Optimal Control Approach to Production Systems with Inventory-Level-Dependent Demand

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Abstract

Demand occasionally depends positively on the amount of displayed stock, especially in the case of novelty or impulse purchase items. Such inventory-level dependence of demand rate has been incorporated into some continuous review inventory control models, but only ones with demand rate which does not vary with time, no shortages, infinite production rate and per-unit-time cost minimization. We propose an optimal control model which relaxes all these assumptions. That model generalizes a known optimal control model by adding inventory-level dependence of demand to the state dynamics. The framework is one of discounted profit maximization. We apply the maximum principle and obtain three possible singular regimes. The special case of time-invariying demand is solved explicitly. The uncapacitated case is analyzed in detail for a "wave-mode" time dependence of demand, and a numerical example is given.

1. Introduction

The Production Control literature typically assumes that demand, whether deterministic or stochastic, or time-varying or not, is exogenous – firm's production/inventory policy cannot affect it. But, it has been observed by marketing practitioners and researchers that displayed stocks can have a stimulating effect on demand (e.g., Corstjens and Doyle 1981). At least two types of stimuli effects of inventory on demand have been identified. The first and more obvious is referred to as the “selective effect”, where more items in inventory provide customers with more to choose from and thus induce them to purchase more. This will happen where units of an item are not identical, a customer may like the feeling of a wide “selection. For certain baked goods, low stocks may raise customers’ perception that the units are “left-overs” and not fresh. The second type of stimulus of display quantity is the “advertising effect”. Large displays often raise the perception among customers that the item is popular in the market, which may signal a good “value” and cause them to purchase more (or more often). Some stores display huge inventory “piles” of products like blank video-tapes, sufficient to meet many years of demand, clearly aiming at stimulating demand. See Gerchak and Wang (1994) and Balakrishnan *et al*. (2000) for a review of relevant literature.

As for incorporating inventory-level dependence into traditional inventory models, Johnson (1968), Gerchak and Wang (1994) and Wang and Gerchak (2000) have done so for stochastic periodic review models, and numerous studies starting with Baker and Urban (1988) through Balakrishnan *et al*. (2000) for deterministic continuous review models. The latter task is complex due to the difficulty of proving joint concavity in lot-size and re-order point, because the holding costs are a solution of a differential equation.

This work proposes a different approach to modeling and analyzing deterministic production/inventory systems with inventory-level dependent demand rate – optimal control (Bensoussan, Crouchy and Proth 1983, Sethi and Thompson 2000, Kogan and Khmelnitsky 2000). The optimal control approach proved its efficiency in many economic, managerial and industrial applications (see, e.g. Khmelnitsky and Caramanis 1998, Zhang, Yin and Boukas 2001). The advantages of this approach in our context are as follows: *i*) it permits using a finite or (discounted) infinite horizon objective; initial and/or terminal inventory level(s) can be specified; *ii*) the production rate can be upper bounded; *iii*) demand rate can be time-varying as well as inventory-level dependent. In case of a shortage, it can decline in the magnitude of the shortage; *iv*) model can be solved to characterize the optimal production policy.

The model generalizes those analyzed by Bensoussan *et al*. (1983, Ch. III), and Kogan and Khmelnitsky (2000, Ch. 3) in adding inventory-level dependence of demand to the state dynamics. Since revenue is not constant anymore, the framework is one of profit maximization; we use an infinite horizon discounted formulation for concreteness. On the other hand, our model does not have setup costs, which were a key ingredient in the EOQ-based formulations. Note that although the resulting inventory depletion pattern is reminiscent of some perishable inventory models, here inventory is depleted by sales, which contribute to revenue.

We apply the maximum principle and obtain three possible singular regimes. The special case of time-invarying demand is solved explicitly. An uncapacitated model with "wave-mode" time dependence of demand is analyzed in detail and a numerical example is provided.

