Coordination and Dynamic Shelf-Space Management of Video Movie Rentals

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Abstract

How should a video rental chain replenish new movies over time? Clearly, any such policy would consist of two key dimensions - the number of copies purchased and when to remove a movie from front shelves and replace it by a newly released one. We first analyze this bi-variate problem for an integrated chain. As for decentralized chains, we show that a (wholesale) price-only contract cannot coordinate such a chain. We then consider a price + revenue sharing contract. Such a contract can achieve coordination, but the unique price and share which are needed may not provide one of the parties with its desired profit (i.e., will violate individual rationality). We thus propose adding a third lever - a license fee (or subsidy) associated with each new movie. Such a contract can coordinate the channel and satisfy the individual rationality requirements. We provide numerical examples which utilize empirical demand data estimated by Lehmann and Weinberg (2000).

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1. Introduction

The video rental industry has recently been the subject of intense interest by the operations management community (Drezner and Pasternack 1999; Pasternack 2001; Lariviere and Cachon 2000; Cachon and Lariviere 2001; Dana and Spier 2000). That interest was generated by a recent change in the type of contract Blockbuster Video has with its supplying movie studios from wholesale-price only to wholesale price + revenue sharing.

Drezner and Pasternack, Pasternack and Lariviere and Cachon modeled the underlying system as single-period, and thus timeless, which is a common practice for modeling perishable/seasonal items offered for sale. But the key characteristic of (new) movie rentals is the temporal pattern of demand. Demand is high initially, but then declines significantly (Lehmann and Weinberg 2000; Dana and Spier 2000, Section 3). That being the case, and as prime shelf space is at a premium, a chain is likely to replace a movie when the demand for it has declined sufficiently. The question is when to do that. That has to be answered jointly with finding the optimal number of copies to buy.

Lehmann and Weinberg (2000; see also Weinberg 2000) established empirically that rental demand for a video typically declines exponentially over time, and that the temporal demand pattern is highly predictable. Thus, we also adopt a deterministic framework similar to theirs. In their model, however, Lehmann and Weinberg assumed that the time a movie remains on the shelf (the cycle’s length) is fixed in advance, independent of the number of copies ordered. For
us, it is a decision variable, optimized jointly with the order size. Lehmann and Weinberg were also interested in the best time for the studio to release a movie in video. We do not address this issue, and indeed make the simplifying assumption that new video releases are frequent enough to enable the execution of the resulting policy.

We first analyze the integrated/centralized model, which itself appears to be new to the operations management literature [a continuous review inventory-level-dependent demand model (Balakrishnan et al. 2000 and references therein) is also a deterministic order-point/order-quantity model, but in the case of rentals holding costs are incurred on all copies throughout the cycle]. We characterize the optimal policy, and the behavior of the decision variables as a function of the production costs and the fixed costs.

We then examine a decentralized system. First, we consider a wholesale-price only contract. As would be expected, a single control/lever cannot coordinate the channel, as a retailer’s behavior has two dimensions (consistent with Ashby’s Law of Requisite Variety (Ashby 1956)). Such a contract will result in buying less movies, and holding them longer, than in an integrated system. We then consider a wholesale price + revenue-sharing contract. Such a contract can achieve coordination, but the unique price and share which are needed may not provide one of the parties with its desired reservation profit (i.e., it will violate individual rationality). Note that in Lariviere and Cachon (2000) these two levers need only control a single-dimensional retailer policy, and since a whole continuum of lever value-pairs achieves that, one can select values which will also satisfy individual rationality (IR). This flexibility is unavailable here. We thus propose adding a third lever, a (per cycle) subsidy or license fee, so that all goals/requirements will be met. A situation with a subsidy is accompanied by a lower revenue-share for the retailer,
and a lower wholesale price than without an IR constraint. A license fee is accompanied by a higher revenue share and a higher wholesale price.

