Analog Matching of Colored Sources to Colored Channels †

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Abstract

Analog (uncoded) transmission provides a simple scheme for communicating a Gaussian source over a Gaussian channel under the mean squared error (MSE) distortion measure. Unfortunately, its performance is usually inferior to the all-digital, separation-based source-channel coding solution, which requires exact knowledge of the channel at the encoder. The loss comes from the fact that except for very special cases, e.g. white source and channel of matching bandwidth (BW), it is impossible to achieve perfect matching of source to channel and channel to source by linear means. We show that by combining prediction and modulo-lattice operations, we can match any colored Gaussian source to any inter-symbol interference (ISI) colored Gaussian noise channel (of possibly different BW), hence we achieve Shannon's optimum attainable performance $R(D) = C$. Furthermore, when the source and channel BWs are equal (but otherwise their spectra are arbitrary), our scheme is asymptotically robust in the sense that for high signal to noise ratio (SNR) the encoder becomes SNR-independent. The derivation is based upon a recent modulo-lattice modulation scheme for transmitting a Wyner-Ziv source over a dirty-paper channel.

keywords: joint source/channel coding, analog transmission, Wyner-Ziv problem, writing on dirty paper, modulo lattice modulation, MMSE estimation, prediction, unknown SNR, broadcast channel, ISI channels, bandwidth expansion/reduction.

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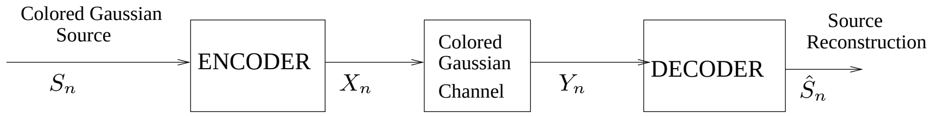


Figure 1: Colored Gaussian Joint Source/Channel Setting

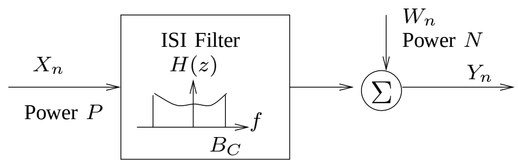


Figure 2: ISI Canonical Model

I. Introduction

Digital transmission of analog sources relies, at least from a theoretical point of view, on Shannon's source-channel separation principle. Being both optimal and easy to implement, digital techniques replace today traditional analog communication even in areas like voice telephony, radio and television. This trend ignores, however, the fact that the separation principle does not hold for communication networks, and in particular for broadcast channels and unknown channels [4], [34], [26]. Indeed, due to both practical and theoretical reasons, *joint* source-channel coding and hybrid digital-analog schemes are constantly receiving attention of researchers in the academia and the industry.

Figure 1 Demonstrates the setting we consider in this paper: Transmission under the MSE distortion criterion, of a general stationary Gaussian source S_n, over a power-constrained linear time-invariant (LTI) filter channel with additive white Gaussian noise (AWGN),

$$Y_n = h_n * X_n + W_n, \tag{1}$$

where X_n and Y_n are the channel input and output, respectively, h_n is the channel filter impulse response, $*$ denotes convolution, and W_n is the noise. See Figure 2. This channel, commonly called in the literature inter-symbol interference (ISI) channel, has also an equivalent ISI-free colored-noise representation, which we shall give in the sequel.

Shannon's joint source-channel coding theorem implies that the optimal (i.e., minimum distortion) performance D^{opt} is given by

$$R\left(D^{opt}\right) = C, \tag{2}$$

where $R(D)$ is the rate-distortion function of the source S_n at MSE distortion D, and $C = C(P)$ is the capacity of the channel (1) at power-constraint P, both given by the well-known water-filling solutions [4]. By Shannon's separation principle, (2) can be achieved by a system consisting of source and channel coding schemes. This system usually requires large delay and complex digital codes. A more serious drawback of the all-digital system is that it suffers from a "threshold effect": if the channel noise turns out to be higher than expected, then the reconstruction will suffer from very large distortion, while if the channel has lower noise than expected, then there is no improvement in the distortion [34], [26], [2].

In contrast, analog communication techniques (like amplitude or frequency modulation [**?**]) are not sensitive to exact channel knowledge at the transmitter. Moreover, in spite of their low complexity and delay, they are sometimes optimal: if we are allowed one channel use per source sample, the source S_n is white (i.e., source samples are i.i.d.), and the channel is ISI-free ($Y_n = h_0 \cdot X_n + W_n$), then a "single-letter" coding scheme achieves the optimum performance of (2), given by

$$D = D^{opt} = \frac{\mathrm{Var}\{S_n\}}{1 + \mathrm{SNR}} \quad , \tag{3}$$

where SNR denotes the channel signal-to-noise ratio; see e.g. [8]. In this scheme, the transmitter consists of multiplication by a constant factor that adjusts the source to the power constraint P, so it is independent of the channel parameters. Only the receiver needs to know the exact channel parameters (the gain h_0 and the power of the noise W_n) to optimally estimate the source from the noisy channel output (by multiplying by the "Wiener coefficient").

For the case of *colored* sources and channels, however, such a simple solution is not available, as single-letter codes are only optimal in very special scenarios [7]. By "colored" we mean that the source spectrum and the frequency response of the channel filter h_n are not flat. A particular case is when the channel bandwidth B_c (i.e., the bandwidth of the filter h_n) is not equal to the source bandwidth B_s, but otherwise they are white (i.e., we are allowed B_c/B_s channel uses per source sample on the average). As it turns out, even if we consider more general linear transmission schemes, [1], still (2) is not achievable in the general colored case. How far do we need to deviate from "analog" transmission in order to achieve optimal performance in the colored case? More importantly, can we still achieve full robustness?

In this work we propose and investigate a semi-analog transmission scheme, based on linear *prediction* with *modulo-lattice arithmetic*. This scheme achieves the optimum performance of (2) for *any* colored source and channel pair, hence we call it the *Analog Matching* scheme. Furthermore, for the matching bandwidth case ($B_c = B_s$), we show that the Analog Matching transmitter is asymptotically robust in the high signal-to-noise ratio (SNR) regime, in the sense that it becomes invariant to the variance of the channel noise. Thus, in this regime, the perfect SNR-invariant matching property of white sources and channels [8] generalizes to the equal-BW colored case.

Previous work on joint source/channel coding for the BW-mismatch/colored setting mostly consists of hybrid digital analog (HDA) solutions, which involve splitting the source or channel into frequency bands, or using a superposition of digital encoders (see [22], [15], [21], [19], [16] and references therein), mostly for the cases of bandwidth expansion ($B_c > B_s$) and bandwidth compression ($B_c < B_s$) with white spectra. Other works [2], [27] treat bandwidth expansion by mapping each source sample to a sequence of channel inputs independently. Most of these solutions, explicitly or implicitly, allocate different power and bandwidth resources to analog and digital source representations, thus they still employ full coding. In contrast, the Analog Matching scheme treats the source and channel in the *time domain*.

The rest of the paper is organized as follows: We start in Section II by demonstrating the basic principles of the Analog Matching scheme. Then in Section III we bring preliminaries regarding sources and channels with memory, as well as modulo-lattice modulation and side-information problems. In Section IV we prove the optimality of the Analog Matching scheme. In Section V we analyze the scheme performance for unknown SNR, and prove its asymptotic robustness. Finally, Section VI contains the conclusion.

II. A Simplified View of the Scheme in the Equal-BW High-SNR Case

To realize where lies the difficulty of matching a colored source to a colored channel, and to demonstrate the basic principles of the Analog Matching scheme, consider an auto-regressive (AR) source model:

$$S_n = Q_n + \sum_{l=1}^{L_S} a_l S_{n-l} \ , \quad -\infty < n < +\infty \tag{4}$$

where the innovation process Q_n is zero-mean white Gaussian with variance σ_Q^2, and where the AR order L_s is in general infinite. For the channel, without loss of generality[1] assume that the filter h_n is causal, monic ($h_0 = 1$) and minimum phase, so (1) can be re-written as

$$Y_n = X_n + \sum_{l=1}^{L_C} h_l X_{n-l} + W_n \ \ , \tag{5}$$

where W_n is zero-mean white Gaussian with variance N, and where the filter length L_c may be infinite (see Figure 2).

[1]Since we can always transform the channel into such using a matched filter at the receiver front, see e.g. /citeProakis83

Let us assume for now that the source bandwidth and the channel bandwidth are equal ($B_S = B_C$), and they both occupy the entire spectrum (This holds, for example, when the AR order (L_s) and the channel filter length (L_c) are finite). In this case, for sufficiently small distortion

$$R(D) = \frac{1}{2} \log \left(\frac{\mathrm{Var}\{Q_n\}}{D} \right)$$

while for small channel noise

$$C \approx \frac{1}{2} \log \left(\mathrm{SNR} \right),$$

where in general

$$\mathrm{SNR} \triangleq \frac{P}{N} \quad . \tag{6}$$

See more on that in Section III. Thus, in the limit where the SNR is high, (2) becomes

$$D^{opt} \approx \frac{\mathrm{Var}\{Q_n\}}{\mathrm{SNR}}, \tag{7}$$

as if we were transmitting a white Gaussian source Q_n over an ISI-free AWGN channel $Y_n = X_n + W_n$. (See (3).)