2. Problem formulation

Let $X(t)$ be the inventory level at time t, and $d(X, t)$ the demand rate at time t when inventory level is X. Let X^+ be the positive portion of X $[X^+ = \max(X,0)]$ and c^+ the holding cost rate per unit. Let X^- be the negative portion of inventory $[X^- = -\min(X,0)]$ and c^- the shortage penalty rate per unit. Let p be the revenue rate per unit and ρ the discount rate. The control is the production rate function $u(t)$, which might be upper bounded by U. Let $C(u)$ be the cost rate corresponding to a production rate u. Then, the problem is to maximize the total profit of the firm (revenue minus inventory and production costs):

$$\text{Max } J = \int_0^\infty e^{-\rho t}\left[pd(X,t) - c^+X^+(t) - c^-X^-(t) - C(u(t))\right]dt \tag{1}$$

s.t.

$$\dot{X}(t) = u(t) - d(X,t), \quad X(0) = X_0, \tag{2}$$

$$0 \le u(t) \le U. \tag{3}$$

Note that production costs can depend on production rate in a non-linear (plausibly, convex) manner, which is particularly important if U is infinite.

3. Capacitated problem with linear production cost

The necessary condition of optimality for the problem (1)-(3) takes the form of the maximum principle, which, for the case of a linear production cost, $C(u) = c_u u$, declares that there exists a continuous co-state function $\psi(t)$ satisfying the co-state equation

$$\dot{\psi}(t) = \frac{\partial d(X,t)}{\partial X}\left(\psi(t) - pe^{-\rho t}\right) + e^{-\rho t}\begin{cases} c^+, & \text{if } X(t) > 0 \\ -c^-, & \text{if } X(t) < 0 \\ [-c^-, c^+], & \text{if } X(t) = 0 \end{cases} \tag{4}$$

Another requirement of optimality is that the maximization of the Hamiltonian function

$$H = e^{-\rho t}\left[pd(X,t) - c^+ X^+(t) - c^- X^-(t) - c_u u(t)\right] + \psi(t)[u(t) - d(X,t)] \tag{5}$$

with respect to $u(t)$ results in

$$u(t) = \begin{cases} U, & \text{if } \psi(t) > c_u e^{-\rho t} \\ 0, & \text{if } \psi(t) < c_u e^{-\rho t} \\ [0, U], & \text{if } \psi(t) = c_u e^{-\rho t} \end{cases} \tag{6}$$

Assumptions on the problem parameters are

- The function $d(X,t)$ is differentiable w.r.t. t and twice differentiable w.r.t X except for X=0; for X=0, the subdifferential of $d(0,t)$, denoted by $\partial d(0,t)$ is a known interval

  $$\partial d(0,t) = [a(t), b(t)],\ a(t) \geq 0,\ b(t) \geq 0.$$

  Here subdifferential $\partial d(0,t)$ is defined as the set $\left\{x^* \mid d(X,t) - d(0,t) \geq Xx^*, \forall X\right\}$.

- $\dfrac{\partial d(X,t)}{\partial X}$ is non-negative for $X \neq 0$.

- $p > c_u$.

The first two lines in (6) uniquely define the production rate as a function of the co-state variable (full production and no production regimes respectively). The third line in (6) presents a singular production regime, for which the optimal control value cannot be obtained by Hamiltonian maximization. To determine $u(t)$ along the singular regime, we let the singular condition $\psi(t) = c_u e^{-\rho t}$ hold in an interval of time. Then, over this interval

$$\dot{\psi}(t) = -\rho c_u e^{-\rho t}\,. \tag{7}$$

By substituting (7) in (4), we obtain three types of singular regimes

SR1: $X(t)=0$ over the singular interval;

SR2: $X(t)>0$ over the singular interval;

SR3: $X(t)<0$ over the singular interval.

Considering the state and co-state equations (2) and (4) under the three singular regimes, we determine the following conditions which are necessary for the regimes occurrence.

The SR1 regime can potentially occur only at those time intervals where

$$(p - c_u)[a(t), b(t)] \cap \left(\rho c_u + [-c^-, c^+]\right) \neq \emptyset \quad \text{and} \quad 0 \leq d(0,t) \leq U.$$

The optimal control is $u(t)=d(0,t)$.