We then provide numerical examples which utilize an empirical demand pattern data from Lehmann and Weinberg (2000). After solving for all relevant contract types for some base parameters, we study the effects of changing these parameters – the parties’ fixed costs, the unit non-financial holding cost and the unit production cost. Their effect on the number of movie copies is less significant than on the duration of holding on to them.

2. Integrated System

Let $\lambda(t)$ be the rental demand rate for a movie $t$ days after its release on video, a decreasing function. The variable cost of producing a copy is $c$. The revenue per rental is $p$ and the unit

![Figure 1: Demand pattern of video rentals over time](image-url)
holding cost is $h(c)$ per day, where $h(c) = h + ic$, $i$ being the interest rate. Note that all copies will be “held” throughout the cycle. The fixed cost of switching to a new movie is $K$. When the movie’s age (from its release in video) reaches $T$ it is replaced by a new one; $Q$ copies are made. These are the two decision variables.

Let $R(Q,T)$ be the rental revenue per cycle (movie) where,

$$R(Q,T) = \begin{cases} p[Q\lambda^{-1}(Q) + \int_{\lambda^{-1}(Q)}^T \lambda(t)dt] & \text{for } T > \lambda^{-1}(Q) \\ pQT & \text{for } T \leq \lambda^{-1}(Q). \end{cases}$$

Then the profit per unit time will be

$$\Pi'(Q,T) = \frac{R(Q,T) - K - cQ - h(c)QT}{T}.$$ \hspace{1cm} (2)

That is the function we wish to maximize w.r.t. $Q$ and $T$. We will denote the optimal $Q$ and $T$ by $Q'$ and $T'$ respectively in this section.

**Proposition 1:** $T' > \lambda^{-1}(Q)$.

**Proof:** For $T \leq \lambda^{-1}(Q)$,

$$\Pi' = pQ - h(c)Q - (K + cQ)/T$$

and $^2 \Pi'_T = (K + cQ)/T^2 > 0$. This implies that $T' > \lambda^{-1}(Q)$. $\|$ $\|$

So, for the rest of the paper, we will focus our attention on $T > \lambda^{-1}(Q)$.

Differentiating w.r.t. $Q$, we obtain

$$\Pi'_Q = (p\lambda^{-1}(Q) - c - h(c)T)/T,$$ \hspace{1cm} (3)

---

1 Our model can also accommodate a $\lambda(t)$ which is initially increasing, as long as $Q$ corresponds to the decreasing portion of $\lambda(t)$ and the demand function is subsequently monotone decreasing.

2 For this paper, $Z_y$ will represent the first derivative of $Z$ with respect to $y$ and $Z_{yy}$ will represent the second derivative.
\[
\Pi^I_T = (TR_T - R + K + cQ)/T^2.
\] (4)

Thus \(\Pi^I_Q\) is zero when

\[ p\lambda^{-1}(Q^I) = c + h(c)T, \]

i.e., when

\[ Q^I(T) = \lambda(A + BT) \] (5)

where \(A = clp\) and \(B = h(c)/p\) [both \(A\) and \(B\) are in \((0, 1)\)]. Since \(T^I > \lambda^{-1}(Q^I) = A + BT^I\) (from Proposition 1 and (5)), i.e., \([p - h(c)]T^I > c\), it implies that the cycle has to be long enough so that the income from at least one copy of the movie will recoup its production costs. Also note that, as \(\lambda\) is a decreasing function, \(Q^I\) is decreasing in \(T\) – as movies are held longer we shall make less copies of them (cf. Lehmann and Weinberg 2000).