It is tempting to try to achieve the performance of (7) by letting the transmitter predict the source (to exploit Q_n from S_n), and invert the channel filter (i.e., cancel the ISI part $\sum_{l=1}^{L_C} h_l X_{n-l}$), thus transform the colored problem into a white one. However, this transformation turns out to be worthless: at the transmitter, inversion of a monic channel filter causes power amplification; while at the receiver, re-generation (re-coloring) of the source from the (white) noisy innovations will amplify the noise. All in all, the overall performance will be the same as if we were transmitting the original AR source directly (with only power matching) over the original ISI channel with no filtering (except for scalar power matching). Such a naive scheme will achieve distortion

$$D \cong \frac{P + \mathrm{Var}\{I_n\}}{P} \cdot \frac{\mathrm{Var}\{S_n\}}{\mathrm{SNR}} = \frac{P + \mathrm{Var}\{I_n\}}{P} \cdot \frac{\mathrm{Var}\{S_n\}}{\mathrm{Var}\{Q_n\}} D^{opt} \quad , \tag{8}$$

which can be arbitrarily far from the optimum, depending upon the source and channel color.

The Analog-Matching scheme circumvent this power/noise-amplification phenomena by employing modulo-arithmetic at the predictors. Unlike the more common configuration in practical systems, the Analog Matching scheme performs source prediction at the *decoder* side while channel inversion at the *encoder* side, as demonstrated below.

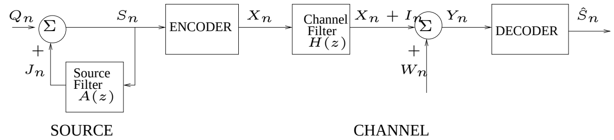


Figure 3: High-SNR Source and Channel Model

Recalling the source model (4) and the channel model (5), and defining the source regression term and the channel ISI term:

$$\begin{aligned} J_n &\triangleq \sum_{l=1}^{L_S} a_l S_{n-l} \\ I_n &\triangleq \sum_{l=1}^{L_C} h_l X_{n-l} \quad , \end{aligned} \tag{9}$$

we have:

$$\begin{aligned} S_n &= Q_n + J_n \quad , \\ Y_n &= X_n + I_n + W_n \quad . \end{aligned} \tag{10}$$

These source and channel models are shown in Figure 3.

The high-SNR variant of the scheme is depicted in Figure 4. The encoder and decoder are given by:

$$X_n = \left[\beta S_n - \tilde{I}_n\right] \bmod \Lambda \tag{11}$$

and

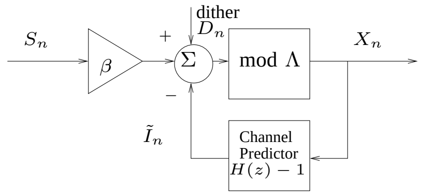
$$\hat{S}_n = \frac{[Y_n - \beta \tilde{J}_n] \bmod \Lambda}{\beta} + \tilde{J}_n \tag{12}$$

respectively, where $\tilde{I}_n$ and $\tilde{J}_n$ are the source and channel predictor outputs. If the source predictor coefficients are taken to be the source AR coefficients a_n and the channel predictor coefficients are taken to be the ISI coefficients h_n, then indeed the predictors are used to “cancel” I_n and J_n, as demonstrated in Figure 5:

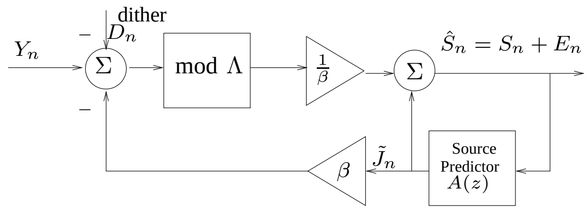
$$\begin{aligned} \tilde{J}_n &= \hat{a}_n * S_n = J_n + a_n * (\hat{S}_n - S_n) \triangleq J_n + a_n * E_n \\ \tilde{I}_n &= (h_n - \delta_n) * X_n = I_n \quad . \end{aligned} \tag{13}$$

In principle, the modulo-Λ block performs a multi-dimensional modulo-lattice operation. However at this stage, for simplicity, we will look at a one-dimensional modulo operation:

$$x \bmod \Lambda \triangleq x - \Delta \, \text{round}\left(\frac{x}{\Delta}\right)$$



(a) Encoder



(b) Decoder

Figure 4: The Analog Matching encoder and decoder at high SNR

where $\Delta > 0$ is the lattice cell size, chosen to be small enough such that the channel power constraint P is satisfied, and $\text{round}(\cdot)$ rounds a real number to the nearest integer.

To that end, Comparing these outputs with (9), we have that Combining (4), (11) and (5), the channel output is:

$$Y_n = [\beta(Q_n + J_n) - \tilde{I}_n] \bmod \Lambda + I_n + W_n \quad .$$

Substituting this in (12) and using the fact that

$$(a \bmod \Lambda + b) \bmod \Lambda = (a + b) \bmod \Lambda$$

we have that:

$$\begin{aligned} \hat{S}_n &= \frac{\left[\beta(Q_n + J_n - \tilde{J}_n) + I_n - \tilde{I}_n + W_n\right] \bmod \Lambda}{\beta} + \tilde{J}_n \\ &= \frac{\left[\beta(Q_n - E_n * a_n) + W_n\right] \bmod \Lambda}{\beta} + \tilde{J}_n \quad . \end{aligned} \tag{14}$$

We see that, up to noise error-dependent terms, the signal fed to the decoder modulo operation is a scaled version of the source innovations Q_n, as described above. Suppose that we choose this scaling factor β to be small enough such that the modulo operation has no effect (i.e. the decoder can reproduce the grid point selected by the encoder), then the reconstruction error is:

$$E_n = \hat{S}_n - S_n = \frac{W_n}{\beta} \quad , \tag{15}$$

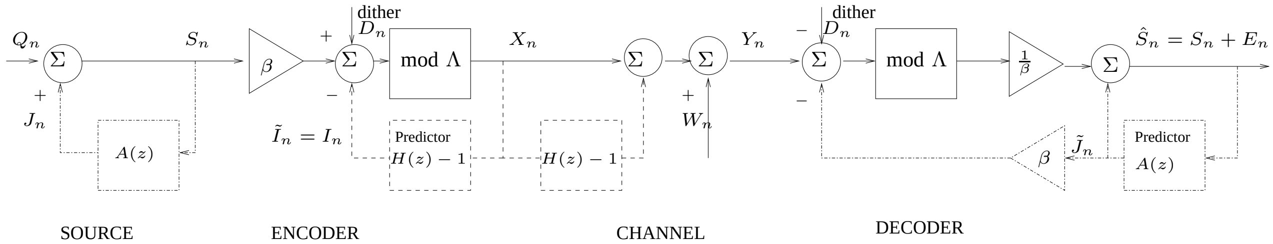


Figure 5: Workings of the High-SNR Scheme. For illustration purposes, we put together the source, encoder, channel and decoder. Dashed lines show the channel ISI, canceled by the channel predictor. Dotted lines show the source memory component, subtracted and then added again using the source predictor.

which is a scaled version of the current noise sample as desired. The resulting performance is:

$$D = \frac{\beta_0^2}{\beta^2} \cdot \frac{\mathrm{Var}\{Q_n\}}{\mathrm{SNR}} \quad , \tag{16}$$

where $\beta_0^2 \triangleq \frac{P}{\mathrm{Var}\{Q_n\}}$. This is identical to the optimum (7), up to the ratio of β's which is a fixed source- and channel-independent loss (as opposed to the loss of a naive scheme, see (8)). We will show in the sequel how even this loss can be eliminated by using a high-dimensional dithered modulo-lattice operation in parallel over interleaved samples, yielding optimum performance.

We can "forget" that the predictor outputs $\tilde{J}_n$ and I_n are produced by the source and channel predictors, and view them as side information (SI) available to the decoder and the encoder, respectively. In this sense, the Analog Matching scheme translates the colored problem into a white SI problem. In fact, it uses ideas of prediction as in precoding [24] and differential pulse code modulation (DPCM) [10], and then treats this joint source/channel SI problem using modulo lattice operations, based upon our recent work [12]. In the context of channel coding, the combination of precoding and nested lattice transmission is optimal for colored Gaussian channels [33, Section VII-B]. As for source coding, there has been much interest in Wyner-Ziv (WZ) video coding, [29], exploiting the dependence between consecutive frames at the decoder rather than at the encoder (see for example [20]). On the more theoretical side, it is shown in [32] that a DPCM-like encoder using prediction to exploit the source memory achieves the Gaussian-quadratic rate-distortion function (RDF). Furthermore, a scheme where prediction is used in the *decoder only* relates to the DPCM scheme the same way that a precoder scheme (with channel prediction at the encoder only) relates to an optimal feed-forward-equalizer / decision-feedback-equalizer (FFE-DFE)

Problem	Conventional Prediction	Side information based solution
Channel coding	FFE/DFE receiver	Dirty paper coding = precoding
Source coding	DPCM compression	WZ video coding
Joint source/ channel coding		Analog matching

Figure 6: Information-Theoretic Time-Domain Solutions to colored Gaussian Source and Channel Problems

scheme [3]. Figure 6 demonstrates the place of the Analog Matching scheme, within information-theoretic time-domain schemes.

III. Preliminaries

In this section we bring preliminaries necessary for the rest of the paper. In Sections III-1 to III-3 we present results connecting the Gaussian-quadratic RDF and the Gaussian channel capacity to prediction, mostly following [32]. In sections III-4 and III-5 we discuss lattices and their application to joint source/channel coding with side information, mostly following [12].

