Poisson and Wiener: Corrections and Additions to Class

10

January 4, 2004

Abstract

See below few corrections to the first class, few additions, and two uncovered topics: Poisson increments process, and Wiener-Levy process.

1 Corrections and Additions

• Counting process versus arrival process: The equivalent events are

\[ \{N(t) < n\} \Leftrightarrow \{T_n > t\} \]

or equivalently \( \{N(t) \geq n\} \Leftrightarrow \{T_n \leq t\} \), but NOT \( \{N(t) \leq n\} \Leftrightarrow \{T_n \geq t\} \) as taught in the first class.

• Probability distribution of first arrival:

\[ \Pr(t < T_1 \leq t + \Delta) = \Pr(N(t) = 0, N(t + \Delta) \geq 1) \]

and not \( \Pr(N(t) = 0, N(t + \Delta) = 1) \) as taught in the first class.

• Joint distribution of three samples (taught in second class but not in the first):
Assume \( t_1 < t_2 < t_3 \) and \( 0 \leq n_1 \leq n_2 \leq n_3 \). Then, by the independent increments property

\[
\begin{align*}
\Pr(N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3) &= \Pr(N(t_1) = n_1, N(t_1, t_2) = n_2 - n_1, N(t_2, t_3) = n_3 - n_2) \\
&= \Pr(N(t_1) = n_1) \cdot \Pr(N(t_1, t_2) = n_2 - n_1 | N(t_1) = n_1) \\
&\cdot \Pr(N(t_2, t_3) = n_3 - n_2 | N(t_1) = n_1, N(t_1, t_2) = n_2 - n_1) \\
&= \Pr(N(t_1) = n_1) \cdot \Pr(N(t_1, t_2) = n_2 - n_1) \cdot \Pr(N(t_2, t_3) = n_3 - n_2)
\end{align*}
\]
2 Poisson Increments

Given a Poisson process \( N(t) \) of rate \( \lambda \), define the increments process

\[
D_\Delta(t) = \frac{N(t) - N(t - \Delta)}{\Delta}.
\]

Note that for small \( \Delta \) this process approximates the derivative of \( N(t) \). An alternative way to write \( D_\Delta(t) \) is as a “random pulse train” located at the arrival times:

\[
\frac{1}{\Delta} \sum_{n=1}^{\infty} P_\Delta(t - T_n)
\]

where \( T_1, T_2, \ldots \) are the arrival times, and \( P_\Delta(t) \) is a pulse of height one and width \( \Delta \). Note that as \( \Delta \to 0 \) this becomes a random impulse train located at the arrival times. Although \( N(t) \) is NOT stationary, the increments process \( D_\Delta(t) \) is stationary (SSS). It is easy to verify that \( D_\Delta(t) \) is WSS with the following properties:

- expectation: \( \mu(t) = \lambda \)
- variance: \( \text{Var}(D_\Delta(t)) = \lambda/\Delta \)
- auto-covariance: \( C(\tau) = \lambda(\Delta - |\tau|)/\Delta^2 \) for \( |\tau| \leq \Delta \), and \( C(\tau) = 0 \) otherwise.
- auto-correlation: \( R(t_1, t_2) = \lambda^2 + C(t_1 - t_2) \).

3 White Noise

As \( \Delta \to 0 \), the auto-covariance function of \( D_\Delta(t) \) becomes \( \lambda \delta(\tau) \), i.e., proportional to a delta function. **Definition**: a zero-mean process whose auto-correlation function is proportional to a delta function is called “white noise” (the reason for “white” will become clear in the class about power spectrum). In a white noise the correlation between any two samples is ZERO. A stronger sense of white noise is when any two samples are statistically independent.

We may conclude that the derivative process of \( N(t) \) is composed of a d.c. component \( \lambda \) and a white noise component multiplied by \( \sqrt{\lambda} \).
4 Wiener-Levy Process

A Wiener process $W(t)$ is a limiting version of a “random walk” (a “drunk walk”), with small but fast increments. Specifically, a Wiener process is the limit as $\Delta \to 0$ of

$$ W_\Delta(t) = \sqrt{\alpha \Delta} \cdot \sum_{n=1}^{\lfloor t/\Delta \rfloor} B_n $$

where $B_1, B_2, \ldots$ are binary i.i.d. with

$$ B_n = \begin{cases} +1, & \text{w.p. } 1/2 \\ -1, & \text{w.p. } 1/2. \end{cases} $$

Note that in multiples of the sampling period, $W_\Delta(t)$ can be written as an auto-regressive process $W_\Delta(n\Delta) = W_\Delta((n-1)\Delta) + \sqrt{\alpha \Delta} \cdot B_n$. It follows that $W(t)$ is Markov and it has the independent increments property (like a Poisson process). We can now easily calculate the second order statistics of $W(t)$:

- $\mu(t) = 0$
- $\text{Var}(W(t)) = \alpha t$
- $R(t_1, t_2) = \alpha \min\{t_1, t_2\}$.

Since for each $t$ the random variable $W(t)$ is the sum of many i.i.d variables, the Central Limit Theorem implies that $W(t)$ has a Gaussian distribution $N(0, \alpha t)$. Furthermore, by the independent increments property it is easy to see that any collection of samples are jointly Gaussian and hence $W(t)$ is a Gaussian process.

Finally, by analyzing the increments process of $W(t)$ (as we did for the increments of $N(t)$ above), we can show that the derivative process of $W(t)$ is a white noise multiplied by $\sqrt{\alpha}$.

5 Summary

The Poisson process $N(t)$ and the Wiener process $W(t)$ have very similar second order statistics, however, their distributions are different (Poisson and Gaussian). Furthermore, their sample functions look very different. For example, while a sample function of $N(t)$ is discontinuous (a step function) with probability one, a sample function of $W(t)$ is continuous with probability one. We conclude that second order statistics do not tell the full story of a random process!!!