The SR2 regime can potentially occur only at those time intervals where

$$0 \leq \dot{\hat{X}}(t) + d(\hat{X}(t), t) \leq U \quad \text{and} \quad \hat{X}(t) > 0\,.$$

The optimal control is $u(t)=\dot{\hat{X}}(t) + d(\hat{X}(t), t)$, where $\hat{X}(t)$ is a solution of equation $\dfrac{\partial d(X,t)}{\partial X} = \dfrac{\rho c_u + c^+}{p - c_u}$.

The SR3 regime can potentially occur only at those time intervals where

$$0 \leq \dot{\tilde{X}}(t) + d(\tilde{X}(t), t) \leq U \quad \text{and} \quad \tilde{X}(t) < 0.$$

The optimal control is $u(t)=\dot{\tilde{X}}(t) + d(\tilde{X}(t), t)$, where $\tilde{X}(t)$ is a solution of equation $\dfrac{\partial d(X,t)}{\partial X} = \dfrac{\rho c_u - c^-}{p - c_u}$.

3.1 Example

Consider a special case when the demand function does not change in time, $d(X,t)=d(X)$. That is the type of demand considered by previous inventory-dependent demand literature though within different economic setting (emphasizing setup cost). Also assume that $\frac{\partial^2 d(X)}{\partial X^2} < 0$ for $X>0$, and that $\rho c_u < c^-$. Then, the SR3 regime does not exist and the SR2 regime exists for only one value of the buffer level, $\hat{X}$, which is constant in time. If both $d(0) \le U$ and $d(\hat{X}) \le U$, then there are two steady state solutions that satisfy the necessary optimality conditions, $X(t) \equiv 0$ and $X(t) \equiv \hat{X}$. In order to compare the objective values of these two solutions, we have to take into account not only the steady state itself, but also how the solution converges to the steady state from the initial value X_0. To be specific, let the steady state be reached at $X(t) \equiv \hat{X}$ and

$$d(X) = \begin{cases} d_{\max} - (d_{\max} - d_0)e^{-\beta X}, & \text{if } X \ge 0 \\ d_0 e^{\alpha X}, & \text{if } X < 0. \end{cases}$$

Such a choice of the $d(X)$ function reflects the "selective" and "advertising" effects of inventory discussed in Section 1, i.e. $d(X)$ increases when X^+ increases. This $d(X)$ indicates also the lost sales effect for shortages, i.e. $d(X)$ decreases when X^- increases. In both cases $d(X)$ converges, either to d_{max} when X^+ goes to infinity, or to zero when X^- goes to infinity.

Now, by integrating the state equation (2), we obtain

$$X(t) = \frac{1}{\beta}\ln\left(e^{\beta(X_0 + At)} + \frac{B}{A}\left(e^{A\beta t} - 1\right)\right),$$

where

$$A = \begin{cases} U - d_{\max}, & \text{if } X_0 < \hat{X} \\ -d_{\max}, & \text{if } X_0 > \hat{X} \end{cases} \quad \text{and } B = d_{\max} - d_0.$$

For the parameters U=10, d_{max}=11, d_0=5, X_0=3, p=5, c_u=3.5, c^+=0.3, β=0.2 and ρ=0.02, we calculated $\hat{X}$ =7.9, $d(\hat{X})$ =9.76, and the time point $\hat{t}$ at which the trajectory enters the singular regime $X(t)=\hat{X}$ is equal to 6.47. The objective value is J=595.19. Similarly, for the other steady state X=0 we found that the steady state level is reached at t=0.47 and the objective value is J=386.24.

If the convergence interval is negligible, then the comparison between the two options is based on the steady state parameters only. In such a case, we find that the solution $X(t) \equiv 0$ is better than the other one, $X(t) \equiv \hat{X}(t)$, if $\frac{d(\hat{X}) - d(0)}{\hat{X}} < \frac{c^+}{p - c_u}$, i.e. if $(p - c_u)[d(\hat{X}) - d(0)] < c^+ \hat{X}$, which is intuitive. Otherwise, the second solution is preferable over the first one.