A. *Spectral Decomposition and Prediction*

The Paley-Wiener condition for a discrete-time spectrum $S(e^{j2\pi f})$ is [25]:

$$\left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\Big(S(e^{j2\pi f})\Big) df \right| < \infty \quad . \tag{17}$$

This condition holds for example if the spectrum $S(e^{j2\pi f})$ is bounded away from zero. Whenever the Paley-Wiener condition holds, the spectrum has a spectral decomposition:

$$S(e^{j2\pi f}) = B(z) B^* \left(\frac{1}{z^*}\right)\bigg|_{z=j2\pi f} P_e\Big(S\Big) \quad , \tag{18}$$

where $B(z)$ is a monic causal filter, and the entropy-power of the spectrum $P_e(S)$ is defined by:

$$P_e(S) \triangleq P_e\Big(S(e^{j2\pi f})\Big) = \exp \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\Big(S(e^{j2\pi f})\Big) df \quad . \tag{19}$$

The *optimal predictor* of a process X_n having a spectrum $S(e^{j2\pi f})$ from its infinite past is $B(z)-1$, a filter with an impulse response satisfying $b_n = 0$ for all $n \leq 0$, with the prediction mean squared error (MSE) being the entropy power:

$$P_e(S) = \mathrm{Var}\{X_n | X_{-\infty}^{n-1}\} \quad , \tag{20}$$

see [25]. The prediction error process can serve as a white innovations process for AR representation of the process. In terms of (4), we have that Q_n is the prediction error of the process S_n from its infinite past, thus

$$\mathrm{Var}\{Q_n\} = P_e(S_S) \quad .$$

We define the *prediction gain* of a spectrum $S(e^{j2\pi f})$ as:

$$\Gamma(S) \triangleq \Gamma\Big(S(e^{j2\pi f})\Big) \triangleq \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} S(e^{j2\pi f}) df}{P_e(S)} = \frac{\mathrm{Var}\{X_n\}}{\mathrm{Var}\{X_n | X_{-\infty}^{n-1}\}} \geq 1 \quad , \tag{21}$$

where the gain equals one if and only if the spectrum is white, i.e. fixed over all frequencies $|f| \leq \frac{1}{2}$. A case of special interest, is where the process is band-limited such that $S(e^{j2\pi f}) = 0 \forall |f| > \frac{B}{2}$ where $B < 1$. In that case, (17) does not hold and the prediction gain is infinite. We re-define, then, the prediction gain of a process band-limited to B as the gain of the process downsampled by $\frac{1}{B}$, i.e.,

$$\Gamma(S) = \frac{\int_{-\frac{B}{2}}^{\frac{B}{2}} S(e^{j2\pi f}) df}{\exp \int_{-\frac{B}{2}}^{\frac{B}{2}} \log\Big(S(e^{j2\pi f})\Big) df} \quad . \tag{22}$$

We will use in the sequel prediction from a noisy version of a process: Suppose that $Y_n = X_n + A_n$, with A_n white with power θ. Then it can be shown that the estimation error is a white process with variance (see e.g. [32]):

$$\mathrm{Var}\{X_n | Y_{-\infty}^{n-1}\} = P_e(S+\theta) - \theta \quad . \tag{23}$$

Note that for any $\theta > 0$, the spectrum $S(e^{j2\pi f}) + \theta$ obeys (17), so that the conditional variance is finite even if X_n is band-limited; In the case $\theta = 0$, (23) collapses to (20).

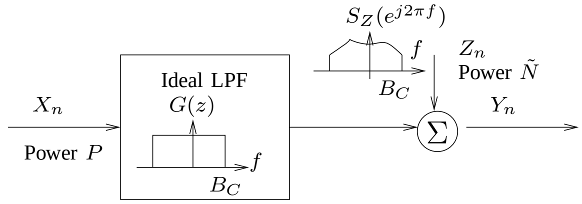


Figure 7: Colored Additive Noise Equivalent Model

B. Water-Filling Solutions and the Shannon Bounds

The rate-distortion function (RDF) for a Gaussian source with spectrum $S_S(e^{j2\pi f})$ with an MSE distortion measure is given by:

$$R(D) = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \log \frac{S_S(e^{j2\pi f})}{D(e^{j2\pi f})} df \quad , \tag{24}$$

where the *distortion spectrum* $D(e^{j2\pi f})$ is given by the reverse water-filling solution: $D(e^{j2\pi f}) = \min\Big(\theta_S, S(e^{j2\pi f})\Big)$ with the *water level* θ_S set by the distortion level D:

$$D = \int_{-1/2}^{1/2} D(e^{j2\pi f}) df \quad .$$

The *Shannon lower bound* (SLB) for the RDF of a source band-limited to B_S is given by:

$$R(D) \geq \frac{B_S}{2} \log \frac{\text{SDR}}{\Gamma_S} \triangleq R_{SLB}(D) \quad , \tag{25}$$

where the signal to distortion ratio is defined as:

$$\text{SDR} \triangleq \frac{\text{Var}\{S_n\}}{D} \tag{26}$$

and $\Gamma_S \triangleq \Gamma(S_S)$ is the source prediction gain (22). This bound is tight for a Gaussian source whenever the distortion level D is low enough such that $D < B_S \min_{|f|\leq B_S} S(e^{j2\pi f})$, and consequently $D(e^{j2\pi f}) = \theta_S = \frac{D}{B_S}$ for all $|f| < B_S$.

For stating the channel capacity, it is convenient to abandon the ISI channel model (5) in favor of an additive (colored) noise equivalent channel (see Figure 7), given by:

$$Y_n = X_n * g_n + Z_n, \tag{27}$$

where g_n is the impulse response of an ideal low pass filter of bandwidth B_C and the noise Z_n has spectrum $S_Z(e^{j2\pi f})$, bandlimited to B_C, and total power

$$\tilde{N} = \int_{-B_C}^{B_C} S_Z(e^{j2\pi f}) df \quad .$$

With respect to this model we define the equivalent signal to noise ratio:

$$\widetilde{\text{SNR}} \triangleq \frac{P}{\tilde{N}} \quad . \tag{28}$$

Note that (5) is equivalent to (27) with noise spectrum $\frac{N}{|H(e^{j2\pi f})|^2}$ for $|f| \leq B_C$. Since we assumed the ISI filter h_n of (5) to be monic, causal and minimum-phase, it follows that the colored noise Z_n can be seen as an AR process with innovations process W_n and prediction filter h_n[2]. Consequently, if we define the channel prediction gain $\Gamma_C = \Gamma(S_Z)$, we have:

$$\Gamma_C = \frac{\tilde{N}}{N} = \frac{\text{SNR}}{\widetilde{\text{SNR}}} \quad .$$

In terms of this channel model, the capacity is given by:

$$C = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1 + \frac{P(e^{j2\pi f})}{S_Z(e^{j2\pi f})}\right) df \quad , \tag{29}$$

where the *channel input spectrum* $P(e^{j2\pi f})$ is given by the water-filling solution: $P(e^{j2\pi f}) = \max\Big(\theta_C - S_Z(e^{j2\pi f}), 0\Big)$ with the *water level* θ_C set by the power constraint P:

$$P = \int_{-1/2}^{1/2} P(e^{j2\pi f}) df \quad .$$

The *Shannon upper bound* (SUB) for the channel capacity is given by:

$$C \leq \frac{B_C}{2} \log\left[\Gamma_C \cdot \left(1 + \widetilde{\text{SNR}}\right)\right] \triangleq C_{SUB} \quad . \tag{30}$$

The bound is tight for a Gaussian channel whenever the equivalent SNR is high enough such that $P \geq B_C \max_{|f| \leq B_C} S_Z(e^{j2\pi f}) - \tilde{N}$ and consequently $S_Z(e^{j2\pi f}) + P(e^{j2\pi f}) = \theta_C = \frac{P + \tilde{N}}{B_C}$.

Combining (25) with (30), we have an the following asymptotically tight upper bound on the Shannon optimum performance (2).

Proposition 1: Let SDR^{opt} be the OPTA performance

$$\text{SDR}^{opt} \triangleq \frac{\text{Var}\{S_n\}}{R^{-1}(C(\widetilde{\text{SNR}}))} \quad , \tag{31}$$

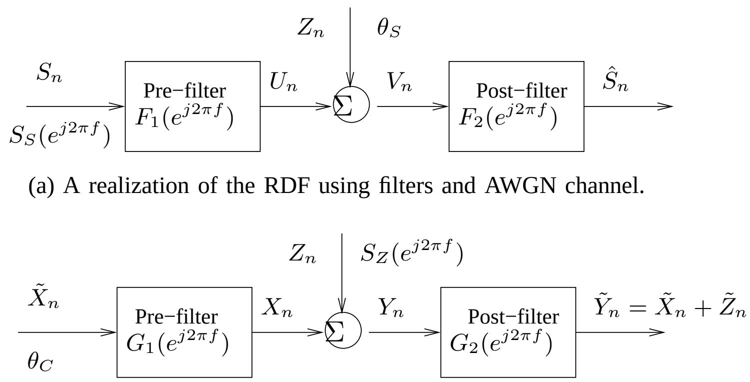
and let the *bandwidth ratio* be

$$\rho \triangleq \frac{B_C}{B_S} \quad . \tag{32}$$

Then:

$$\text{SDR}^{opt} \leq \Gamma_S \Gamma_C (1 + \widetilde{\text{SNR}})^{\rho} \quad , \tag{33}$$

[2]In the bandlimited case $B_C < 1$, this refers to a downsampled version of the signals and of the filter.



(a) A realization of the RDF using filters and AWGN channel.

(b) A realization of the channel capacity using filters and white Gaussian input.

