spaces A_1 and A_2 is a third space whose dimension is the sum of the dimensions of the constituent subspaces, A_1 and A_2, and it spans a space of vectors that can be expressed as a direct sum of two vectors $\boldsymbol{a}_1 \in A_1$ and $\boldsymbol{a}_2 \in A_2$. Namely,

$$A_1 \oplus A_2 = \{\boldsymbol{a}_1 + \boldsymbol{a}_2 : \boldsymbol{a}_1 \in A_1, \boldsymbol{a}_2 \in A_2\}.$$

Definition 5: An n-*dimensional hyperbox* is a polytope in $\mathbb{R}^n$ that includes all the points $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ provided that $a_i \le x_i < b_i$, $\forall 1 \le i \le n$, for some set of n constant pairs (a_i, b_i). We also denote $l_i = b_i - a_i$.

Definition 6: An n-dimensional lattice is called an *orthogonal lattice* (e.g., [22]) if it is the $\mathbb{Z}$-span of n mutually orthogonal vectors $\{v_1, v_2, \cdots, v_n\}$, i.e., $v_i \cdot v_j = 0$, whenever $i \ne j$.

By definition, an orthogonal lattice is a direct sum of n one-dimensional lattices $\{l_i\mathbb{Z}, 1 \le i \le n\}$, where $\mathbb{Z}$ denotes the set of integer numbers. An orthogonal lattice is associated with a distance set of the constituent one-dimensional lattices. This distance set will be denoted by $\boldsymbol{l}$, $\boldsymbol{l} \triangleq \{l_i, 1 \le i \le n\}$, and the corresponding lattice will be denoted by $\boldsymbol{l}\mathbb{Z} = l_1\mathbb{Z} \oplus l_2\mathbb{Z} \oplus \cdots \oplus l_n\mathbb{Z}$. Let B be a Voronoi cell of a lattice. We say that $B + \sum_{i=1}^n r_i \cdot \boldsymbol{e}_i$ is a shifted version of a Voronoi cell (with respect to a given coordinate system), where $\{r_i\}$ are n arbitrary real numbers, and $\boldsymbol{e}_i$ is a unit vector along the ith coordinate.

The traditional definition of a periodic packing P in $\mathbb{R}^n$ (cf. [5, p. 8]), pertains to a packing that comprises n *linearly independent* vectors $\{v_1, v_2, \cdots, v_n\}$ such that

$$P + v_i = P, \qquad \forall i \in \{1, 2, \cdots, n\}.$$

However, along this work, we use a more narrow definition:

Definition 7: An n-dimensional packing P is said to be *periodic* if there exist n *mutually orthogonal* vectors $\{v_1, v_2, \cdots, v_n\}$ such that

$$P + v_i = P, \qquad \forall i \in \{1, 2, \cdots, n\}.$$

Consequently, a periodic packing comprises an orthogonal sublattice. The primitive n-dimensional orthogonal lattice included in P with respect to a given coordinate system will be called the *orthogonal sublattice*. A shifted version of a Voronoi cell of this lattice (with respect to the coordinate system associated with the constituent n one-dimensional lattices) will be called a *fundamental hyperbox* of the packing. The minimum lengths of the vectors in the above definition determine the fundamental hyperbox. The set of packing points, Q, included in this region will be referred to as a *fundamental module*. A fundamental module Q is actually a set of *glue vectors* (translate representatives), $P \triangleq Q + \boldsymbol{l}\mathbb{Z}$, that is, $P = \{\boldsymbol{q} + \boldsymbol{l}\mathbb{Z} : \boldsymbol{q} \in Q\}$.

The coordinate system whose axes are collinear to the vectors $\{v_1, v_2, \cdots, v_n\}$ of Definition 7, along with the dimensions of the fundamental hyperbox $\{l_i = b_i - a_i\}$ are sufficient descriptive parameters of the fundamental hyperbox. In particular, a lattice L has a finite trellis diagram if and only if it comprises an orthogonal sublattice (e.g., [22]), that is, it is periodic (under Definition 7). The foregoing fundamental hyperbox should be distinguished from the Voronoi cell. For instance, the Voronoi cell of the two-dimensional hexagonal lattice are hexagons whereas the fundamental hyperbox is rectangular by definition, and it includes at least two lattice points. All useful constructions of nonlattice packings, i.e., Constructions A, B, and C which are based on codes (e.g., [5]) and packings built up by layers comply with the above definition of periodic packings.

In particular, when the dimensions of the fundamental hyperbox in all n coordinates are identical, $l_i = m$, $\forall 1 \le i \le n$, then the packing will be denoted by $P = Q + m\mathbb{Z}^n$, where $\mathbb{Z}^n$ is the n-*dimensional cubic lattice* or the *integer lattice*. In the sequel, we denote the fundamental module by the set of its packing points, Q, or by the pair $(Q, \boldsymbol{l})$, for expedience. The additional parameter vector $\boldsymbol{l}$ denotes the distance set of the corresponding orthogonal sublattice. In many cases it may be convenient to represent the fundamental module as a set of codewords. The notation $(n, k, d)_L$ will stand for a binary linear code of length n, dimension k, and minimum Hamming distance d, and the triple (n, M, d) will denote a binary nonlinear code of length n, M codewords, and minimum Hamming distance d. For convenience, unless otherwise stated, all the packings along this work are scaled to have minimum distance 1.

The (fundamental) *volume* of a periodic packing $V(P)$ is the quotient of the volume of the fundamental periodic region (fundamental hyperbox) and the total number of packing points included in it. The volume of the fundamental hyperbox, $\prod_{i=1}^n l_i$, is M times larger than the volume of the packing, where M is the total number of packing points included in the hyperbox.