Substituting (5) into (2), we express the objective in terms of \(T\):

\[
\Pi^I[Q^I(T)] = \left\{ p\int_{A+BT}^T \lambda(t)dt - K \right\}/T.
\] (6)

Since \(R_T = p\lambda(T)\), the derivative of (6) w.r.t. \(T\) is

\[
\Pi^I_T[Q^I(T)] = p(T\lambda(T) - (A + BT)\lambda(A + BT) - \int_{A+BT}^T \lambda(t)dt + K/p + A\lambda(A + BT))/T^2
\]

\[ = \left[p\int_{A+BT}^T t\lambda(t)dt + K + pA\lambda(A + BT)\right]/T^2. \] (7)

And, after simplification,

\[
\Pi^I_{TT} = \left\{ p\lambda^I(A + BT)\left[\frac{\lambda^I(T)}{\lambda^I(A + BT)} - B^2\right] - 2\Pi^I_T \right\}/T. \] (8)

**Proposition 2:** If \(\lambda^I(t)/\lambda^I(A + Bt)\) is monotone decreasing in \(t\) and if \(T^B\) satisfies

\[
\lambda^I(T^B)/\lambda^I(A + BT^B) = B^2, \]

then there is an unique \(T^I\) in \([A/(1 - B), T^B]\) maximizing \(\Pi^I\).
**Proof:** \( \Pi^I (A / (1 - B)) = -K / T < 0 \) and \( \Pi^I (A/(1-B)) > 0 \) [from Proposition 1].

We assume that there is some feasible region of \( t \) for where \( \Pi^I > 0 \).

\[
\lim_{T \to \infty} \Pi^I [Q^I (T)] = \lim_{T \to \infty} [ p^T_{A+BT} \lambda(t) dt - K] / T.
\]

Since \( \lambda(t) \) is decreasing,

\[
\int_{A+BT}^T \lambda(T) dt < \int_{A+BT}^T \lambda(t) dt < \int_{A+BT}^T \lambda(A+BT) dt
\]

(Note each of the areas in Figure 2)

\[
\Rightarrow [ T(1-B) - A] \lambda(T) < \int_{A+BT}^T \lambda(t) dt < [ T(1-B) - A] \lambda(A+BT)
\]

\[
\Rightarrow [(1-B) - A/T] \lambda(T) - K/T < \Pi^I (T) < [(1-B) - A/T] \lambda(A+BT) - K/T.
\]

If we assume that \( \lambda(t) \to 0 \) as \( T \to \infty \), both

\[
[(1-B) - A/T] \lambda(T) - K/T \quad \text{and} \quad [(1-B) - A/T] \lambda(A+BT) - K/T
\]

will tend towards zero as \( T \to \infty \) implying that \( \lim_{T \to \infty} \Pi^I (T) = 0 \).

Let \( T^I \) be the value which satisfies \( \Pi^I (T^I) = 0 \). It is then easy to see that

\[
\Pi^I_{TT} (T^I) \leq 0^B \quad \text{if} \quad T^I \leq T^B
\]

and

\[
\Pi^I_{TT} (T^I) > 0 \quad \text{if} \quad T^I > T^B.
\]

**Figure 2: Areas in Proposition 2**
Hence we see that there is an unique \( T^I < T^B \) that maximizes \( \Pi^I \) (since \( T^I > T^B \) will be in the convex region and as \( T \to \infty, \Pi^I \to \infty \).

**Note:** For \( \lambda(t) = ae^{-bt}, \frac{\lambda_I(t)}{\lambda_J(A + Bt)} = e^{Ab} \cdot e^{-b(1-B)} \) which is monotone decreasing in \( t \) and \( T^B = -\ln(B^2 / e^{Ab})/b(1-B) \).

For \( \lambda(t) = a - bt, \frac{\lambda_I(t)}{\lambda_J(A + Bt)} = 1 \) and hence \( \frac{\lambda_I(t)}{\lambda_J(A + Bt)} - B^2 < 0 \). This implies that for any \( T^I \) satisfying \( \Pi^I_T = 0 \) we have \( \Pi^I_{TT} < 0 \), i.e., there will be no optimal cycle in the convex region.

Now we have the following comparative statics. In the following propositions, \( Q \) and \( T \) represent \( Q^I \) and \( T^I \), respectively.