Figure 8: Realizations of the RDF and capacity

with equality if and only if the SLB and SUB both hold with equality[3]. Furthermore if the noise spectrum is held fixed while the power constraint P is taken to infinity:

$$\lim_{\widetilde{\text{SNR}}\to\infty} \frac{\text{SDR}^{opt}}{\widetilde{\text{SNR}}^{\rho}} = \Gamma_S \Gamma_C \quad . \tag{34}$$

C. Predictive Presentation of the Gaussian RDF and Capacity

Not only the SLB and SUB in (25) and (30) can be written in predictive forms, but also the rate-distortion function and channel capacity, in the Gaussian case. These predictive forms are given in terms of the realizations depicted in Figure 8.

For source coding, let $F_1(e^{j2\pi f})$ be some filter with amplitude response satisfying

$$|F_1(e^{j2\pi f})|^2 = 1 - \frac{D(e^{j2\pi f})}{S_S(e^{j2\pi f})} \quad , \tag{35}$$

where $D(e^{j2\pi f})$ is the distortion spectrum materializing the water-filling solution (24). We call $F_1(e^{j2\pi f})$ and $F_2(e^{j2\pi f}) = F_1^*(e^{j2\pi f})$ the pre- and post-filters for the source S [31].

Proposition 2: The pre/post filtered AWGN depicted in Figure 8a satisfies:

$$R(D) = \frac{1}{2} \log \left(1 + \frac{\text{Var}\{U_n | V_{-\infty}^{n-1}\}}{\text{Var}\{Z_n\}} \right) \quad ,$$

[3]The SLB and SUB never strictly hold if $S_S(e^{j2\pi f})$ is not bounded away from zero, or $S_Z(e^{j2\pi f})$ is not everywhere finite. However, they do hold asymptotically if these spectra satisfy the Paley-Wiener condition.

where $\mathrm{Var}\{Z_n\} = \theta_S$.

This Proposition is a direct consequence of (23). It is due to [32], where this form is used to establish the optimality of a DPCM-like scheme, where the prediction error of U_n from the past samples of U_n is being quantized and the quantizer is equivalent to an AWGN. Note that in the limit of low distortion the filters vanish, prediction from U_n is equivalent to prediction from V_n, and we go back to (25). Defining the source Wiener coefficient

$$\alpha_S = 1 - \exp\left(2R(D)\right) \quad , \tag{36}$$

the Proposition implies that

$$\mathrm{Var}\{U_n | V_{-\infty}^{n-1}\} = \frac{\alpha_S}{1-\alpha_S}\theta_S \quad . \tag{37}$$

For channel coding, let $G_1(e^{j2\pi f})$ be some filter with amplitude response satisfying

$$|G_1(e^{j2\pi f})|^2 = \frac{P(e^{j2\pi f})}{\theta_C} \quad , \tag{38}$$

where $P(e^{j2\pi f})$ and θ_C are the channel input spectrum and water level materializing the water-filling solution (29). $F_1(e^{j2\pi f})$ is usually referred to as the channel shaping filter, but motivated by the the similarity with the solution to the source problem we call it a channel pre-filter. At the channel output we place $G_2(e^{j2\pi f}) = G_1^*(e^{j2\pi f})$, known as a matched filter, which we call a channel post-filter.

Proposition 3: In the pre/post filtered colored-noise channel depicted in Figure 8b, let the input $\tilde{X}_n$ be white and define $\tilde{Z}_n = \tilde{Y}_n - \tilde{X}_n$. Then the channel satisfies:

$$C = \frac{1}{2}\log\left(\frac{\mathrm{Var}\{\tilde{X}_n\}}{\mathrm{Var}\{\tilde{Z}_n|\tilde{Z}_{-\infty}^{n-1}\}}\right)$$

where $\mathrm{Var}\{\tilde{X}_n\} = \theta_C$.

This Proposition is again due to [32], following the analysis in [6]. It is used to establish the optimality of a scheme based upon noise prediction, where the decoder uses past decisions in order to evaluate the linear filtering error, and then subtracts the prediction of this error in order to achieve capacity. It can also be shown to be equivalent to the better known MMSE FFE-DFE solution [3]. Note that in the limit of low noise the filters vanish, prediction from $\tilde{Z}_n$ is equivalent to prediction from Z_n, and we go back to (30). defining the channel Wiener coefficient

$$\alpha_C = 1 - \exp\left(2C\right) \quad , \tag{39}$$

The Proposition implies that

$$\mathrm{Var}\{\tilde{Z}_n|\tilde{Z}_{-\infty}^{n-1}\} = \frac{1-\alpha_C}{\alpha_C}\theta_C \quad . \tag{40}$$

The predictive forms described above are highly attractive as the basis for coding schemes, since the filters and predictors take care of the source or channel memory, allowing to use the design of generic optimal codebooks for *white* sources and channels, regardless of the actual spectra, without compromising optimality. See e.g. [9], [32].

D. Good Lattices for Quantization and Channel Coding

Let Λ be a K-dimensional lattice, defined by the generator matrix $G \in \mathbb{R}^{K\times K}$. The lattice includes all points $\{\mathbf{l} = G\cdot \mathbf{i} : \mathbf{i} \in \mathbb{Z}^K\}$ where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. The nearest neighbor quantizer associated with Λ is defined by

$$Q(\mathbf{x}) = \arg\min_{\mathbf{l}\in\Lambda} \|\mathbf{x}-\mathbf{l}\| \quad .$$

Let the basic Voronoi cell of Λ be

$$\mathcal{V}_0 = \{\mathbf{x} : Q(\mathbf{x}) = \mathbf{0}\} \quad ,$$

while the second moment of a lattice per dimension is given by the variance of a uniform distribution over the basic Voronoi cell:

$$\sigma^2(\Lambda) = \frac{1}{K}\cdot\frac{\int_{\mathcal{V}_0}\|\mathbf{x}\|^2 d\mathbf{x}}{\int_{\mathcal{V}_0} d\mathbf{x}} \quad . \tag{41}$$

The modulo-lattice operation is defined by:

$$\mathbf{x} \bmod \Lambda = \mathbf{x} - Q(\mathbf{x}) \quad .$$

We say that *correct decoding* of a vector $\mathbf{x}$ by a lattice Λ occurs, whenever

$$\mathbf{x} \bmod \Lambda = \mathbf{x} \quad , \tag{42}$$

For a dither vector $\mathbf{d}$ which is independent of $\mathbf{x}$ and uniformly distributed over the basic Voronoi cell $\mathcal{V}_0$, $[\mathbf{x}+\mathbf{d}] \bmod \Lambda$ is uniformly distributed over $\mathcal{V}_0$ as well, and independent of $\mathbf{x}$ [30].

We will assume the use of lattices which are simultaneously good for source coding (MSE quantization) and for AWGN channel coding [5]. Roughly speaking, a sequence of K-dimensional lattices is *good for MSE quantization* if the second moment of these lattices tends to this of a ball of the same volume, as K grows. A sequence of lattices is *good for AWGN channel coding* if the probability of correct decoding (42) of a Gaussian i.i.d. vector with element variance smaller than the variance of a ball having the same volume as the lattice basic cell, approaches zero for large K. There exists a sequence of

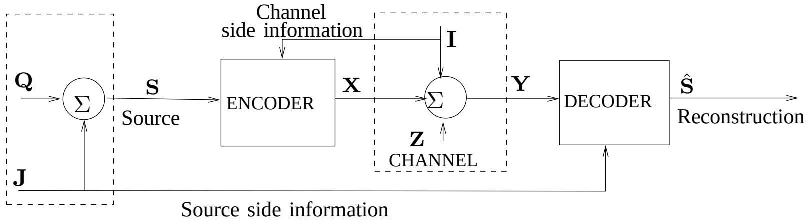


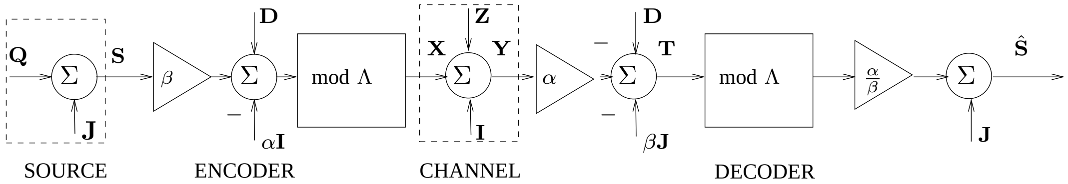
Figure 9: The Wyner-Ziv / Dirty Paper Coding Problem

lattices satisfying both properties simultaneously, thus for these lattices, correct decoding holds with high probability for Gaussian i.i.d. vectors with element variance smaller than $\sigma^2(\Lambda)$, for large enough K. This property also holds when the Gaussian vector is replaced by a linear combination of Gaussian and "self noise" (uniformly distributed over the lattice basic cell) components, see [12, Proposition 1] for an exact statement. We also assume that these lattices have the property that the second moment along each coordinate of a uniform distribution over the basic lattice cell is identical, and it is equal to the lattice second moment $\sigma^2(\Lambda)$. This is proven for lattices which are good for source coding in [30].

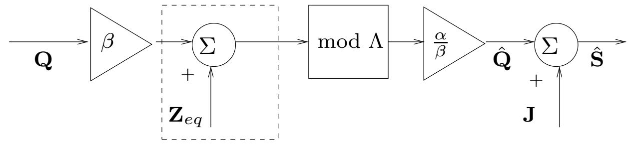
E. Coding for the Joint WZ/DPC Problem using Modulo-Lattice Modulation

The lattices discussed above can be used for achieving the optimum performance in the joint source/channel Gaussian Wyner-Ziv/Dirty Paper coding[4], depicted in Figure 9. In that problem, the source is the sum of an unknown i.i.d. Gaussian component Q_n and an arbitrary component J_n known at the decoder, while the channel noise is the sum of an unknown i.i.d. Gaussian component Z_n and an arbitrary component I_n known at the encoder. In [12] the MLM scheme of Figure 10a is shown to be optimal for suitable α and β. This is done showing first equivalence to the modulo-additive channel of Figure 10b, and then, for good lattices, asymptotic equivalence with high probability to the real-additive channel of Figure 10c. The output-power constraint in that last channel reflects the element variance condition in order to ensure correct decoding (42) of the vector $\beta Q_n + \mathbf{Z}_e q_n$ with high probability. When this holds, the dithered modulo-lattice operation at the encoder and the decoder perfectly cancel each other. This way, the MLM scheme asymptotically translates the SI problem to the simple problem of transmitting the unknown source component Q_n over an AWGN, where the interference I_n is not present.