The *coding gain* of an n-dimensional packing P is defined as $\gamma(P) = d^2(P)/[V(P)]^{2/n}$ where $d(P)$ is the minimum Euclidean distance between any two packing points. The supremum coding gain of any i-dimensional *packing* is denoted herein by Γ_i, and γ_i will denote *Hermite's constant*, i.e., the maximum coding gain of a lattice packing in $\mathbb{R}^n$.

Example 3.1: Fig. 1 depicts the hexagonal lattice generated by

$$\begin{bmatrix} 1 & 0 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}.$$

We illustrate the fundamental hyperbox (rectangle) of the lattice in two possible coordinate systems. The fundamental hyperboxes are shown in dashed lines, and the Voronoi cell of the lattice is delineated by a solid line. In the coordinate system for which the generator matrix is given, the volume (area) of the hyperbox is $1 \cdot \sqrt{3}$, and it comprises two lattice points. The volume of the fundamental hyperbox in the proposed rotated coordinate system is $\sqrt{7} \cdot \sqrt{21}$, and it includes 14 lattice points. The corresponding generator matrix for

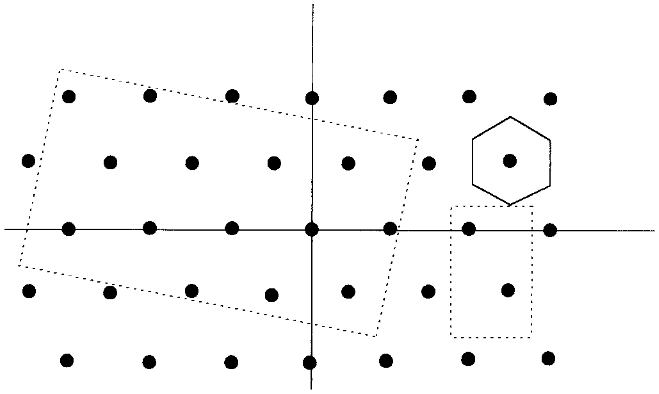

Fig. 1. Fundamental modules of the hexagonal packing in two different coordinate systems and the Voronoi cell.

the representation via the second fundamental hyperbox is

$$\begin{bmatrix} \sqrt{7}/14 & 3\sqrt{21}/14 \\ 5\sqrt{7}/14 & \sqrt{21}/14 \end{bmatrix}.$$

Example 3.2: The Schläfli lattice D_4 is the densest lattice in four dimensions. Its coding gain is $2^{1/2}$. Two possible constructions of this lattice are

I) $D_4 = 2^{-1/2}(4,3,2)_L + 2^{1/2}\mathbb{Z}^4$

II) $D_4 = 2^{-1}(4,1,4)_L + \mathbb{Z}^4$.

In the first coordinate system, the volume of the fundamental hyperbox is 4. The fundamental hyperbox in the second coordinate system is four times smaller, and it comprises just two packing points.

In the sequel, we basically adhere to the nomenclature of [8]. We construct a trellis diagram for the packing by using a fundamental module. This construction is a generalization of the known construction for lattice packings [6]. In a given coordinate system, the label code (the fundamental module) of a periodic packing is a block code over arbitrary finite alphabet sets at the different coordinates. The state space and the state-transition space of the trellis for the packing are identical to those of its fundamental module. Therefore, the bounds of [18], and in particular, the weighted coordinates bound of the present work also apply to periodic packings.

A fundamental module in a periodic packing also defines a fundamental hyperbox. In turn, this hyperbox constitutes the coordinate system in which the packing is depicted. In a given coordinate system, i.e., for a given fundamental module, a trellis diagram for a periodic packing is defined by ordering these coordinates into a nested sequence of vector spaces. We order the n one-dimensional spaces associated with the fundamental hyperbox, and accordingly we denote them by $\{W_i, 1 \le i \le n\}$. We define the nested sequence of increasing subspaces

$$V_0 = \{0\}, \quad V_i = V_{i-1} \oplus W_i, \quad 1 \le i \le n.$$

Clearly, $V_0 \subset V_1 \cdots \subset V_n = \mathbb{R}^n$.

We also denote

$$U_J \triangleq W_{j_1} \oplus W_{j_2} \oplus \cdots \oplus W_{j_{|J|}}$$

where $J = \{j_1, j_2, \cdots, j_{|J|}\}$.

The projection $P_J(\boldsymbol{x})$ of a vector $\boldsymbol{x} \in \mathbb{R}^n$ onto the $|J|$-dimensional space U_J is the image of $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ under the map $P_J : \mathbb{R}^n \to U_J$ defined by

$$P_J(\boldsymbol{x}) = (x_{j_1}, x_{j_2}, \cdots, x_{j_{|J|}}).$$

The projection of a vector $\boldsymbol{x}$ onto W_i will therefore be denoted by $P_i(\boldsymbol{x})$. Let D be a set of points in $\mathbb{R}^n$, then we define $P_J(D) \triangleq \{P_J(\boldsymbol{x}) : \boldsymbol{x} \in D\}$.

Under a given decomposition of $\mathbb{R}^n$ as above, we denote by $P_J(P \mid \boldsymbol{c})$ a *cross section* of P

$$P_J(P \mid \boldsymbol{c}) \triangleq \{P_J(\boldsymbol{x}) : \boldsymbol{x} \in P \text{ and } P_{I-J}(\boldsymbol{x}) = \boldsymbol{c}\}$$

where $\boldsymbol{c}$ is an $(n - |J|)$-dimensional vector. Obviously, in any coordinate system where P is periodic, $P_J(P \mid \boldsymbol{c})$, is also a periodic packing.