4. Uncapacitated problem with non-linear production cost

In this section we assume that the production cost function $C(u)$ is strictly convex and differentiable. In such a case, the production costs are increasing sharply, so a high production rate is not likely to occur, thereby allowing us to drop the capacity constraint. The maximization of the Hamiltonian results in

$$u(t) = \begin{cases} 0, & \text{if } \psi(\text{t}) \le e^{-\rho t} C'(0) \\ [C']^{-1}(e^{\rho t}\psi(t)), & \text{if } \psi(\text{t}) > e^{-\rho t} C'(0) \end{cases} \tag{8}$$

where $\psi(t)$ satisfies the same co-state equation (4). The demand function is assumed to be differentiable w.r.t. X, except for X=0, and differentiable w.r.t. t except for a finite number of time points. Unlike in the previous (linear, capacitated) case, here there is only one type of singularity, at X=0. If the singular regime occurs in an interval of time, then over that interval

$$u(t) = d(0,t), \quad \psi(t) = e^{-\rho t} C'(d(0,t)).$$

The necessary condition for the regime to occur is

$$(p - C'(d(0,t)))[a(t), b(t)] \cap (\rho C'(d(0,t)) - C''(d(0,t))\dot{d}(0,t) + [-c^-, c^+]) \ne \text{Ø}. \tag{9}$$

4.1 Solution method

Consider a "wave-mode" demand, i.e. for any X, $d(X,t)$ increases over an initial time interval and $d(X,t)$ drops down thereafter (see examples in Figure 1), which is rather plausible in many situations. The “wave-mode” demand can, for example, reflect a typical product life cycle, which consists of four major segments: start-up, rapid growth, maturation and decline (see Nahmias (1997)). For such cases, the number of time intervals over which condition (9) holds, is at most three. That is because for large positive and large negative values of $\dot{d}(0,t)$, condition (9) does not hold (for example, at the jump points in Figure 1b).

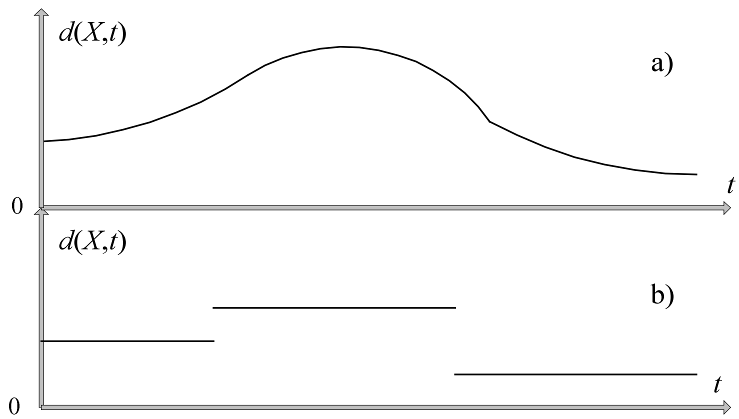

Figure 1. Examples of a "wave-mode" demand.

Without loss of generality, we assume that the number of time intervals over which (9) holds is exactly three. If there are less, then the solution method simplifies. Denote the time intervals over which (9) holds by τ_1, τ_2 and τ_3. Since the time horizon is not bounded, the final regime must be singular. Otherwise, the solution is not bounded, i.e. $|X(t)|$ goes to infinity. However, the two intermediate intervals τ_1 and τ_2 may or may not have singularity. Therefore, the problem of finding the optimal solution becomes a combinatorial one. Denote the starting time point by τ_0, i.e. $\tau_0=\{0\}$, and by $\tau_0 \to \tau_{i_1} \to \ldots \to \tau_{i_j} \to \tau_3$, $0 \le j \le 2$ the solution which does have singularity within the intervals τ_{i_k}, k=1,…,j, and does not have singularity within other intervals.