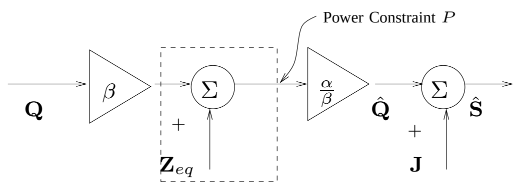
[4]An alternative form of this scheme may be obtained by replacing the lattice with a random code and using mutual information considerations, see [28].



(a) Analog Wyner-Ziv / Dirty-Paper Coding Scheme.



(b) Equivalent modulo-lattice channel. The equivalent noise $\mathbf{Z}_{eq} = \alpha\mathbf{Z} - (1-\alpha)\mathbf{X}$ is independent of $\mathbf{Q}$.



(c) Asymptotic equivalent real-additive channel for "good" lattices.

Figure 10: Equivalent Channels for the WZ/DPC Coding Scheme

IV. Optimality of the AM Scheme

In this section we provide analysis of the performance of the Analog Matching scheme, depicted in Figure 11. The high-SNR variant of Section II (see Figure 4), is a special case of the general scheme where the pre/post filters are taken to be scalar factors and Λ is taken to be a scalar lattice. In terms of the quantities defined in Section III, the performance of this variant in the high-SNR regime (16) can be re-written as:

$$\mathrm{SDR} \cong \frac{\beta^2}{\beta_0^2}\Gamma_S\Gamma_C\widetilde{\mathrm{SNR}} \quad .$$

Comparing this to Proposition 1 we see that the scheme is indeed asymptotically optimal up to the factor of β's. As promised in Section II, we show how this factor may be eliminated, and moreover, we show that by choosing the filters of the Analog Matching scheme to be optimal in the MMSE sense, it can approach the optimal performance (31) for any SNR.

For proving the optimality of the scheme, we need high lattice dimension. We assume for now that we

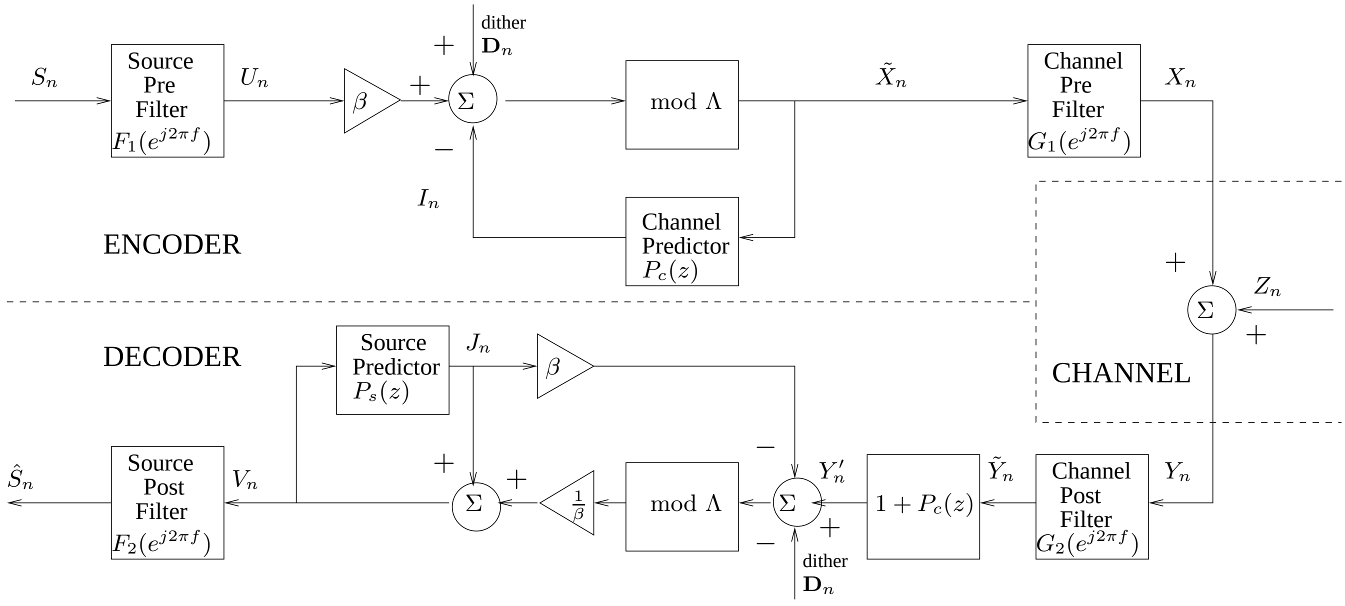


Figure 11: The Analog Matching Scheme

have K independent identical source-channel pairs in parallel[5], which allows a K-dimensional dithered modulo-lattice operation across these pairs. Other operations are done independently in parallel. To simplify notation we will omit the index k of the source/channel pair ($k = 1, 2, \ldots, K$), and use scalar notation meaning *any* of the K pairs; We will denote by bold by bold letters K-dimensional vectors, for the modulo-lattice operation. Subscripts denote time instants. Under this notation, the AM encoder is given by:

$$\begin{aligned}
U_n &= f_{1_n} * S_n \\
\tilde{\mathbf{X}}_n &= [\beta \mathbf{U}_n - \mathbf{I}_n + \mathbf{D}_n] \bmod \Lambda \\
I_n &= \sum_{m=1}^{\infty} p_{C_m} \tilde{X}_{n-m} \\
X_n &= g_{1_n} * \tilde{X}_n \quad ,
\end{aligned} \tag{43}$$

[5]We will discuss in the sequel how this leads to optimality for a single source and a single channel.

while the decoder is given by:

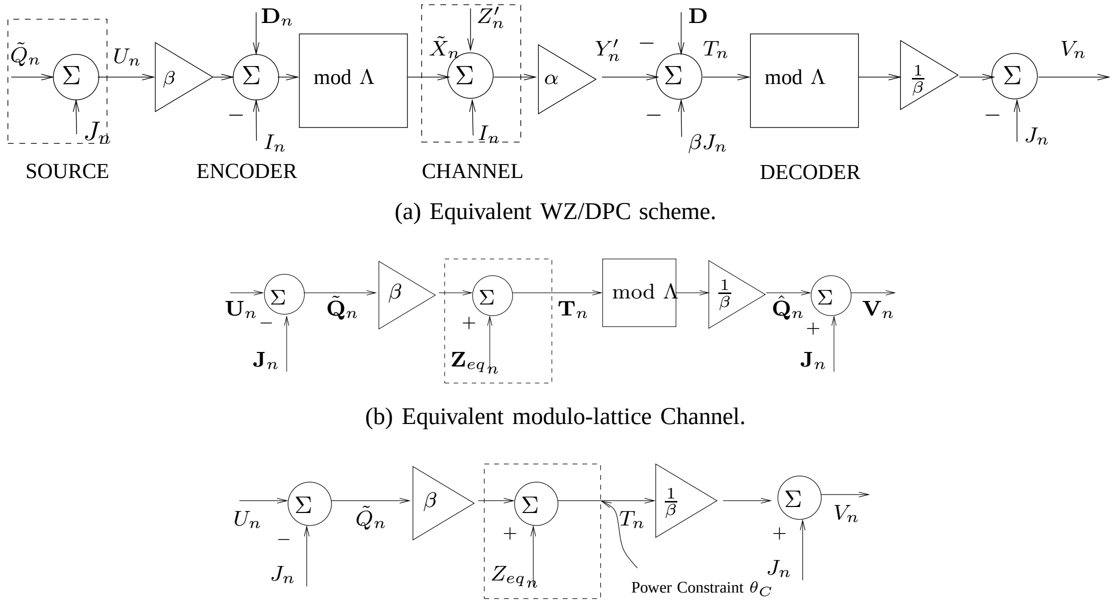
$$\begin{aligned}
\tilde{Y}_n &= g_{2n} * Y_n \\
Y'_n &= \tilde{Y}_n + \sum_{m=1}^{\infty} p_{C_m} \tilde{Y}_{n-m} \\
\mathbf{V}_n &= \frac{1}{\beta}\left[\tilde{\mathbf{Y}}_n - \beta \mathbf{J}_n - \mathbf{D}_n\right] \bmod \Lambda + \mathbf{J}_n \\
J_n &= \sum_{m=1}^{\infty} p_{S_k} V_{n-k} \\
\hat{S}_n &= f_{2n} * V_n \quad ,
\end{aligned} \tag{44}$$

where $*$ denotes convolution, and for each filter h_n denotes the impulse response of the corresponding frequency response $H(e^{j2\pi f})$. Each of the K parallel channels is given by the equivalent colored noise model[6] (5).