A *trellis diagram* $T = (\Sigma, A, E)$ for a periodic packing P is an edge-labeled directed graph consisting of $n+1$ levels. The set of vertices, or *states*, Σ, is the union of disjoint sets Σ_i, the set of states at level i. The label alphabet is denoted by A, and E is a set of ordered triples $(\sigma_{i-1}, \alpha, \sigma_i)$, $1 \le i \le n$, where $\sigma_{i-1} \in \Sigma_{i-1}$, $\sigma_i \in \Sigma_i$, and $\alpha \in \mathbb{R}$. The diagram is defined by the decomposition of the periodic packing into a fundamental module Q and the orthogonal sublattice $\boldsymbol{l}\mathbb{Z} : P \triangleq Q + \boldsymbol{l}\mathbb{Z}$. Every packing point $\boldsymbol{x} \in P$ passes through a *state sequence* $\sigma(\boldsymbol{q}) = \{\sigma_1(\boldsymbol{q}), \sigma_2(\boldsymbol{q}), \cdots, \sigma_n(\boldsymbol{q})\}$, where $\boldsymbol{x} - \boldsymbol{q} \in \boldsymbol{l}\mathbb{Z}$, and $\sigma_i(\boldsymbol{q}) \in \Sigma_i$. An edge between two adjacent states $\sigma_{i-1}(\boldsymbol{q})$ and $\sigma_i(\boldsymbol{q})$ is labeled by $q_i + l_i\mathbb{Z}$. There is thus an infinite set of *parallel transitions*, i.e., an infinite number of packing points that pass through each state sequence $\sigma(\boldsymbol{q})$. This set is a translate of the orthogonal sublattice of P, namely, $\boldsymbol{q} + \boldsymbol{l}\mathbb{Z}$.

In the framework of system theory, the orthogonal sublattice of a periodic packing is a generalization of the parallel transition code of a linear system. From this viewpoint, the orthogonal sublattice represents the *nondynamical component* of the system. The trellis diagram for P may be viewed as an efficient way of representing the fundamental module. This diagram, with respect to the translate set Q, is the same as that of Q, except that every edge in the latter diagram is replaced by an infinite number of parallel edges connecting the same state pair. When the trellis representation for Q does not include parallel transitions, then it comprises $|Q|$ state sequences (nonparallel length-n paths). Thus the dynamics of the trellis diagram for P is governed by the set of translate representatives in a given coordinate system.

The state complexity of the trellis diagram at level i, $s_i(P)$, is the logarithm of the vertex count at this level, i.e., $s_i(P) \triangleq \log|\Sigma_i|$. The sequence $\boldsymbol{s}(P) = \{s_i(P), 0 \le i \le n\}$ is the *state complexity profile* of P. Let $B_{i,j}$ denote the set of branches (paths) between states at indices i and $j > i$. Each transition between states in a trellis for a packing represents an infinite number of packing points. Thus we refer to the *branch complexity* of the label code, i.e., the state-pair complexity. Each branch is described by the triple $(\sigma_i, P_{[i+1,j]}(\boldsymbol{q}), \sigma_j)$ where $\sigma_i \in \Sigma_i$, $\sigma_j \in \Sigma_j$, and $\boldsymbol{q} \in Q$, and we define the branch complexity as the log-cardinality of the set $B_{i,j}$, $b_{i,j}(P) \triangleq \log|B_{i,j}|$.

Example 3.3: In three dimensions, the *hexagonal close-packing* (*hcp*) may be defined as a (nonlattice) laminated packing. This packing has the same density as the densest lattice packing in three dimensions, the *face-centered cubic* (*fcc*) *lattice*. The *hcp* is constructed by stacking layers of the two-dimensional *hexagonal lattice* A_2 and its translate $A_2 + (1/2, 1/\sqrt{12})$ alternately, where A_2 is spanned by the vectors $(1,0)$ and $(1/2, \sqrt{3}/2)$. The distance between the stacked layers is $\sqrt{2/3}$. This packing is hence spanned by the lattice L and its translate $L + (1/2, 1/\sqrt{12}, \sqrt{2/3})$, where L is the $\mathbb{Z}$-span of the vectors $\{(1,0,0), (1/2, \sqrt{3}/2, 0), (0,0,2\sqrt{2/3})\}$. Thus the orthogonal sublattice of the *hcp* in the above coordinate system is spanned by $\{(1,0,0), (0,\sqrt{3},0), (0,0,2\sqrt{2/3})\}$. The volume of this version of the *hcp* is $1 \cdot \sqrt{3} \cdot 2\sqrt{2/3}/4 = 1/\sqrt{2}$, and its coding gain is $\gamma(hcp) = 2^{1/3}$. The *hcp* is spanned by four translates of

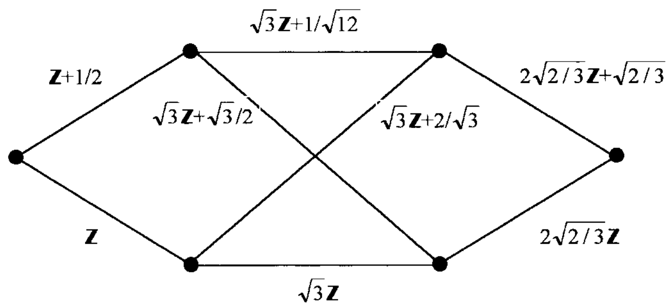

Fig. 2. Trellis diagram for the hexagonal close-packing.

the above orthogonal sublattice whose representatives (which do not form a group code) are

$$\{(0,0,0),(1/2,\sqrt{3}/2,0),(1/2,1/\sqrt{12},\sqrt{2/3}),(0,2/\sqrt{3},\sqrt{2/3})\}.$$

These four points are embodied in a single fundamental hyperbox of the *hcp*, and they are a fundamental module of the *hcp* thereupon. The corresponding trellis representation for the *hcp* is given in Fig. 2.