There are four possible solutions

$$\tau_0 \to \tau_1 \to \tau_2 \to \tau_3, \quad \tau_0 \to \tau_1 \to \tau_3, \quad \tau_0 \to \tau_2 \to \tau_3, \quad \tau_0 \to \tau_3. \tag{10}$$

For these solutions the state-co-state dynamics over the singular arcs, denoted by τ_i, is known to be $X(t)=0$ and $\psi(t)=c_u e^{-\rho t}$. The state-co-state dynamics over the regular arcs, which link either two singular arcs or τ_0 with a singular arc, is determined from the following state and co-state equations:

$$\dot{X}(t) = -d(X,t) + \begin{cases} 0, & \text{if } \psi(\text{t}) \le e^{-\rho t} C'(0) \\ [C']^{-1}(e^{\rho t}\psi(t)), & \text{if } \psi(\text{t}) > e^{-\rho t} C'(0) \end{cases} \tag{11}$$

$$\dot{\psi}(t) = \frac{\partial d(X,t)}{\partial X}\left(\psi(t) - pe^{-\rho t}\right) + e^{-\rho t}\begin{cases} c^+, & \text{if } X(t) \ge 0 \\ -c^-, & \text{if } X(t) < 0 \end{cases}. \tag{12}$$

To link a pair of τ's, the method makes use of a shooting procedure for solving two-point boundary-value problems (Sethi and Thompson, 2000.). Specifically, to link τ_0 with τ_i, *i*=1,2,3, the procedure looks for such value of $\psi(0)$ that allows the solution of the system (11)-(12) to enter the singular arc at a time point $t_i^{in} \in \tau_i$. Similarly, to link τ_i with τ_n, $1 \le i < n \le 3$, the procedure looks for such point $t_i^{out} \in \tau_i$ of exiting singularity at τ_i that allows the solution of the system (11)-(12) to enter the singular arc at a time point $t_n^{in} \in \tau_n$. A solution $\tau_0 \to \tau_{i_1} \to ...\tau_{i_j} \to \tau_3$, $0 \le j \le 2$ for which $t_{i_k}^{in} \le t_{i_k}^{out}$, *k*=1,…,*j* is locally optimal, since it satisfies the necessary conditions of optimality. If, among four possible solutions (10) there are two or more local optima, the solution with the least objective (1) is the globally optimal one.

4.2 Example

The parameters of the problem were chosen as:

X_0=3, p=5, ρ=0.02, α=0.1, β=0.2, c^+=0.3, c^-=1, $C(u) = 3u + 0.3u^2$.

To exemplify the solution method, we choose the *d*(*X*,*t*) function in such a way that it has a “wave” pattern as a function of *t* for each *X*, as discussed in Section 4.1, and it has a monotone increasing pattern as a function of *X* for each *t*, as discussed in Section 3.1. Formally,

$$d(X,t)=\begin{cases} d_{\max}(t)-(d_{\max}(t)-d_0(t))e^{-\beta X}, & \text{if } X \ge 0 \\ d_0(t)e^{\alpha X}, & \text{if } X < 0 \end{cases}, \quad \text{where}$$

$$d_{\max}(t)=\begin{cases} 7, & \text{if } 0 \le t < 4 \\ 11, & \text{if } 4 \le t < 7 \\ 4, & \text{if } 7 \le t \end{cases} \qquad d_0(t)=\begin{cases} 3, & \text{if } 0 \le t < 4 \\ 5, & \text{if } 4 \le t < 7 \\ 2, & \text{if } 7 \le t \end{cases}$$

For the adopted parameters, the singular regime condition (9) holds at three intervals: $\tau_1=(0,4)$, $\tau_2=(4,7)$ and $\tau_3=(7,\infty)$. By shooting, we tried to link each pair of the intervals. It was found that the pairs $\tau_0 \to \tau_3$ and $\tau_1 \to \tau_3$ cannot be linked. In spite of the fact that all the pairs in the trajectory $\tau_0 \to \tau_1 \to \tau_2 \to \tau_3$ are linked, the trajectory itself does not exist. This is because $t_1^{in} > t_1^{out}$. Thus, the only trajectory which does exist is $\tau_0 \to \tau_2 \to \tau_3$ where both pairs of intervals are linked and $t_2^{in} < t_2^{out}$. Figures 2-4 present the optimal solution.

