Convergence Analysis of Turbo-Decoding of Serially Concatenated Product Codes

Amir Krause, Assaf Sella and Yair Be'ery
Department of Electrical Engineering - Systems
Tel-Aviv University
Tel-Aviv, Israel
e-mail: {amirkr; asella; ybeery}@eng.tau.ac.il

Abstract — **The recently presented geometric interpretation of turbo-decoding has founded a framework for the analysis of decoding parallel-concatenated codes. We extend this analytical basis for the case of decoding serially concatenated codes, and focus on product codes (i.e., product codes with checks on checks). For this case, the extrinsic information should be calculated not only for the information bits, but also for the check bits, and we extend the theory accordingly. We show how the analysis tools can be adopted, and use them to investigate the convergence of product codes with check on checks: we derive a general form for the update equations, as well as expressions for the Jacobian and stability matrices. We show that these matrices can be viewed as a generalization of the corresponding matrices of parallel-concatenated product codes.**

The geometric interpretation of turbo-decoding [1] has founded a framework for the analysis of this algorithm. Based on it, the convergence of turbo-decoding of parallel-concatenated product codes (PCPC) was analyzed [2], [5]. Here we extend the analysis to the case of serially concatenated product codes (SCPC).

Let C_R and C_C be an $(n_y, k_c, d_r), (n_z, k_r, d_c)$ linear codes, respectively. A linear $(n_y n_z, k_r k_c)$ product code can be formed by arranging the information bits in a k_rxk_c rectangular array, and encoding each row and column using C_R and C_C, respectively.

Several turbo-decoding algorithms were proposed for this scheme ([3], [4]). Common to all of them is the calculation of extrinsic information not only for the information bits (as in PCPC), but also for the check bits. It can be easily shown that otherwise the contributions of the check on checks portion are independent on the iterative process (such algorithms are degenerate). Hence, we first extend Richardson's theory for this case, and derive a generic form of the update equations.

Denote with $P_{\tilde{x}|x}, P_{\tilde{y}|y}, P_{\tilde{z}|z}$ and $P_{\tilde{w}|w}$ the log-densities of the received code portions (information bits, checks on rows, checks on columns, and checks on checks, respectively). Let $Q^{(m)}_{dec,portion}$ denote the extrinsic information calculated by the *dec* decoder (*dec*={R,C} for the {row, column} decoder), for *portion*=$\{x, y, z, w\}$ (at the m-th iteration). The new update equations become:

$$[Q^{(m)}_{R,x}, Q^{(m)}_{R,y}] \leftarrow \widetilde{\pi}(P^{C_R}_{\tilde{x}|x} + P^{C_R}_{\tilde{y}|y} + Q^{(m-1)}_{C,x} + Q^{(m-1)}_{C,y}) - (P^{C_R}_{\tilde{x}|x} + P^{C_R}_{\tilde{y}|y} + Q^{(m-1)}_{C,x} + Q^{(m-1)}_{C,y}),$$

$$[Q^{(m)}_{R,z}, Q^{(m)}_{R,w}] \leftarrow \widetilde{\pi}(P^{C_R}_{\tilde{z}|z} + P^{C_R}_{\tilde{w}|w} + Q^{(m-1)}_{C,z} + Q^{(m-1)}_{C,w}) - (P^{C_R}_{\tilde{z}|z} + P^{C_R}_{\tilde{w}|w} + Q^{(m-1)}_{C,z} + Q^{(m-1)}_{C,w}),$$

$$[Q^{(m)}_{C,x}, Q^{(m)}_{C,z}] \leftarrow \widetilde{\pi}(P^{C_C}_{\tilde{x}|x} + P^{C_C}_{\tilde{z}|z} + Q^{(m)}_{R,x} + Q^{(m)}_{R,z}) - (P^{C_C}_{\tilde{x}|x} + P^{C_C}_{\tilde{z}|z} + Q^{(m)}_{R,x} + Q^{(m)}_{R,z}),$$

$$[Q^{(m)}_{C,y}, Q^{(m)}_{C,w}] \leftarrow \widetilde{\pi}(P^{C_C}_{\tilde{y}|y} + P^{C_C}_{\tilde{w}|w} + Q^{(m)}_{R,y} + Q^{(m)}_{R,w}) - (P^{C_C}_{\tilde{y}|y} + P^{C_C}_{\tilde{w}|w} + Q^{(m)}_{R,y} + Q^{(m)}_{R,w}).$$

In the various decoding algorithms some of the Q's are set to zero or multiplied by restraining factors.

The new stability matrices are then derived. For example, the stability matrix of the row decoder is generated by perturbing Q_C to $Q_C + \delta_C$. We get that Q_R deviates by:

$$\delta_R = \begin{pmatrix} J^{C_R}_{x,y} - I & 0 \\ 0 & J^{C_R}_{z,w} - I \end{pmatrix} \begin{bmatrix} \delta_{C,x}, \delta_{C,y} \\ \delta_{C,z}, \delta_{C,w} \end{bmatrix} = S^R \delta_C,$$

where J denotes the Jacobian matrix. $J^{C_R}_{x,y} - I$ can be interpreted as the stability matrix of the information (systematic) portion of the row decoder (resembling S^R of PCPC), and $J^{C_R}_{z,w} - I$ - the stability matrix of the checks portion (both are block diagonal matrices). Hence, the stability matrix of SCPC is a generalization of the stability matrix of PCPC.

The Jacobian matrix is the main analysis tool. For the case of SCPC, we show that it can be derived as a "natural" extension of the expression in [1] (where i and j correspond to either an information bit or a check bit):

$$(J_P)_{i,j} = \frac{\sum_{b \in H_i \cap H_j} p(b)}{\sum_{b \in H_i} p(b)} - \frac{\sum_{b \in \bar{H}_i \cap H_j} p(b)}{\sum_{b \in \bar{H}_i} p(b)} \qquad 1 \le i, j \le n.$$

In our talk we present simulation results for the stability matrices of Hamming $[(7,4,3)]^2$ and Golay $[(24,12,8)]^2$ serially concatenated product codes.

References

[1] T. Richardson, "The geometry of turbo-decoding dynamics," *IEEE Trans. on Information Theory* , vol. 46, no. 1, pp. 9-23, Jan. 2000.

[2] A. Sella and Y. Be'ery, "Convergence analysis of turbo-decoding of product codes," *IEEE Trans. on Information Theory* , vol. 47, no. 2, pp. 723-735, February 2001.

[3] S. Benedetto, D. Divsalar, G. Montorsi, F. Pollara, "Serial concatenation of interleaved codes: performance analysis, design, and iterative decoding," *IEEE Trans. on Information Theory* , vol. 44, no. 3, pp. 909-926, May 1998.

[4] R. Pyndiah, "Near-optimum decoding of product codes: block turbo codes", *IEEE Trans. on Communications* , vol. 46, pp. 1003-1010, Aug. 1998.

[5] D. Agrawal and A. Vardy, "The turbo decoding algorithm and its phase trajectories," *IEEE Trans. on Information Theory* , vol. 47, no. 2, pp. 699-722, February 2001.

0-7803-7123-2/01/$10.00 ©2001 IEEE