IV. Entropy/Dimension Profiles and Bounds on Trellis Complexity of Periodic Packings

The entropy/dimension profile (EDP) of a periodic packing is analogous to the *entropy/length profile* of block codes [18]. Heuristically, the unconditional profiles measure the entropy, viz., the amount of randomness, per unit volume, of a projection of the packing points onto lower-dimensional subspaces. The conditional entropy profiles measure the randomness, per unit volume, of cross sections of the packing points. Along the following definitions, we concentrate on the fundamental module of a periodic packing, and hence we confine the scope of our results to a given (unordered) coordinate system. The various entropy profiles of the fundamental module also measure the density of the corresponding entropy profiles of the whole packing. Some of the results of this section utilize the EDP to extend results, especially of [8], to nonlattice periodic packings.

The bounds apply to both lattice and nonlattice periodic packings in a given, but not necessarily ordered, coordinate system. Hence for lattices the DLP of [8] which applies to any coordinate system is more general than the corresponding EDP presented in this section. Nevertheless, it seems that for nonlattice periodic packings there is a "natural" coordinate system to define the trellis for the packing, and this degree of freedom of rotating the coordinate system while maintaining a periodic packing structure does not exist. Furthermore, the use of the EDP does not restrict the generality of the bounds on trellis complexity.

We define the different entropy/dimension profiles of a periodic packing with respect to a fundamental module Q and a weight profile

$$\boldsymbol{w}=\{w_i, 1\le i\le n\}: \boldsymbol{g}'(P,\boldsymbol{w}),\boldsymbol{g}(P,\boldsymbol{w}),\boldsymbol{h}'(P,\boldsymbol{w}),\boldsymbol{h}(P,\boldsymbol{w})$$

as the corresponding weighted entropy/length profiles of the fundamental module. One useful weight set is

$$-\log\boldsymbol{l}=\{-\log l_1,-\log l_2,\cdots,-\log l_n\}$$

where $\boldsymbol{l}$ is the distance set of the orthogonal sublattice. Clearly, $g_i'(P,-\log\boldsymbol{l})$ is the density of the entropy of the projections of the packing points per unit volume, and similarly $g_i(P,-\log\boldsymbol{l})$ is the entropy density of cross sections of the packing. Moreover, for lattices $g_i'(P,-\log\boldsymbol{l})$ reduces to the *inverse-ordered density/length profile* [8], and $g_i(P,-\log\boldsymbol{l})$ coincides with the *ordered density/length profile*. We recall that

$$V(P)=\frac{1}{M}\cdot\prod_{j=1}^{n} l_j$$

where M is the cardinality of the fundamental module. Thus we get for this choice of a weight set

$$\begin{aligned}g_i'(P,-\log\boldsymbol{l})&=H(X_{i-})-\sum_{j=1}^{i}\log l_j\\&=-\log V(P)-H(X_{i+}\mid X_{i-})+\sum_{j=i+1}^{n}\log l_j.\end{aligned}\tag{4.1}$$

Likewise, using Lemma 1, it is easily shown that

$$h_i(P,-\log\boldsymbol{l})+h_{n-i}'(P,-\log\boldsymbol{l})=-\log V(P).\tag{4.2}$$

Another special choice of a weight set is the zero set, $\boldsymbol{w}=\boldsymbol{0}$. The different profiles under this choice, unlike the DLP for a lattice, are invariant under scaling of the packing by a different factor in each dimension. Indeed, such transformations preserve the structure of the trellis of the packing, though they may change its coding gain. The zero set of weights is useful to discuss the branch complexity of a packing. The branch complexity profile cannot be defined or bounded by the profiles under a nonzero weight set. In particular, the branch complexity of a lattice cannot be expressed in terms of DLP.

Lemma 2: Let P be a periodic packing, then

$$h_1(P,-\log\boldsymbol{l})\le-\log[d(P)].\tag{4.3}$$

Proof: We assume without loss of generality that

$$h_1(P)=H(X_1\mid X_{[2,n]}).$$

We denote

$$K\triangleq\max_{y\in P_{[2,n]}(P)}|\{(x,\boldsymbol{y}):(x,\boldsymbol{y})\in Q\}|$$

where Q is a fundamental module of P and

$$p(\boldsymbol{y})=\frac{|\{(x,\boldsymbol{y}):(x,\boldsymbol{y})\in Q\}|}{M},\qquad M=|Q|.$$

Thus clearly $l_1\ge Kd(P)$.

$$\begin{aligned}h_1(P,-\log\boldsymbol{l})&=\sum_{y\in X_{[2,n]}}p(\boldsymbol{y})H(X_1\mid\boldsymbol{y})-\log l_1\\&\le\log K-\log l_1\le-\log[d(P)].\end{aligned}\qquad\square$$

Using similar argumentation one can readily verify that $h_i(P,-\log\boldsymbol{l})$ is not larger than the log-density of the densest i-dimensional cross section of P

$$P_J(P\mid\boldsymbol{c})\triangleq\{\boldsymbol{x}:\boldsymbol{x}\in P,P_{I-J}(\boldsymbol{x})=\boldsymbol{c},|J|=i\}$$

in the underlying coordinate system.

Lemma 3: Let P be a periodic packing and let $(Q,\boldsymbol{l})$ be a fundamental module of P inducing a decomposition of $\mathbb{R}^n$ into a direct sum of n one-dimensional subspaces W_i : $V_0=\{0\}$, $V_i=V_{i-1}\oplus W_i$, $1\le i\le n$, then

$$h_i(P,-\log\boldsymbol{l})\le-\min_{J,c}\{\log[V(P_J(P\mid\boldsymbol{c}))]:|J|=i,\boldsymbol{c}\in P_{I-J}(P)\}.\tag{4.4}$$

For lattice packings there is a coordinate system in which the bounds of Lemmas 2 and 3 are met with equality. Likewise, it follows from the above inequality that $h_i(P,-\log\boldsymbol{l})$ is not larger than the density of the densest cross section of the packing in *any* coordinate system.