**Proposition 3:** \( dQ^I/dc < 0 \) and \( dT^I/dc > 0 \).

**Proof:** Total differentiation of \( \Pi^I_T = 0 \) [from (7)] with respect to \( c \) gives,

\[
d \Pi^I_T/dc = (d \Pi^I_T/dT)T_c + (d \Pi^I_T/dA)A_c + (d \Pi^I_T/db)B_c = 0.
\]

\[
\Rightarrow T_c = p \left\{ A_c \left[ BT\lambda_I(A + BT) - \lambda(A + BT) \right] + B_c \cdot T^2 B\lambda_I(A + BT) \right\}/T^2 \Pi^I_{TT}.
\]

Since, \( \Pi^I_{TT} < 0, A_c > 0, B_c > 0, B > 0, \lambda_I(\cdot) < 0 \) and \( \lambda(\cdot) > 0 \), we conclude that \( T_c > 0 \).

From \( \Pi^I_Q = 0 \) [in equation (5)] we have,

\[
Q_c = \lambda_I(A + BT) \frac{\partial}{\partial c} (A + BT) = [(1 + iT) + h(c)T_c] \lambda_I(A + BT)/p < 0.
\]

Making less copies of expensive movies and holding on to them longer agrees with intuition.
Proposition 4: \(\frac{dT}{dK} > 0, \frac{dQ}{dK} < 0\) and \(\frac{d\Pi}{dK} < 0\).

Proof: Differentiating \(\Pi_T^I = 0\) with respect to \(K\), we have,

\[
T_K \Pi_T^I + 1/T^2 = 0 \Rightarrow T_K = -\Pi_T^I/T^2 > 0 \quad \therefore \Pi_T^I < 0.
\]

Then, from (5), \(Q_k = d\lambda(A + BT)/dK = T_K \cdot B\lambda(A + BT) < 0\).

Also, from (6), \(\Pi_K^I = T_K \Pi_T^I - 1/T = -1/T < 0\).

When fixed costs are high, it makes sense to incur them less often. Since \(T\) thus increases with \(K\), the system will reduce \(Q\) in order to partly compensate for the increase in holding cost per cycle (since, as argued on top of page 5, \(Q\) is decreasing in \(T\)).

3. Decentralized System: Wholesale-Price-Only Contract

The video rental chain (store) purchases each copy of the movie at a wholesale price of \(w\) per unit, rents them at \(p\) per unit, has a set-up cost of \(K_R\) per cycle (when switching movies) and holding cost of \(h(w)\) per unit per time, where \(h(w) = h + iw\) (where we assumed that the retailer’s capital cost rate is same as of integrated system). The studio has a production cost of \(c\) per unit, and set-up cost of \(K_M\) per cycle. Presumably \(K_R + K_M\) equal the integrated firm’s fixed costs.

Then,

\[
\text{Retailer’s profit} = \Pi^R = [R(Q, T) - K_R - wQ - h(w)QT]/T, \quad (9)
\]

and

\[
\text{Manufacturer’s profit} = \Pi^M = [(w - c)Q - K_M + (w - c)iQT]/T, \quad (10)
\]
where we assume that the manufacturer invests the revenue of \((w - c)Q\) it receives at beginning of cycle at interest rate \(i\) (same as the retailer’s cost of capital) for the whole cycle (cf. Boyaci and Gallego 2000). With that assumption, \(\Pi^R + \Pi^M = \Pi^f\). Note that if we assume that \(h(c) = h(w) = h\), then \(\Pi^R + \Pi^M = \Pi^f\) holds without the manufacturer investment assumption. However, with \(h(c) = h + ic\) and \(h(w) = h + iw\) (where \(w \neq c\)) and without the investment assumption, \(\Pi^R + \Pi^M \neq \Pi^f\) and \(w\) then becomes a decision variable, in addition to \(Q\) and \(T\), even in the “combined” system.

