Poisson and Wiener: Corrections and Additions to Class 10

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Abstract

See below few corrections to the first class, few additions, and two uncovered topics: Poisson increments process, and Wiener-Levy process.

1 Corrections and Additions

- Counting process versus arrival process: The equivalent events are

$$\{N(t) < n\} \Leftrightarrow \{T_n > t\}$$

or equivalently $\{N(t) \geq n\} \Leftrightarrow \{T_n \leq t\}$, but NOT $\{N(t) \leq n\} \Leftrightarrow \{T_n \geq t\}$ as taught in the first class.

- Probability distribution of first arrival:

$$\Pr(t < T_1 \leq t + \Delta) = \Pr(N(t) = 0, N(t + \Delta) \geq 1)$$

and not $\Pr(N(t) = 0, N(t + \Delta) = 1)$ as taught in the first class.

- Joint distribution of three samples (taught in second class but not in the first): Assume $t_1 < t_2 < t_3$ and $0 \leq n_1 \leq n_2 \leq n_3$. Then, by the independent increments property

$$\Pr(N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3) \tag{1}$$
$$= \Pr(N(t_1) = n_1, N(t_1, t_2) = n_2 - n_1, N(t_2, t_3) = n_3 - n_2) \tag{2}$$
$$= \Pr(N(t_1) = n_1) \cdot \Pr(N(t_1, t_2) = n_2 - n_1 | N(t_1) = n_1) \tag{3}$$
$$\cdot \Pr(N(t_2, t_3) = n_3 - n_2 | N(t_1) = n_1, N(t_1, t_2) = n_2 - n_1) \tag{4}$$
$$= \Pr(N(t_1) = n_1) \cdot \Pr(N(t_1, t_2) = n_2 - n_1) \cdot \Pr(N(t_2, t_3) = n_3 - n_2) \tag{5}$$

2 Poisson Increments

Given a Poisson process $N(t)$ of rate λ, define the increments process

$$D_\Delta(t) = \frac{N(t) - N(t-\Delta)}{\Delta}.$$

Note that for small Δ this process approximates the derivative of $N(t)$. An alternative way to write $D_\Delta(t)$ is as a "random pulse train" located at the arrival times:

$$\frac{1}{\Delta}\sum_{n=1}^{\infty} P_\Delta(t - T_n)$$

where $T_1, T_2, \ldots$ are the arrival times, and $P_\Delta(t)$ is a pulse of height one and width Δ. Note that as $\Delta \to 0$ this becomes a random impulse train located at the arrival times. Although $N(t)$ is NOT stationary, the increments process $D_\Delta(t)$ is stationary (SSS). It is easy to verify that $D_\Delta(t)$ is WSS with the following properties:

- expectation: $\mu(t) = \lambda$
- variance: $\mathrm{Var}\,(D_\Delta(t)) = \lambda/\Delta$
- auto-covariance: $C(\tau) = \lambda(\Delta - |\tau|)/\Delta^2$ for $|\tau| \leq \Delta$, and $C(\tau) = 0$ otherwise.
- auto-correlation: $R(t_1, t_2) = \lambda^2 + C(t_1 - t_2)$.

3 White Noise

As $\Delta \to 0$, the auto-covariance function of $D_\Delta(t)$ becomes $\lambda\delta(\tau)$, i.e., proportional to a delta function. **Definition:** a zero-mean process whose auto-correlation function is proportional to a delta function is called "white noise" (the reason for "white" will become clear in the class about power spectrum). In a white noise the correlation between any two samples is ZERO. A stronger sense of white noise is when any two samples are statistically independent.

We may conclude that the derivative process of $N(t)$ is composed of a d.c. component λ and a white noise component multiplied by $\sqrt{\lambda}$.

4 Wiener-Levy Process

A Wiener process $W(t)$ is a limiting version of a "random walk" (a "drunk walk"), with small but fast increments. Specifically, a Wiener process is the limit as $\Delta \to 0$ of

$$W_\Delta(t) = \sqrt{\alpha\Delta} \cdot \sum_{n=1}^{\lfloor t/\Delta \rfloor} B_n$$

where $B_1, B_2, \ldots$ are binary i.i.d. with

$$B_n = \begin{cases} +1, & w.p.\ 1/2 \\ -1, & w.p.\ 1/2. \end{cases}$$

Note that in multiples of the sampling period, $W_\Delta(t)$ can be written as an auto-regressive process $W_\Delta(n\Delta) = W_\Delta((n-1)\Delta) + \sqrt{\alpha\Delta} \cdot B_n$. It follows that $W(t)$ is Markov and it has the independent increments property (like a Poisson process). We can now easily calculate the second order statistics of $W(t)$:

- $\mu(t) = 0$
- $\mathrm{Var}\,(W(t)) = \alpha t$
- $R(t_1, t_2) = \alpha \min\{t_1, t_2\}$.

Since for each t the random variable $W(t)$ is the sum of many i.i.d variables, the Central Limit Theorem implies that $W(t)$ has a Gaussian distribution $N(0, \alpha t)$. Furthermore, by the independent increments property it is easy to see that any collection of samples are jointly Gaussian and hence $W(t)$ is a Gaussian process.

Finally, by analyzing the increments process of $W(t)$ (as we did for the increments of $N(t)$ above), we can show that the derivative process of $W(t)$ is a white noise multiplied by $\sqrt{\alpha}$.

5 Summary

The Poisson process $N(t)$ and the Wiener process $W(t)$ have very similar second order statistics, however, their distributions are different (Poisson and Gaussian). Furthermore, their sample functions look very different. For example, while a sample function of $N(t)$ is discontinuous (a step function) with probability one, a sample function of $W(t)$ is continuous with probability one. We conclude that second order statistics do not tell the full story of a random process!!!