Lemma 4: Let P be a periodic packing, then

$$h_i(P, -\log \boldsymbol{l}) \le i/2 \cdot \log[\Gamma_i / d^2(P)]. \tag{4.5}$$

It should be noted, however, that the values of Γ_i are known only for the first two dimensions, and bounded in higher dimensions. For a lattice L, Γ_i is replaced by Hermite's constant γ_i, and Lemma 4 reads [8]

$$h_i(L, -\log \boldsymbol{l}) \le i/2 \cdot \log[\gamma_i / d^2(L)]. \tag{4.6}$$

Example 4.1: A possible fundamental module of the *hcp* of Example 3.3 is

$$\{(0,0,0), (1/2, \sqrt{3}/2, 0), (1/2, 1/\sqrt{12}, \sqrt{2/3}), (0, 2/\sqrt{3}, \sqrt{2/3})\}.$$

In this coordinate system and for the weight set $\{-\log \boldsymbol{l}\}$, the ordered conditional EDP, coincides with the unordered profile yielding,

$$\boldsymbol{h}(hcp, -\log \boldsymbol{l}) = \{0, 0, \log(2/\sqrt{3}), 0.5 \cdot \log 2\}.$$

This profile meets the maximum attainable profile of three-dimensional lattices of inequality (4.6).

In a given coordinate system, a lattice has a unique minimal biproper trellis [14] which simultaneously minimizes several measures of trellis complexity. Nonlattice periodic packings may not have such a minimal scheme. The minimization of the trellis diagram with respect to different criteria may result in different trellises. In the rest of this section, we pursue the methodology of [18], and generalize the bounds on the state and branch complexity to periodic packings. In fact, the latter bounds, unlike the state complexity bounds, cannot be formulated in terms of geometrical properties such as cross sections and projections of the packing in lower-dimensional subspaces. We bound the state complexity profile of periodic packings under any order of a given coordinate system. Actually, this problem has not yet been studied. We show that the weighted coordinates bound is a suitable tool to derive such bounds. The bounds in this section are based on the theorems established in [18] and Section II of the present work, and hence the proofs of most of the claims in this section are omitted. Finally, we apply our bounds to Constructions B and C for packings.

The bounds on the state complexity profile of [18, Theorem 4] and Theorem 1 of the present work are easily generalized to periodic packings. Given an ordered coordinate system for $\mathbb{R}^n$ defined by the following decomposition into a direct sum of n one-dimensional subspaces, $\{W_i, i = 1, 2, \cdots, n\}$, $V_0 = \{0\}$, $V_i = V_{i-1} \oplus W_i$, $1 \le i \le n$, an underlying fundamental module Q, and an arbitrary weight set $\boldsymbol{w}$, the state complexity profile (of any trellis representation) of a periodic packing P is bounded by

$$s_i(P) \ge I(X_{i-}; X_{i+}) = g_i'(P, \boldsymbol{w}) - g_i(P, \boldsymbol{w}), \qquad 0 \le i \le n. \tag{4.7}$$

Lattice packings meet this bound with equality. The state complexity profile of P in this coordinate system under any ordering of the coordinates is bounded by

$$s_i(P) \ge h_i'(P, \boldsymbol{w}) - h_i(P, \boldsymbol{w}). \tag{4.8}$$

In particular, for the weight set $\boldsymbol{w} = \{-\log \boldsymbol{l}\}$, we get

$$\begin{aligned} s_i(P) &\ge -\log V(P) - h_i(P, -\log \boldsymbol{l}) - h_{n-i}(P, -\log \boldsymbol{l}) \\ &= \log V(P) + h_i'(P, -\log \boldsymbol{l}) + h_{n-i}'(P, -\log \boldsymbol{l}), \\ &\qquad 0 \le i \le n. \end{aligned} \tag{4.9}$$

The next corollary follows from (4.9) and the bounds on the conditional EDP in Lemma 4. It generalizes a result of [8] for lattices to periodic packings.

Corollary 1: The state complexity profile of a periodic packing P in $\mathbb{R}^n$ in any coordinate system (in which it admits a periodic structure) is bounded by

$$s_i(P) \ge (n/2) \cdot \{\log \gamma(P) - (i/n) \cdot \log \Gamma_i - (n-i)/n \cdot \log \Gamma_{n-i}\}. \tag{4.10}$$

Example 4.2: For $n = 3$ we use the following values: $\Gamma_1 = \gamma_1 = 1$, $\Gamma_2 = \gamma_2 = 2/\sqrt{3}$, $\gamma_3 = 2^{1/3}$. Using the above corollary we have that the minimum possible state profile of any periodic packing in three dimensions whose coding gain is γ_3 is $\{1, 2, 2, 1\}$ states. Consequently, the state complexity profile of the *hcp* as depicted in Fig. 2 is minimal.

Example 2.2 (Continued): The group code of this example is isomorphic to the label code of a version of the laminated lattice Λ_4 corresponding to the generator matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 1/\sqrt{3} & \sqrt{2/3} & 0 \\ 0 & 0 & \sqrt{3/2} & 1/\sqrt{2} \end{bmatrix}.$$

It is easily verified that the weight set which was chosen for this code also improves upon the bound induced by the DLP of this lattice $\boldsymbol{h}(\Lambda_4, -\log \boldsymbol{l})$. However, the above coordinate system is not a good one for Λ_4. The minimal trellis for this lattice (in a different coordinate system) comprises only two paths [1].

The maximum possible coding gain in any dimension is upper-bounded by several bounds, e.g., Lindsey's bound [16] and Rogers' bound [19]. Two other well-known asymptotic theorems for sufficiently large values of n are the Minkowski–Hlawka theorem (e.g., [5]) and the Kabatyanskii–Levenshtein theorem [11]. The Minkowski–Hlawka lower bound states that there exist lattices with coding gain satisfying $2\pi e \gamma_n \ge n$. The Kabatyanskii–Levenshtein upper bound means that the densest (lattice or nonlattice) packings satisfy $2\pi e \Gamma_n \le 1.744n$. The use of these bounds in conjunction with Lemma 4 enables the extension of a theorem by Blake and Tarokh [4] concerning the trellis complexity of the densest *lattice* packings in $\mathbb{R}^n$, also to nonlattice periodic packings.

