Convergence Analysis of Turbo-Decoding of Product Codes

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Abstract — **Geometric interpretation of turbo-decoding has founded an analytical basis, and provided tools for the analysis of this algorithm. Based on this geometric framework, we extend the analytical results for turbo-decoding of product codes, and show how analysis tools can be practically adopted for this case. Specifically, we investigate the algorithm's stability and its convergence rate. We present new results concerning the structure and properties of stability matrices of the algorithm, and develop upper bounds on the algorithm's convergence rate. We prove that for any 2 x 2 (information bits) product codes, there is a unique and stable fixed point. For the general case, we present sufficient conditions for stability. The interpretation of these conditions provides an insight to the behavior of the decoding algorithm.**

I. INTRODUCTION

Turbo codes, first introduced in 1993 [1], are considered as one of the most important developments in coding theory in recent years. Although simulation and practical results generally show excellent performance, there is a lack of theoretical basis for explaining the results and providing tools for their analysis. Recently, a new approach [2] of geometric interpretation to the decoding algorithm has managed to reveal interesting features of the decoding process. Based on it, we extend the analytical results, and use simulations to gain a deeper understanding of the turbo-decoding of product codes.

II. PRODUCT CODES TURBO-DECODING

A product code (without checks on checks) turbo encoder uses block encoders, and a rows to columns interleaver. The information bits are arranged in k_r rows and k_c columns. The i-th row (x^{r_i}) enters a (n_y, k_c, d_r) block encoder and forms a row code word y^i. The i-th column (x^{c_i}) enters a (n_z, k_r, d_c) block encoder and forms a column code word z^i (where d_r and d_c are the minimal distances of the row and column codes, respectively).

Let P_x, P_y and P_z represent the log-densities corresponding to the posterior densities $p(\widetilde{x}|x), p(\widetilde{y}|x)$ and $p(\widetilde{z}|x)$, respectively. Let Q_y, Q_z denote the extrinsic information from the rows and columns decoders, respectively. In [2] it is shown that the stability of the decoding algorithm is determined by the stability of S:

$$S = S^R S^C = (J_{P_x+P_y+Q_z} - I)(J_{P_x+Q_y+P_z} - I), \qquad (1)$$

and the general expression for the Jacobian matrix is given. Using the independence of the decoding of different rows (or columns), we develop an explicit expression for J_P. E.g. for the Jacobian of the rows decoding - $(J^R)_{i,j}$ we get ($P = P_x + P_y + Q_z$):

$$\begin{cases} e^P(x_j = 1|x_i = 1) - e^P(x_j = 1|x_i = 0) & x_i, x_j \in x^{r_a} \\ 0 & x_i \in x^{r_a}, x_j \in x^{r_b} \end{cases} \qquad (2)$$

The brute-force calculation complexity of a J^R element is $o(2^{k_c-1})$, also, note that it is a diagonal block matrix, whose i-th block ($J^{R,i}$) is the Jacobian matrix of the i-th row decoding.

We show that for general values of k_r and k_c , S is a block matrix, where each block ($S^{i,j}$) is a k_c x k_c matrix, with an all zeros diagonal. The main diagonal of S is the zero matrix ($S^{i,i} = 0$). For $k_r = k_c = 2$ we get:

$$S = \begin{pmatrix} & & & a_{1,2}b_{2,4} \\ & & a_{2,1}b_{1,3} & \\ & a_{3,4}b_{4,2} & & \\ a_{4,3}b_{3,1} & & & \end{pmatrix}, \qquad (3)$$

where $a_{i,j} = (S^R)_{i,j}$ and $b_{i,j} = (S^C)_{i,j}$.

Theorem 1: The fixed point of *any* product code turbo-decoder with $k_r = k_c = 2$ is always stable.

Proof: From (2) we deduce that the absolute value of each element of J^R is less or equal to 1, hence, $|a_{i,j}| < 1$. The same holds for $b_{i,j}$. Therefore, the eigenvalues of S are inside the unit circle, and S is stable (regardless of the SNR or the rows or columns encoders).

For the general case, we develop in [3] an upper bound for the maximal eigenvalue of S (which governs the convergence rate in the vicinity of the fixed point), and sufficient conditions for the stability of the decoding algorithm. The basic component in these conditions is the product of the posterior dependence between two bits in a row, and the sum of the posterior dependencies between one of these bits and all the bits in its column. Hence, small column posterior dependencies (i.e. successful columns decoding) can compensate for a large value of inter-row bit dependence (i.e. unsuccessful row decoding) and vice versa.

In our talk we present simulation results for the stability matrices of Hamming $[(7,4,3)]^2$ and Golay $[(24,12,8)]^2$ product codes. Further analysis of the results is made using distribution histograms of the complete eigenvalues spread, at the algorithm's fixed-point.

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