Let \(T^w\) and \(Q^w\) be the solution to \(\Pi^R_T = 0\) and \(\Pi^R_Q = 0\). They represent the retailer’s optimal policy as a function of the wholesale price \(w\). Note that the structural similarity between \(\Pi^R\) and \(\Pi^f\) implies that Propositions 2, 3 and 4 are also applicable for \(\Pi^R\). In particular, the higher the wholesale price, the fewer copies will the retailer order and the longer it will hold on to them before switching. If the retailer’s fixed cost of ordering is high, it will order infrequently and small quantities.

Proposition 5: \(T^w > T^f\) and \(Q^w < Q^f\).

Proof: \(T^f \Pi^R_Q(Q^f, T^f) = p \lambda^{-1}(Q^f) - w - hf - iwT^f\)

\[= T^f \Pi^f_Q(Q^f, T^f) - (w - c)(1 + iT^f) \quad \text{[from (3)]} \]

\[= -(w - c)(1 + iT^f) < 0. \quad \text{[since \(\Pi^f_Q(Q^f, T^f) = 0\)]} \]

Since \(Q^f\) is in the decreasing region of \(\Pi^R\), \(Q^w < Q^f\).

\[(T^f)^2 \cdot \Pi^R_T(Q^f, T^f) = T^f R(Q^f, T^f) - R(Q^f, T^f) + K_R + wQ^f\]

\[= (T^f)^2 \cdot \Pi^f_T(Q^f, T^f) - K_R + (w - c)Q^f \quad \text{[since \(K = K_R + K_M\)]} \]
\[\begin{align*}
&= -K_M + (w - c)Q^i \\
&> -K_M + (w - c)Q^w \\
\text{Hence, if the manufacturer’s profit for a wholesale-price-only contract with } i = 0 \text{ is positive (which is entirely plausible), } -K_M + (w - c)Q^w > 0 \text{ and then } T^I \text{ is in the increasing region of } \Pi^R, \text{ implying that } T^W > T^I.
\end{align*}\]

Proposition 5 implies that for any \( w \), a wholesale price only contract cannot coordinate the system. So, even if \( \Pi^M(Q, T) \) is concave in \( w \) and the manufacturer offers \( w^* \) that will maximize \( \Pi^M \), it will not be able to coordinate the system.

4. Revenue-Sharing-with-Wholesale-Price Contract

Let the share of the revenue that the retailer will receive be \( \theta (0 \leq \theta \leq 1) \) while the manufacturer will receive \( 1 - \theta \). Note that under such revenue sharing \( (R-S) \) contract \( w \) can be less than \( c \).

The retailer’s profit with such a \( R-S \) contract is,
\[\Pi^{RS} = [\theta R(Q, T) - K_R - wQ - h(w)QT]/T, \quad (11)\]
and manufacturer’s profit is,
\[\Pi^{MS} = [(1 - \theta)R(Q, T) - K_M + (w - c)Q + (w - c)iQT]/T. \quad (12)\]

If the manufacturer invests (borrows) the initial revenue (shortfall) from wholesale price, then
\[\Pi^{RS} + \Pi^{MS} = \Pi^I.\]

For a fixed \( \theta \), Propositions 2, 3 and 4 apply to \( \Pi^{RS} \).
Differentiating $\prod^{RS}$ with respect to $Q$ and $T$ we have,

$$T^2 \prod^{RS}_T = \theta TR_T - \theta R + K_R + wQ = 0,$$

(13)

$$T \prod^{RS}_Q = \theta p \hat{\lambda}^{-1}(Q) - w - h(w)T = 0.$$  

(14)

Let $\theta^*$ and $w^*$ be the values of the coordinating levers that result in $Q^I$ and $T^I$ as the solutions of (13) and (14). Since $p \hat{\lambda}^{-1}(Q^I) = c + h(c)T^I$ from (5) and $T^I R_T = R - (K_R + K_M) - cQ^I$ from $\prod^I_T = 0$ in (4), we can obtain the following solution:

$$\theta^* = 1 - \frac{K_M}{K_R + K_M - hQ^I T^I/(1 + iT^I)},$$

(15)

$$w^* = \theta^* c - \frac{(1 - \theta^*) h T^I}{1 + iT^I},$$

(16)

and on simplification we have,

$$\prod^{RS}(\theta^*, w^*) = \theta^* \prod^I - \frac{hQ^I (1 - \theta^*)}{1 + iT^I}.$$  