Corollary 2: Let P be an n-dimensional periodic packing whose coding gain $\gamma(P)$ satisfies $2\pi e \gamma(P) \ge 0.872n$. We denote by $|\Sigma_i|$ the vertex count of a trellis representation for P at level i, then each such representation satisfies the following inequality:

$$\frac{1}{n} \sum_{i=1}^{n} |\Sigma_i| \ge z^{\pi e \gamma(P)/1.744} \tag{4.11}$$

for any z, $1 < z < 4\pi e \gamma(P)/1.744n$.

Proof: Using Corollary 1 which bounds the state complexity profile by functions of the maximum coding gain of any packing in the corresponding dimension, one can repeat the proof of [4] along the same claims. The following observations should be noted.

- For sufficiently large n, the Minkowski–Hlawka lower bound ensures the existence of packings whose coding gains satisfy the requirement $2\pi e \gamma(P) \ge n$.
- In the above equation, z is larger than a unit and thus, given a sequence of periodic packings $\{P_n, n = 1, 2, \cdots,\}$, where n stands for the dimension of each packing, then the average vertex count of this sequence grows at least exponentially as a function of $\gamma(P_n)$. □

Theorem 2 can be rewritten as a bound on the state complexity of periodic packings, using the EDP measures for a given fundamental module. The upper bound on the state complexity under a given coordinate system ([18, Theorem 7]) applies as is to periodic packings.

The total number of trellis edges is usually regarded as a more accurate measure of the computational complexity of the Viterbi algorithm. Moreover, the maximum state-pair complexity, unlike the maximum state complexity, cannot be reduced by sectionalization [7]. Thus for decoding purposes, the branch complexity profile has a more practical importance than the state complexity profile. This profile is well-defined for lattices, and bounded for nonlattice periodic packings, by some entropy properties of the fundamental module. The bounds of [8, Theorems 8–11] on the branch complexity of block codes, can be applied to the branch complexity of the label code for the packing in a given coordinate system.

Block codes are used to construct dense periodic packings. We use the bounds devised in this section to address the relations between the trellis complexity of packings obtained by Constructions B and C [5] and the trellises of the constituent codes. These results are limited to the coordinate systems induced by the codes that define the corresponding packings. In the following results, we do not scale the packings to have minimum distance 1. Detailed description of these constructions can be found in [5]. All the logarithms here onwards are taken to the *binary* base.

Construction B—Let C be an (n, M, d) binary code with the property that the weight of each codeword is even. We construct a periodic packing $P(C)$ in $\mathbb{R}^n$

$$P(C) = \left\{ \boldsymbol{x} = (x_1, x_2, \cdots, x_n) : \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \bmod 2 \in C, \sum_{i=1}^{n} x_i \equiv 0 \pmod 4 \right\}.$$

A fundamental hyperbox for this construction is an n-dimensional hypercube whose volume is 4^n. The fundamental module comprises $2^{n-1}M$ points of the form $\boldsymbol{c} + \boldsymbol{r}$ such that $\boldsymbol{c} \in C$, and $\boldsymbol{r}$ is a permutation of the vector $2^t 0^{n-t}$ for some even number t if the weight of $\boldsymbol{c}$ is divisible by 4, and t is odd if the weight of $\boldsymbol{c}$ is not divisible by 4. A lattice packing is obtained if and only if C is a linear code.

The different entropy/dimension profiles of the packing satisfy

$$\begin{aligned} g_i'(P(C), \mathbf{0}) &= i + g_i'(C, \mathbf{0}), && 1 \le i \le n-1 \\ g_i(P(C), \mathbf{0}) &= (i-1) + g_i(C, \mathbf{0}), && 1 \le i \le n-1 \\ h_i'(P(C), \mathbf{0}) &= i + h_i'(C, \mathbf{0}), && 1 \le i \le n-1 \\ h_i(P(C), \mathbf{0}) &= (i-1) + h_i(C, \mathbf{0}), && 1 \le i \le n-1. \end{aligned} \quad (4.12)$$

Likewise,

$$g_n'(P(C), \mathbf{0}) = g_n(P(C), \mathbf{0}) = h_n'(P(C), \mathbf{0}) = h_n(P(C), \mathbf{0}) = \log M + (n-1).$$

We denote the mutual information between the past and the future portions of the code at index i by $I_C(X_{i-}; X_{i+})$. Using (4.12) we obtain

$$\begin{aligned} s_i(P(C)) &\ge I_C(X_{i-}; X_{i+}) + 1, \qquad 1 \le i \le n-1 \\ b_{i,j}(P(C)) &\ge I_C(X_{j-}; X_{i+}) + j - i + 1, \qquad 1 \le i < j \le n-1. \end{aligned} \quad (4.13)$$

When a linear code C is used to define a lattice $L(C)$, we have the equality

$$s_i[L(C)] = g_i'[L(C), \mathbf{0}] - g_i[L(C), \mathbf{0}] = s_i(C) + 1, \qquad 1 \le i \le n-1. \quad (4.14)$$

The branch complexity of the trellis for a lattice constructed in this way thus satisfies

$$b_{i,j}(L(C)) = g_j'(L(C), \mathbf{0}) - g_i(L(C), \mathbf{0}) = b_{i,j}(C) + j - i + 1, \qquad 1 \le i < j \le n-1. \quad (4.15)$$

Thus for this case, if the trellis for the linear code C comprises $|E|$ edges, then the total number of edges in the trellis for the lattice generated by Construction B is $4 \cdot |E| - 8$.