The optimality proof shows that at each time instant the scheme is equivalent to a joint source/channel side-information (SI) scheme, and then applies the Modulo-Lattice Modulation (MLM) analysis of such schemes presented in Section III-5. The key to the proof is showing that the correct decoding event (42) holds, thus the modulo-lattice operation at the decoder exactly cancels the corresponding operation at the encoder. This is an event which involves all the source/channel pairs, and its analysis requires verifying the signal distribution, see Section III-4. Once this holds, the rest of the analysis is *scalar*, i.e. we can treat each of the K source/channel pairs separately, and *quadratic*, i.e. we can ignore the distribution of signals and deal with variances only. In this scalar quadratic exposition, we find that with a choice of MMSE filters, the scheme materializes the capacity realization of Proposition 3, nested inside the RDF realization of Proposition 2; The channel error, scaled down by factor β, serves as the AWGN in the RDF realization. The calculations in the lemmas below, showing step by step equivalence to the channels in Figure 12, result in approaching the optimum performance (31). Throughout the proof we use θ_C according to (29), and θ_S according to (24) at a distortion level corresponding with the optimum (31). We also use $\alpha = \alpha_S = \alpha_C$ (36),(39).

We start by showing that the Analog Matching scheme is equivalent at each time instant to the WZ/DPC scheme of Section III-5. This equivalence is feasible, since I_n and J_n are constructed in the encoder and

[6] Note that when moving from the high-SNR scheme to the general optimal scheme, we find it convenient to replace the ISI channel (27) by the colored noise channel. For equal source and channel BW, these are interchangeable as discussed in Section III-2. Consequently we also look at the channel predictor $P_C(e^{j2\pi f})$ as a part of a noise-predictor, rather than a Tomlinson precoder, see [6], [32].



(a) Equivalent WZ/DPC scheme.

(b) Equivalent modulo-lattice Channel.

(c) Asymptotic equivalent scalar additive channel for good lattices.

Figure 12: Equivalent Channels for the Analog Matching Scheme

the decoder using *past* values of $\tilde{X}_n$ and V_n, respectively, thus at any time instant they can be seen as side information. Specifically, the equivalent scheme is shown in Figure 12a, which is identical to the scheme of Figure 10a, with the substitutions summerized in the following table:

AM	$\mathbf{U}_n$	$\mathbf{V}_n$	$\tilde{\mathbf{Q}}_n$	$\mathbf{J}_n$	$\tilde{\mathbf{X}}_n$	$\mathbf{Y}'_n$	$\mathbf{Z}'_n$	$\mathbf{I}_n$	$\mathbf{D}_n$	θ_C
WZ/DPC	$\mathbf{S}$	$\frac{1}{\alpha}\hat{\mathbf{S}}$	$\mathbf{Q}$	$\frac{1}{\alpha}\mathbf{J}$	$\mathbf{X}$	$\alpha\mathbf{Y}$	$\mathbf{Z}$	$\alpha\mathbf{I}$	$\mathbf{D}$	P

It remains to show that an optimum choice of filters indeed results in a channel where the unknown noise component is white, and evaluate its variance.

Lemma 1: **(Equivalent side-information scheme)** If we choose $G_1(e^{j2\pi f})$ and $G_2(e^{j2\pi f})$ according to (38) and we choose $P_C(e^{j2\pi f})$ as the minus of the optimal predictor of the spectrum

$$S_{\tilde{Z}}(e^{j2\pi f}) = \Big(1 - |G_1(e^{j2\pi f})|^2\Big)^2 \theta_C + |G_1(e^{j2\pi f})|^2 S_Z(e^{j2\pi f}) \quad ,$$

then

$$Z'_n \triangleq \frac{Y'_n - I_n}{\alpha} - \tilde{X}_n$$

is a white process, independent of all U_n, with variance

$$\mathrm{Var}\{Z'_n\} = \frac{1-\alpha}{\alpha}\theta_C \quad .$$

Proof: By the properties of the modulo-lattice operation, $\tilde{X}_n$ is a white process of variance $\sigma^2(\Lambda) = \theta_C$. Now the channel from $\tilde{X}_n$ to $\tilde{Y}_n$ is identical to the channel in Proposition 3, thus we have that:

$$Y'_n = (\tilde{X}_n + \tilde{Z}_n) * (\delta_n - p_{C_n}) = \tilde{X}_n + I_n + Z''_n \quad ,$$

where $Z''_n = \tilde{Z}_n * (\delta_n + p_{C_n})$ and $\tilde{Z}_n$ has spectrum $S_{\tilde{Z}}(e^{j2\pi f})$. Since $-P_C(e^{j2\pi f})$ is the optimum predictor of the process $\tilde{Z}_n$, we have that Z''_n is a white process, with variance $\frac{1-\alpha}{\alpha}\theta_C$ according to (40). Now since $\tilde{Y}_n = Y'_n - I_n$ is an optimum estimator for $\tilde{X}_n$ given the channel output, we have by the orthogonality principle that the estimation error is uncorrelated with the process Y'_n, resulting in an additive backward channel (see e.g. [32]):

$$\tilde{X}_n = Y'_n - I_n - Z''_n \quad .$$

Reverting back to a forward channel, we have

$$Y'_n = \alpha(\tilde{X}_n + Z'_n) + I_n \quad ,$$

where Z'_n is white with the same variance as Z''_n ■

Using this result, the equivalence to the channel of Figure 12b is immediate, as follows.

Lemma 2: **(Equivalent modulo-lattice channel)** Under the same conditions of Lemma 1, the first modulo-Λ operation can be dropped, resulting in an equivalent channel:

$$\mathbf{V}_n = \frac{1}{\beta}\left[\beta(\mathbf{U}_n - \mathbf{J}_n) + \mathbf{Z}_{eq_n}\right] \bmod \Lambda + \mathbf{J}_n \quad ,$$

where Z_{eq_n} is white additive noise with variance $(1-\alpha)\theta_C$.

Proof: We apply Lemma 1 and identify the Analog Matching scheme with the the WZ/DPC scheme as described above. Seeing that α is indeed the Wiener coefficient of the channel from $\tilde{X}_n$ to Y'_n, we can use [12, Lemma 1] to arrive at the desired channel with $\mathrm{Var}\{Z_{eq_n} = \alpha \,\mathrm{Var}\{Z'_n\} = (1-\alpha)\theta_C$ ■

Assuming that correct decoding holds, the modulo-lattice operation can be dropped, and we have the scalar additive channel of Figure 12c:

$$V_n = U_n + \frac{Z_{eq_n}}{\beta} \quad . \tag{45}$$

We now show, that if correct decoding held at all past times, then the variance of the signal at the input of the decoder modulo-lattice operation can be bounded from above.

Lemma 3: Under the conditions of Lemma 2, and if in addition (45) held for all past instances $n-1, n-2, \ldots$, then the variance of

$$T_n \triangleq \beta(U_n - J_n) + Z_{eq_n}$$

satisfies:

$$\mathrm{Var}\{T_n\} = (1-\delta)\theta_C$$

where $\delta = \delta\left(S_U(e^{j2\pi f}), \theta_C, \beta\right) > 0$, provided that

$$\beta^2 < \beta_0^2 \triangleq (1-\alpha)\frac{\Theta_C}{\Theta_S} \quad , \tag{46}$$

$F_1(e^{j2\pi f})$ is chosen according to (35) and $P_S(e^{j2\pi f})$ is the optimum predictor of the spectrum

$$S_V(e^{j2\pi f}) = \beta^2 S_U(e^{j2\pi f}) + (1-\alpha)\theta_C \quad .$$

Proof: We note that $P_S(e^{j2\pi f})$ is the optimum predictor of the process T_n. Under our assumption, $V_n = T_n$ for all past instances, thus it is also the optimum predictor of V_n. Consequently, $U_n - J_n$ is white, with variance

$$\begin{aligned}
\mathrm{Var}\{U_n - J_n\} &= \mathrm{Var}\{U_n | V_{n-1}, V_{n-2}, \ldots\} \\
&\overset{(a)}{=} P_e\Big(S_U + \frac{\mathrm{Var}\{Z_{eq_n}\}}{\beta^2}\Big) - \frac{\mathrm{Var}\{Z_{eq_n}\}}{\beta^2} \\
&= P_e\Big(S_U + \frac{\beta_0^2}{\beta^2}\theta_S\Big) - \frac{\beta_0^2}{\beta^2}\theta_S \\
&< \frac{\beta_0^2}{\beta^2}\Big[P_e(S_U + \theta_S) - \theta_S\Big] \\
&\overset{(b)}{=} \frac{\beta_0^2}{\beta^2} \cdot \frac{\alpha}{1-\alpha}\theta_S \\
&= \frac{\alpha\theta_C}{\beta^2} \quad ,
\end{aligned}$$

where (a) holds by (23) and (b) holds by applying the same in the opposite direction, combined with (37). By the whiteness of Z_{eq_n} and its independence of all U_n, we have that $U_n - J_n$ is independent of Z_{eq_n}, thus the variance of T_n is given by

$$\mathrm{Var}\{T_n\} = \beta^2\,\mathrm{Var}\{U_n - J_n\} + \mathrm{Var}\{Z_{eq_n}\} < \theta_C \quad . \tag{47}$$

The margin from θ_C depends on the margin in the inequality in the chain above, which depends only on $S_U(e^{j2\pi f})$, θ_C and β, and is strictly positive for all $\beta < \beta_0$ as required ■

We now note, that if we could choose $\beta = \beta_0$, and if (45) holds at all times, then by setting $G_2(e^{j2\pi f}) = G_1^*(e^{j2\pi f})$ we have the equivalent channel of Proposition 2, thus we achieve the optimum (31). The following shows in what sense we can approach this.