(17)

If $h = 0$, then $\theta^* = K_R / (K_R + K_M)$, $w^* = \theta^* c$ and $\prod^{RS}(\theta^*, w^*) = \theta^* \prod^I$.

5. Revenue-Sharing-with-Wholesale-Price and License Fee Contract

Note that with the $R-S$ contract of Section 4, depending on the relative values of $K_R$ and $K_M$, the retailer’s/manufacturer’s profit might be unacceptably small and the parties might not be willing to enter into such contract. In order to satisfy the IR of both parties (i.e., so that they will not be worse off than in status-quo), we introduce a new control lever $S$, where $S$ is the subsidy (license fee) paid (received) by the manufacturer per cycle (i.e., per new movie).

With $S$, the retailer’s profit with such $R-S$ contract is,
\[ \Pi^{RS} = [\theta_s R(Q, T) - K_R + S - wQ - h(w)QT] / T. \] (18)

and manufacturer's profit is,
\[ \Pi^{MS} = [(1 - \theta_s) R(Q, T) - K_M - S + (w - c)Q + (w - c)iQT] / T. \] (19)

Following the same technique as in Section 4 we can show that,
\[ \theta_s^* = 1 - \frac{K_M + S}{K_R + K_M - hQ^I T^I / (1 + iT^I)}, \] (20)

\[ w_s^* = \theta_s^* c - \frac{(1 - \theta_s^*) hT^I}{1 + iT^I} \] (21)

and
\[ \Pi^{RS}(\theta_s^*, w_s^*) = \theta_s^* \Pi^I - \frac{hQ^I}{1 + iT^I} (1 - \theta_s^*) \] (22)

By varying the value of \( S \) it is now possible to satisfy IR for both parties.

**Proposition 6:** \( d \Pi^{RS} / dS < 0, \ d \theta_s^* / dS < 0 \) and \( dw_s^* / dS < 0 \).

**Proof:** Differentiating (20) w.r.t. \( S \), we have \( d \theta_s^* / dS < 0 \). Differentiating (21) and (22) w.r.t. \( S \) and using \( d \theta_s^* / dS < 0 \), it is clear that \( dw_s^* / dS < 0 \) and \( d \Pi^{RS} / dS < 0 \). ||

The incorporation of \( S \) affects not only IR but also \( w \) and \( \theta \). If we assume the IR level for the retailer to be equal to \( \Pi^R \), then for \( \Pi^{RS} = \Pi^R \), the retailer's IR is satisfied exactly and hence \( S = 0 \), implying that \( \theta_s^* = \theta^* \) and \( w_s^* = w^* \) [comparing (15) with (20) and (16) with (21)]. If \( \Pi^{RS} > \Pi^R \), then \( S > 0 \), i.e., the manufacturer pays a subsidy to the retailer for each new movie. Hence, from Proposition 6, \( \theta_s^* < \theta^* \) and \( w_s^* < w^* \). If \( \Pi^{RS} < \Pi^R \), then \( S < 0 \), i.e., the retailer has...
to pay a license fee to the manufacturer for each new movie, and then from Proposition 6, \( \theta_s^* > \theta^* \) and \( w_s^* > w^* \).

With the subsidy, the effective price paid by the retailer = \( w_s^* + (S / Q^I) \). But it is important to remember that this effective price is only an ex-post result. In the ex-ante model it is necessary to consider the fixed fee \( S \) separately from the per-unit price for the subsequent optimization to give proper results.