Example 4.3: In 12 dimensions the $(12, 24, 6)$ nonlinear code is used to obtain the nonlattice packing L_{12} by Construction B. The coding gain of this packing is 3.56 dB. The coding gain of the densest known packing in 12 dimensions, the Coxeter–Todd lattice K_{12}, is 3.63 dB. Yet L_{12} is denser than any known laminated lattice in 12 dimensions. The EDP of L_{12}, evaluated according to (4.12), is

$$\boldsymbol{h}(L_{12}, \mathbf{0}) = \{0, 0, 1, 2, 3, 4, 5\tfrac{1}{6}, 6\tfrac{1}{6}, 7\tfrac{2}{3}, 9.58, 11.58, 13.58, 15.58\} \quad \text{[bits]}.$$

This profile provides the following bound on the state complexity profile of the packing:

$$\boldsymbol{s}(L_{12}) \ge \log\{1, 4, 8, 16, 31, 43, 39, 43, 31, 16, 8, 4, 1\}.$$

This profile is fairly tight. The minimal trellis for the packing has the following vertex count: $\{1, 4, 8, 16, 32, 44, 42, 44, 32, 16, 8, 4, 1\}$. This trellis is achieved for a code that consists of the following codewords and their complements (see the bottom of this page).

Construction C—Let $C_m(n, M_m, d_m)$, $m = 0, 1, \cdots, K$, be $K+1$ binary codes with the property that the weight of each codeword is even and $d_m = u \cdot 4^{K-m}$ where u is either 1 or 2. We construct a periodic packing $P(C_0, C_1, \cdots, C_K)$ in $\mathbb{R}^n$

$$P(C) = \{\boldsymbol{x} = (x_1, x_2, \cdots, x_n) : \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \bmod 2 \in C_0, 2^{-m}[(\boldsymbol{x} - \boldsymbol{x} \bmod 2^m) \bmod 2^{m+1}] \in C_m\}.$$

A fundamental hyperbox of the packing is a

$$2^{K+1} \times 2^{K+1} \times \cdots \times 2^{K+1}$$

hypercube. The fundamental module Q comprises $\prod_{m=0}^{K} M_m$ points.

$$Q = \left\{ \sum_{m=0}^{K} 2^m \cdot \boldsymbol{c}_m : \boldsymbol{c}_m \in C_m, \ \forall m \right\}.$$

In general, this construction results in a nonlattice packing even when the constituent codes are linear. In particular, when the weight of each codeword of C_0 is divisible by 4, $d_0 = 8$, and C_1 is the even weight linear code $(n, n-1, 2)_L$, then Construction C reduces to a version of Construction B.

In order to evaluate the ordered EDP of the packing we decompose the points of the fundamental module as follows:

$$\boldsymbol{q} = \sum_{m=0}^{K} \Psi_m$$

$$\Psi_0 \triangleq \boldsymbol{q} \bmod 2, \quad \Psi_m \triangleq (\boldsymbol{q} - \boldsymbol{q} \bmod 2^m) \bmod 2^{m+1}. \quad (4.16)$$

000000000000 000011110110 000101111001 001001001111 010010011101 010111000011
010100101110 011000110011 011011101000 011101010100 001110011010 001110100101

We make the fundamental module into a probability space. The distinct $K+1$ summands in the above decomposition are statistically independent. Consequently,

$$g_i'(P(C_0, C_1, \cdots, C_K), \mathbf{0}) = \sum_{m=0}^{K} g_i'(C_m, \mathbf{0}), \qquad 0 \le i \le n$$
$$g_i(P(C_0, C_1, \cdots, C_K), \mathbf{0}) = \sum_{m=0}^{K} g_i(C_m, \mathbf{0}), \qquad 0 \le i \le n. \tag{4.17}$$

Thus we have

$$I_Q(X_{i-}; X_{i+}) = \sum_{m=0}^{K} I_{C_m}(X_{i-}; X_{i+}), \qquad 0 \le i \le n$$
$$I_Q(X_{j-}; X_{i+}) = \sum_{m=0}^{K} I_{C_m}(X_{j-}; X_{i+}), \qquad 0 \le i < j \le n. \tag{4.18}$$

When the constituent codes are *linear*, the resulting packing is a nonlattice packing in general. For this case of construction by linear codes we have the following corollary.

Corollary 3: Let P be a periodic packing constructed from $K+1$ linear block codes $C_0, C_1, \cdots, C_K$ via Construction C, then P has a *minimal* trellis whose state and branch complexity profiles are given by

$$\boldsymbol{s}(P) = \sum_{m=0}^{K} \boldsymbol{s}(C_m)$$
$$b_{i,j}(P) = \sum_{m=0}^{K} b_{i,j}(C_m). \tag{4.19}$$

Proof: On the one hand, we have from (4.18) that $\boldsymbol{s}(P)$ and $b_{i,j}(P)$ are not smaller than the respective sums on the right-hand side of (4.19). Conversely, the Shannon product ([15], [21]) of the trellises of the constituent codes is a (not necessarily minimal) trellis for the packing whose fundamental module is constructed as a direct sum of $K+1$ codes. The complexity of the trellis product meets the right-hand side of (4.19). Hence it is the minimal trellis for the packing. □

V. Conclusion

In this correspondence, we have presented weighted entropy profiles and a new bound on the state complexity profile of block codes. This bound elaborates on the dimension/length profiles and the entropy/length profiles, and generalizes these ideas to block codes whose components are drawn from alphabets of different sizes and, in particular, to group codes. These ideas were applied to generalize the construction of a trellis diagram for lattices and some related bounds on trellis complexity to periodic packings. However, this generalization for packings is applicable only to a given coordinate system.