Lemma 4: (**Steady-state behavior of the Analog Matching scheme**) Let $p_e(K)$ be the probability that (45) does not hold in the present instance, in an Analog Matching scheme using a lattice of dimension K. Then

$$\lim_{K\to\infty} p_e(K) = 0 \quad ,$$

under the conditions of Lemma 3, provided that $\Lambda = \Lambda_K$ is taken from a sequence of lattices simultaneously good for source and channel coding in the sense of [12, Proposition 1].

Proof: The channel of Lemma 2 is equivalent to the (45) if correct decoding (42) holds for $\mathbf{T}_n$. By [12, Proposition 1], the probability approaches zero for large K if the input of the modulo-lattice operation is a combination of Gaussian and uniform (over the basic lattice cell) components[7], and if the power of T_n is less than $\sigma^2(\Lambda) = \theta_C$. The first condition holds since T_n is a combination of Z_{eq_n}, U_n and past values of V_n, and these, by our assumptions, are composed of the Gaussian processes U_n and Z_n and of the process $\tilde{X}_n$ uniformly distributed over the basic lattice cell, all passed through linear filters. The second condition holds by Lemma 3. $p_e(K)$ does not depend on n, since the margin δ in Lemma 3 depends only on the spectra which are fixed. ■

Now we translate the conditional result above, to an optimality claim for blocks of any finite length. We assume steady-state behavior, in the sense that the filters' state is correct.

Definition 1: We say that the Analog Matching scheme is *correctly initialized* at time instance n, if all signals at all times $n-1, n-2, \ldots$ take values according to the correct decoding assumption

Theorem 1: (**Asymptotic optimality of the Analog Matching scheme**) For any $\epsilon > 0$, the Analog Matching scheme can achieve

$$\text{SDR} \geq \text{SDR}^{opt} - \epsilon \quad ,$$

[7]Actually, [12] only discusses the combination of a Gaussian vector with a single uniform component, but the extension to multiple uniform components is straightforward.

where SDR^{opt} was defined in (31), in transmitting source blocks of sufficient length N, provided that the lattice dimension K is high enough and that at the start of transmission the scheme is correctly initialized.

Proof: We assume that the scheme is correctly initialized at time $n = 1$. Let $N' = N + M$ be the number of channel uses made. Now as long as correct decoding holds, we have that $V_n = U_n + \frac{Z_{eq_n}}{\beta}$, where the additive noise term has variance

$$\frac{(1-\alpha)\theta_C}{\beta^2} = \frac{\beta_0^2}{\beta^2}\theta_S \triangleq \theta_S + \epsilon_1(\beta) \quad .$$

By Proposition 2, if correct decoding always holds and $\epsilon_1 = 0$, SDR $=$ SDR^{opt} exactly. Now we decide upon some finite N', and set $V_n = 0$ for all $n > N'$. This adds some distortion $\epsilon_2(N')$, but for large enough excess number of uses M the effect vanishes. If at any instance n correct decoding does not hold anymore, V_n starts taking arbitrary values, bounded by the maximum magnitude of $\mathcal{V}_0$ divided by β. We bound the performance then, by assuming that if that happens, then throughout the whole block V_n deviates by some v_{max}. Finally we can bound the probability of such an event: Defining $p_e(K, N')$ as the probability that correct decoding fails in *any* of $n = 1, 2, \ldots, N'$ we have by Lemma 4 and by the union bound that

$$p_e(K, N') \le p_e(K) \quad .$$

Taking all these effects into account, we have that:

$$D \le D^{opt} + \epsilon_1(\beta) + \epsilon_2(N') + v_{max}^2 \cdot N' p_e(K) \quad .$$

By taking β close enough to β_0 we can make ϵ_1 as small as desired. By taking large enough M we can make ϵ_2 as small as desired. Finally, by choosing K large enough we can make the last term small enough for any finite choice of β and N'. We have made M extra channel uses, but for large enough N the optimum performance for N' uses approaches this of N uses. ■

From the idealized scheme to implementation:

We now discuss how the scheme can be implemented with finite filters, how the correct initialization assumption may be dropped, and how the scheme may be used for a single source/channel pair.

1. **Filter length.** If we constrain ourself to filters of finite length, we may not be able to implement the optimum filters. However, it is possible to show, that both the effect on the correct decoding condition and on the final distortion can be made as small as desired, since the additional signal errors due to the filters truncation can be all made to have arbitrarily variance by taking long enough filters. In the sequel we assume the filters all have length L.

2. **Initialization.** After taking finite-length filters, we note that correct initialization now only involves a *finite* history of the scheme. Consequently, we can create this state by adding a finite number of channel uses. Now we may create a valid state for the channel predictor $P_C(e^{j2\pi f})$ by transmitting L values $\tilde{X}_n = 0$, see [9]. For the source predictor the situation is more involved, since in absence of past values of V_n, the decoder cannot reduce the source power to the innovations power, and correct decoding may not hold. This can be solved by de-activating the predictor for the first L values of U_n, and transmitting them with lower β such that (47) holds without subtracting J_n. Now in order to achieve the desired estimation error for these first values of V_n, one simply repeats the same values of U_n a number of times according to the (finite) ratio of β's. If the block length N is long enough relative to L, the number of excess channel uses becomes insignificant.

3. **Single source/channel pair.** A pair of interleaver/de-interleaver can serve to emulate K parallel sources, as done in [9] for an FFE-DFE receiver, and extended to lattice operations in [33].

General Remarks:

1. **Composition of the noise.** We bring here a short account of how noises at different points at the scheme are composed. For simplicity, we substitute here $\delta = 0$ in the margin of Lemma 3. The channel equivalent noise,

$$Z_{eq_n} = \alpha {Z'}_n - (1-\alpha)\tilde{X}_n$$

is an MMSE linear combination of an "unbiased slicer error" term ${Z'}_n$ and a "self-noise" term $\tilde{X}_n$. Both terms are white, and ${Z'}_n$ itself is composed of future channel inputs ("residual ISI") and channel noise. The ISI and the noise are weighted at each frequency according to the value of the channel post-filter, which is in turn set by the channel SNR at that frequency. Namely, if we denote the spectrum of the channel noise and self noise components as $S_{noise}(e^{j2\pi f})$ and $S_{ISI}(e^{j2\pi f})$, we have that:

$$\frac{S_{noise}(e^{j2\pi f})}{S_{ISI}(e^{j2\pi f})} = \frac{|G_2(e^{j2\pi f})|^2 S_Z(e^{j2\pi f})}{\left|1 - G_1(e^{j2\pi f})G_2(e^{j2\pi f})\right|^2 \theta_C} = \frac{P(e^{j2\pi f})S_Z(e^{j2\pi f})}{\left(\theta_C - P(e^{j2\pi f})\right)^2} = \frac{P(e^{j2\pi f})}{S_Z(e^{j2\pi f})} \quad ,$$

where the last equality holds if the SUB is satisfied with equality. The total reconstruction error is then a pre/post filtered version of the channel equivalent noise, thus again it is composed by a channel equivalent noise component and a bias component. The ratio between these components can be computed, to find that at each frequency the higher source spectrum is, the larger the part of the channel equivalent noise is.

2. **Composition of the signal at the decoder modulo-lattice output.** We turn to consider the signal $Z_{eq_n} + \beta Q_n$, where $Q_n \triangleq U_n - J_n$ is the source prediction error or "innovations" process. This signal should satisfy the output power constraint (47), thus substituting again $\delta = 0$, it has a total power θ_C.

These two components are mutually independent, and both white. The innovations process occupies a portion α of this power. This is reflected in the zooming coefficient, since we have:

$$\beta_0^2 = (1-\alpha)\frac{\theta_C}{\theta_S} = \alpha\frac{\theta_C}{\mathrm{Var}\{Q_n\}} \quad .$$

Thus β plays the role of amplifying the source (noisy) innovations process to the lattice power, in order to enhance the signal to distortion ratio, but the $\alpha < 1$ factor is still needed for "leaving room for the noise". In high SNR $\alpha \to 1$ and this effect vanishes. See [12] for further discussion.

Examples: Bandwidth Expansion and Compression

At this point, we present the special cases of *bandwidth expansion* and *bandwidth compression*, and see how the analog matching scheme specializes to these cases. In these cases the source and the channel are both white, but with different bandwidth (BW). The source and channel prediction gains are both one, and the optimum condition (31) becomes:

$$\mathrm{SDR}^{opt} = \left(1+\widetilde{\mathrm{SNR}}\right)^{\rho} \quad , \tag{48}$$

where the bandwidth ratio ρ was defined in (32).

For bandwidth expansion, we choose to work with a sampling rate corresponding with the channel bandwidth, thus in our discrete-time model the channel is white, but the source is band-limited to a frequency of $\frac{1}{2\rho}$. As a result, the channel predictor $P_c(z)$ vanishes and the channel pre- and post-filters become the scalar Wiener factor α. The source water-filling solution allocates all the distortion to the in-band frequencies, thus we have $\theta_S = \rho D$ and the source pre- and post-filters become both ideal low-pass filters of width $\frac{1}{2\rho}$ and height

$$\sqrt{1-\frac{1}{\mathrm{SDR}^{opt}}} = \sqrt{1-\left(1+\widetilde{\mathrm{SNR}}\right)^{-\rho}} \quad . \tag{49}$$

As the source is band-limited, the source predictor is non-trivial and depends on the distortion level. The resulting prediction error of U_n has variance

$$\mathrm{Var}\{U_n|V_{-\infty}^{n-1}\} = \frac{\rho\sigma_S^2}{\left(1+\widetilde{\mathrm{SNR}}\right)^{\rho-1}} \quad ,$$

and the resulting distortion achieves the optimum (48).