6. Example

For this example, we assume the rental period (unit of time) to be 3 days. Based on industry average data of demand rate for videos from Lehmann and Weinberg (2000) and assuming that the approximate number of Blockbuster video stores was 5000 in 1998 (source: Blockbuster video) it seems we can express the demand function for the entire chain as \( \lambda(t) = (# \text{ of stores}) \times v_t e^{-v_j t} \) = \( (5000 \times 26.99 \times 3/7) e^{-(0.5 \times 3/7) t} = 57835.7 e^{-0.02143 t} \) (note that for Lehmann and Weinberg the unit of time is a week and the unit of analysis is a single store, not, as for us, the entire chain). We assume that the values for the other parameters are as follows: \( i = 20\% \times 3/365 = .1644\% \), \( p = 5 \), \( h = .01644 \), \( c = 5 \), \( K_R = 50,000 \), \( K_M = 250,000 \). Some of these values were selected rather arbitrarily, so more extensive computational results which vary them will be provided subsequently.

Based on this data the optimal choices for the integrated system will be, \( Q^I = 56,518 \) (i.e., about eleven copies per store which constitutes 97.7% of the initial demand) and \( T^I = 15.3 \) (about 46 days). The resulting optimal profit is \( \Pi^I = 206,854 \).
For a decentralized system with wholesale-price-only contract, $Q^w = 35,420$, $T^w = 59.2$, $w^* = 103.4$, $\Pi^M = 60,324$ and $\Pi^R = 74,659$. Note that $\Pi^R + \Pi^M$ is almost 35% less than $\Pi'$. $Q^w$ satisfies only 61.2% of the initial demand implying that many customers will be dissatisfied.

For a decentralized system with R-S and price contract, $\theta^* = .126$, $w^* = .416$, $\Pi_{RS} = 25,319$ and $\Pi_{MS} = 181,536$. Note that with this contract, by construction, $\Pi_{RS} + \Pi_{MS} = \Pi'$. However, $\Pi_{RS} < \Pi^R$ and $\Pi_{MS} > \Pi^M$. So, though this contract is beneficial for the manufacturer and achieves coordination, the retailer might not accept it. While the wholesale price is quite low, the retailer’s share in revenue is low as well.

If we introduce a subsidy (license fee), we can show that with a license fee of 67,948 paid by the retailer (i.e., $S = -67,948$), $\theta^*_s = 0.364$ and $w^*_s = 1.66$, the system can be coordinated (i.e, optimal $Q = Q'$ and optimal $T = T'$). With this contract, $\Pi_{RS_s} = 74,659 = \Pi^R$ and $\Pi_{MS_s} = 132,196 > \Pi^M = 60,324$. So, the manufacturer can increase its profit by almost 220% compared to optimal wholesale-price-only contract by using revenue sharing and subsidy simultaneously while satisfying the supplier’s IR (which was assumed to equal the profit it could obtain from an optimal wholesale-price-only contract). Although the retailer pays a substantial license fee, the corresponding (much) higher revenue share more than compensates him for that. The effective wholesale price paid by the retailer in this case is approximately 2.86. However note that if $\theta = 0.364$ and $w = 2.86$ in a revenue sharing with wholesale price contract (without $S$), $Q^{RS} = 55,703 < Q'$, $T^{RS} = 15.3$, $\Pi^{RS} = 74,579 < \Pi_{RS_s} = \Pi^R$ and $\Pi_{MS} = 132,184 < \Pi_{MS_s}$.

For sensitivity analysis, we focus our attention on the following four parameters: $K_R$, $K_M$, $\ell$ and $c$, since the demand function is taken from empirical data and the values of $p$ and $i$ are realistic. The results are shown in Table 1. The base case refers to the example explained above.
For sensitivity analysis of each parameter, we change only that particular parameter value keeping the values of other parameters at the base case level. We highlight in bold the base case in each data set. The following observations can be made from the sensitivity analysis:

- As $K_R$ increases, $S$ increases - for low values of $K_R$, $S$ is negative and it becomes positive for high values. This is intuitive since as the retailer spends more on set-up cost, the manufacturer will subsidize it more (or at least lower the license fee). On the other hand, as $K_M$ increases, $S$ decreases (becoming more negative for higher $K_M$). The higher the manufacturer’s fixed costs, the larger the license fee it will “require”.