A legitimate question that may arise at this stage is how hard is it to compute the proposed bound and how does this complexity compare to the complexity of deriving the minimal trellis? In this section, we analyze the computational complexity of the bound, and compare it to the complexity of the derivation of the minimal trellis. For the class of rectangular codes several algorithms were suggested to find the minimal biproper trellis [17], [20], [26]. The computational complexity of the algorithms proposed in [20] and [26] for an (n, M)-code C is $O(nM \log M)$. The algorithm of Lucas *et al*. [17] is more efficient, and the derivation of the trellis representation of a rectangular code (under a given permutation) according to this algorithm can be carried out in a $O(nM)$ time algorithm. The complexity of deriving the state complexity at a single index is $O(M \log M)$. These algorithms apply to *rectangular* codes only and under a given symbol permutation.

The total number of operations required to compute the ordered ELP with all-zero weights for the entire state complexity profile of an (n, M)-code C under a given permutation is upper-bounded by $O(nM)$. The complexity of computing the ELP at a single level is $O(M \log M)$.

There is no general efficient algorithm for generating a minimal trellis over all possible symbol permutations. Moreover, it is well known [10] that even the problem of finding a coordinate permutation that minimizes the vertex count at a given level of the minimal trellis for a binary linear code is NP-complete. Consequently, the problem of finding a minimal trellis under any symbol permutation involves a repetition of the algorithm for a given symbol order for all possible permutations (roughly, $n!$ permutation). In order to compute the unordered ELP (with zero weights) one should also check all permutations. Also, both the computation of the state complexity of the minimal trellis or the ELP bound at level i require $O(\binom{n}{i} M \log M)$ operations.

Thus the computation of the weighted coordinates bound is a very hard problem. The bulk of the computation concerns the evaluation of the ELP with zero weights. Once this profile is computed, it may be easily checked whether the use of nonequal weights may improve the bound. When the WELP provides a tighter bound then the choice of the weight set is not unique. For a given index i, the WELP bound provides a tighter bound only when there is a coordinate (or coordinates), say f_j, such that for any J, $|J| = i$, $f_j \in J$, the unweighted ELP of the code satisfies $H(X_J \mid X_{I-J}) < h_i(C)$, and also for each K, $|K| = n-i$, $f_j \in K$, $H(X_K \mid X_{I-K}) < h_{n-i}(C)$. Clearly, in this case the bound on the state complexity at level i, will be improved by assigning a suitable (positive) weight to f_j. Alternatively, an improvement may be achieved when there is a coordinate, say f_m, which is comprised in any set J, $|J| = i$, for which $H(X_J \mid X_{I-J}) = h_i(C)$, and also in any set K, $|K| = n-i$, satisfying $H(X_K \mid X_{I-K}) = h_{n-i}(C)$. In the latter case, a negative weight is assigned to f_m. When the use of weights improves the bound, then the nonzero weights should be assigned values that are of the same order of the differences between the values of the ELP under the different permutations. For example, one can start from values that are substantially smaller than these differences and increase them by this step until a looser bound is achieved. During this procedue a tighter bound will be found. In particular, for linear codes over a fixed alphabet set the nonzero weights may be increased in units steps.

To conclude, the computational complexity of the derivation of the minimal trellis for group codes under any symbol order breaks even with that of the proposed bound. It was illustrated in Example 2.2 that the new bound does improve upon the DLP/ELP bound for group codes. The bound will also exhibit an improvement for nongroup codes whose symbols are taken form alphabets of different sizes. When the latter codes are not rectangular, there is no algorithm to derive a minimal trellis. However, our examination of short codes over a *common* alphabet suggests that the use of nonzero weights does not achieve an improvement for codes with good parameters, i.e., codes that have the best minimum distance for a given length and cardinality. Thus we believe that the main contribution of the bounds is to Euclidean-space codes.

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The Preparata and Goethals Codes: Trellis Complexity and Twisted Squaring Constructions

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***Abstract*—The trellis complexity of the Preparata and Goethals codes is examined. It is shown that at least for a given set of permutations these codes are rectangular. Upper bounds on the state complexity profiles of the Preparata and Goethals codes are given. The upper bounds on the state complexity of the Preparata and Goethals codes are determined by the DLP of the extended primitive double- and triple-error-correcting BCH codes, respectively. A twisted squaring construction for the Preparata and Goethals codes is given, based on the double- and triple-error-correcting extended primitive BCH codes, respectively.**

***Index Terms*— BCH codes, Goethals codes, Preparata codes, trellis complexity, twisted squaring construction.**

I. INTRODUCTION

We examine the trellis complexity of two interesting families of nonlinear codes: the Preparata codes and the Goethals codes. The Preparata codes [17], [1], [5] are binary nonlinear codes of length 2^{m+1} (for odd $m \geq 3$) and with minimum distance six that contain the maximum possible number of codewords for these parameters. These codes contain twice as many codewords as the corresponding extended primitive double-error-correcting BCH codes. For odd $m \geq 5$ the Goethals codes [17], [1] are nonlinear binary codes having the same length as the Preparata codes, and with minimum distance eight. These codes contain four times the number of codewords of the corresponding extended primitive triple-error-correcting BCH codes.

Trellis diagrams of block codes are employed for efficient soft-decision decoding. The theory of trellis diagrams of nonlinear block codes was considered in [8], [18], [16], [15], [14], [19], [21], and [22]. The structure and complexity of the trellis representations of specific nonlinear codes were given in [8], [16], [19] for relatively small codes, and in [20] for the Kerdock and Delsarte–Goethals codes.

Generally speaking, a nonlinear code under a given coordinate ordering may not have a trellis representation minimizing the vertex count at all indices simultaneously [15], [14]. However, when a nonlinear code satisfies certain conditions (namely, it is *rectangular*), it admits a unique *biproper* trellis representation. This biproper trellis minimizes both the edge and vertex counts at all indices simultaneously [15], [14]. Furthermore, as proved in [23] and [21] the biproper trellis minimizes also the total number of addition-equivalent operations required to perform the Viterbi algorithm.

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