5. Conclusions

The optimal control approach allowed us to formulate problems of inventory/production control at considerable level of generality. In particular, time dependence (as well as inventory level dependence) of demand is a feature which existing inventory models do not have. We acknowledge, however, that our "continuous production" formulation does not account for setup costs, the key ingredient of EOQ-like models in literature. We provided specific insights, analytical and numerical, into two scenarios which may be viewed as capturing a similar reality: capacitated production with linear production costs and uncapacitated production with convex production costs.

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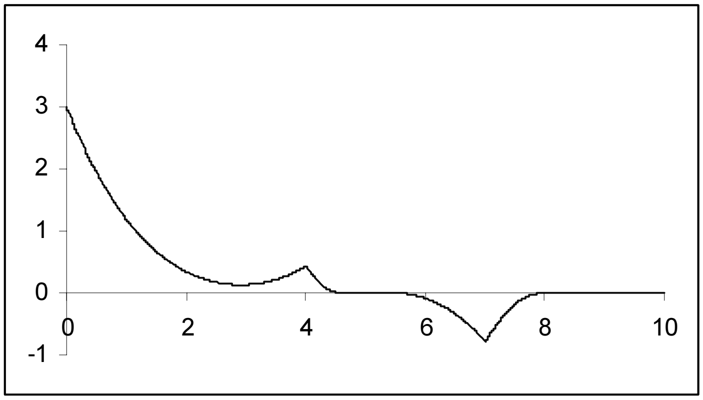

Figure 2. Inventory level $X(t)$.

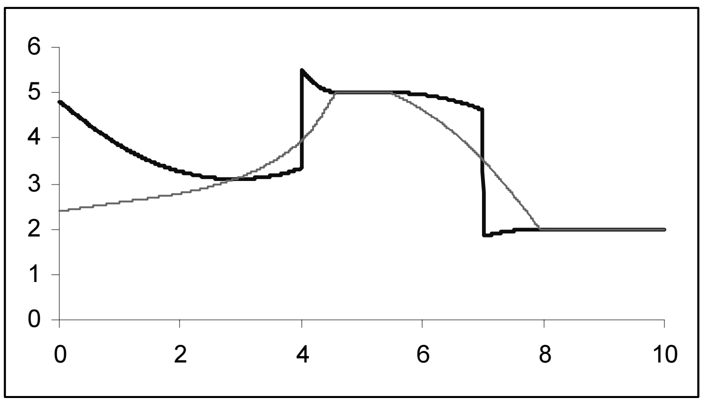

Figure 3. Demand $d(X,t)$ (bold line) and production rate $u(t)$ (thin line).

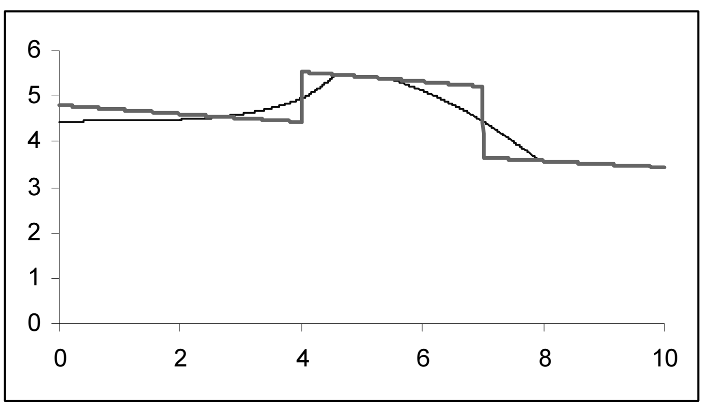

Figure 4. Singular co-state dynamics $e^{-\rho t}$ *Cost'*($d(0,t)$) (bold line) and co-state variable $\psi(t)$ (thin line).