- As $h$ increases, $S$ becomes more negative, i.e., the retailer pays a higher license fee per cycle. That may seem surprising if viewed in isolation, but we note that, at the same time, the retailer’s share of the profit increases and the wholesale price decreases. That combination maintains the retailer’s profit at its required level, while the manufacturer’s profit declines.

- It might also seem strange that for high values of $K_M$ or $h$, $w^*$ is negative (note that for the models of Sections 4 and 5 there is no restriction on the sign of $w$; however, for Section 3 we must have $w > c$). A negative $w$ can be thought of as a form of per-unit subsidy from the manufacturer to the retailer, accompanied simultaneously by a decrease in the retailer’s share in revenue.

- As $c$ increases, as expected, the wholesale price increases. Simultaneously, the retailer’s share in revenue increases, but it has to now pay a higher license fee to the manufacturer.

- The incorporation of $S$ not only enables meeting of retailer's IR, but also affects the sensitivity of the optimal variable values towards the parameters (e.g., as $K_M$ increases, $\theta^*$ decreases while $\theta_s^*$ increases).
- It seems that the effect of the parameters on \( T^I \) is much more significant than on \( Q^I \). As for \( T^w \) and \( Q^w \), neither is significantly affected.

- The effect of \( K_R \) and \( K_M \) is primarily on \( S \). While \( h \) affects \( S \), \( \theta_s \) and \( w_s^* \), the effect of \( c \) is primarily on \( w_s^* \).

7. Concluding Remarks

The main contributions of this paper are as follows. 1. We propose and analyze a dynamic bivariate model of video rental replenishments. This model appears to be new even in the context of an integrated channel. 2. We show that a price-only contract cannot coordinate a decentralized chain. 3. We show that a price + RS contract can coordinate the channel, but IR may not be satisfied. 4. We propose to add a license fee (or subsidy) to the contract as a third lever, thereby satisfying all the requirements. 5. We provide comparative statics and numerical insights for all contracts.

It may appear that by proposing a deterministic model of video rental operations we are moving "backwards" vis-à-vis the predominantly stochastic literature. But we believe such is not the case, for the following reasons. 1. Since most of the uncertainty concerning a new movie's popularity is realized before it is released in video, by the time of release on video its demand can be forecasted rather accurately. Indeed, Lehmann and Weinberg (2000) argue that even the subsequent demand dynamics can also be well predicted. 2. We utilize our deterministic framework to perform a more detailed temporal analysis of a movie's shelf "career", unlike the static (newsvendor) models proposed before. Our integrated setting itself gives rise to a new bivariate inventory model. 3. We pay close attention to the nature and contracting-over holding
costs, something that was partly addressed by some authors (e.g., Boyaci and Gallego 2000; Buzacott and Zhang 2001) but not in a video rental context. 4. The contracts we discuss are richer - they may include license fees (or subsidies), revenue-shares and wholesale prices. 5. Revenue-sharing and other forms of consignment are usually associated with situations of demand uncertainty and/or information asymmetry, where they provide means of sharing risk. As our model is deterministic and information is symmetrical, the observation that revenue-sharing can nevertheless be useful seems interesting.

Future research wishing to incorporate stochastic aspects into the model should perhaps focus on the timing of availability of newly released video movies (note that Lehmann and Weinberg considered that to be the studio’s decision variable). If a new movie is available “early”, that does not directly affect our model, but if no new movie is available at time $T$, the assumptions underlying our model cease to hold. Other uncertainties, like demand pattern, could be incorporated in principle, but it is not clear whether they would enrich the setting much.

References


Table 1: Sensitivity Analysis

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