For bandwidth compression, the sampling rate reflects the source bandwidth, thus the source is white but the channel is band-limited to a frequency of $B_C = \frac{\rho}{2}$. In this case the source predictor becomes redundant, and the pre- and post-filters become a constant factor equal to (49). The channel pre- and post-filters are ideal low-pass filter of width $\frac{\rho}{2}$ and unit height. The channel predictor is the SNR-dependent

DFE. Again this results in achieving the optimum distortion (48). It is interesting to note, that in this case the outband part of the channel error $\tilde{Z}_n$ is entirely ISI (a filtered version of the channel inputs), while the inband part is composed of both channel noise and ISI, and tends to be all channel noise at high SNR.

V. Unknown SNR

So far we have assumed in our analysis that both the encoder and decoder know the source and channel statistics. In many practical communications scenarios, however, the encoder does not know the channel, or equivalently, it needs to send the same message to different users having different channels. Sometimes it is assumed that the channel filter $H_0(e^{j2\pi f})$ is given, but the noise level N is only known to satisfy $N \leq N_0$ for some given N_0. For this special case, and specifically the broadcast bandwidth expansion and compression problems, see [22], [15], [21], [19].

Throughout this section, we demonstrate that the key factor in asymptotic behavior for high SNR is the bandwidth ratio ρ (32). We start in Section V-1 by proving a basic lemma regarding achievable performance when the encoder is not optimal for the actual channel. In the rest of the section we utilize this result: In Section V-2 to show asymptotic optimality for unknown SNR in the case $\rho = 1$, then in Section V-3 we show achievable performance for the special cases of (white) BW expansion and compression, and finally in Section V-4 we discuss general spectra in the high-SNR limit.

A. Basic Lemma for Unknown SNR

We prove a result which is valid for the transmission of a colored source over a degraded colored Gaussian broadcast channel: We assume that the channel is given by (5), where B_C is known but the noise spectrum $S_Z(e^{j2\pi f})$ is unknown, except that it is bounded from above by some spectrum $S_{Z0}(e^{j2\pi f})$ everywhere. We then use an Analog Matching encoder optimal for $S_{Z0}(e^{j2\pi f})$, as in Theorem 1, but optimize the decoder for the actual noise spectrum. Correct decoding under $S_{Z0}(e^{j2\pi f})$ ensures correct decoding under $S_Z(e^{j2\pi f})$, thus the problem reduces to a *linear* estimation problem, as will be evident in the proof.

For this worst channel $S_{Z0}(e^{j2\pi f})$ and for optimal distortion (31), we find the water-filling solutions (24),(29), resulting in the source and channel water levels θ_S and θ_C respectively, and in a *source-channel passband* $\mathcal{F}_0$, which is the intersection of the inband frequencies of the source and channel water-filling solutions:

$$\begin{aligned}\mathcal{F}_S &= \{f : S_S(e^{j2\pi f}) \geq \theta_S\} \quad , \\ \mathcal{F}_C &= \{f : S_{Z_0}(e^{j2\pi f}) \leq \theta_C\} \quad , \\ \mathcal{F}_0 &= \mathcal{F}_S \cap \mathcal{F}_C \quad . \end{aligned} \tag{50}$$

Under this notation we have the following.

Lemma 5: For any noise spectrum $S_{Z0}(e^{j2\pi f})$, exists a single encoder, such that for any equivalent noise spectrum

$$S_Z(e^{j2\pi f}) \leq S_{Z0}(e^{j2\pi f}) \ \forall \ f \in \mathcal{F}_C \quad , \tag{51}$$

a suitable decoder can arbitrarily approach:

$$D = \int_{-\frac{1}{2}}^{\frac{1}{2}} D(e^{j2\pi f}) df \quad ,$$

where the distortion spectrum $D(e^{j2\pi f})$ satisfies:

$$D(e^{j2\pi f}) = \left\{ \begin{array}{ll} \frac{S_S(e^{j2\pi f})}{1+\Phi(e^{j2\pi f})}, & \text{if } f \in \mathcal{F}_0 \\ \min\Big(S_S(e^{j2\pi f}), \theta_S\Big), & \text{otherwise} \end{array} \right\} \quad , \tag{52}$$

with

$$\Phi(e^{j2\pi f}) = \frac{S_{Z_0}(e^{j2\pi f})}{S_Z(e^{j2\pi f})} \left[1 - \frac{S_{Z_0}(e^{j2\pi f}) - S_Z(e^{j2\pi f})}{\Theta_C}\right] \frac{S_S(e^{j2\pi f}) - \theta_S}{\theta_S} \quad .$$

Proof: We work with the optimum Analog Matching encoder for the noise spectrum $S_{Z0}(e^{j2\pi f})$. At the decoder, we note that for any choice of the channel post-filter $G_2(e^{j2\pi f})$, we have that the equivalent noise Z_{eq_n} is the noise $\tilde{Z}_n \triangleq \tilde{Y}_n - \tilde{X}_n$ passed through the filter $1 + P_C(e^{j2\pi f})$. Consequently, this noise has spectrum:

$$S_{eq}(e^{j2\pi f}) = S_{\tilde{Z}}(e^{j2\pi f})|1 + P_C(e^{j2\pi f})|^2 \quad .$$

The filter $G_2(e^{j2\pi f})$ should, therefore, be the Wiener filter which minimizes $S_{\tilde{Z}}(e^{j2\pi f})$ at each frequency. This filter achieves a noise spectrum

$$S_{\tilde{Z}}(e^{j2\pi f}) = \frac{\theta_C - S_{Z0}(e^{j2\pi f})}{\theta_C - S_{Z0}(e^{j2\pi f}) + S_Z(e^{j2\pi f})} S_Z(e^{j2\pi f})$$

inside $\mathcal{F}_C$, and θ_C outside. Denoting the variance of the (white) equivalent noise in the case $S_{Z0}(e^{j2\pi f}) = S_Z(e^{j2\pi f})$ as we have that σ_{eq}^2, we find that:

$$|1 + P_C(e^{j2\pi f})|^2 = \frac{\sigma_{eq}^2 \theta_C}{(\theta_C - S_{Z0}(e^{j2\pi f}))S_{Z0}(e^{j2\pi f})}$$

inside $\mathcal{F}_C$, and $\frac{\sigma_{eq}^2}{\theta_C}$ outside. We conclude that we have equivalent channel noise with spectrum

$$S_{eq}(e^{j2\pi f}) = \frac{S_Z(e^{j2\pi f})}{S_{Z0}(e^{j2\pi f})} \cdot \frac{\theta_C}{\theta_C - S_{Z0}(e^{j2\pi f}) + S_Z(e^{j2\pi f})}\sigma_{eq}^2 = \frac{S_S(e^{j2\pi f}) - \Theta_S}{\Phi(e^{j2\pi f})\theta_S}\sigma_{eq}^2$$

inside $\mathcal{F}_C$, and σ_{eq}^2 outside. Now, since this spectrum is everywhere upper-bounded by σ_{eq}^2, we need not worry about correct decoding. We have now at the source post-filter input the source, corrupted by an additive noise $\frac{Z_{ekn}}{\beta}$, with spectrum arbitrarily close to

$$\frac{S_{eq}(e^{j2\pi f})}{\beta_0^2} = \frac{S_S(e^{j2\pi f}) - \Theta_S}{\Phi(e^{j2\pi f})}$$

inside $\mathcal{F}_C$, and Θ_S outside. Now again we face optimal linear filtering, and we replace the source post-filter $F_2(e^{j2\pi f})$ by the Wiener filter for the source, to arrive at the desired result ■

Remarks:

1. Outside the source-channel passband $\mathcal{F}_0$, there is no gain when the noise spectrum density is lower than expected. Inside $\mathcal{F}_0$, the distortion spectrum is strictly monotonously decreasing in $S_Z(e^{j2\pi f})$, but the dependence is never stronger than inversely proportional. It follows, that the overall SDR is at most linear with the SNR. This is to be expected, since all the gain comes from linear estimation.

2. In the unmatched case modulation may change performance. That is, swapping source frequency bands before the analog matching encoder will change $\mathcal{F}_0$ and $\Phi(e^{j2\pi f})$, resulting in different performance as $S_Z(e^{j2\pi f})$ varies. It can be shown that the best robustness is achieved when $S_S(e^{j2\pi f})$ is monotonously decreasing in $S_Z(e^{j2\pi f})$.

3. The degraded channel condition (51) is not necessary. A tighter condition for correct decoding to hold can be stated in terms of $S_S(e^{j2\pi f})$, $S_{Z0}(e^{j2\pi f})$ and $S_Z(e^{j2\pi f})$, though it is cumbersome.

B. Asymptotic Optimality for equal BW

We prove asymptotic optimality in the sense that, if in the channel (27), the ISI filter is known but the SNR is only known to be above some SNR_0, then a single encoder can simultaneously approach optimality for any such SNR, in the limit that the minimum SNR is high. This follows directly from Lemma 5, noting that this is equivalent to the channel (5) with $S_Z(e^{j2\pi f}) = \frac{\widetilde{\text{SNR}_0}}{\widetilde{\text{SNR}}} \cdot S_{Z0}(e^{j2\pi f})$.

Theorem 2: **(Robustness at high SNR):** Assume we are allowed one channel use per source sample, and let the source and channel be given by (4) and (5), respectively. Then, exists an <u>SNR-independent</u> encoder that for any $\delta > 0$ achieves distortion

$$\text{SDR}(\text{SNR}) \geq (1-\delta)\text{SDR}^{opt}(\text{SNR})$$

for sufficiently large (but finite) SNR, i.e., for all $\text{SNR} \geq \text{SNR}_0(\delta)$.

Proof: We apply Lemma 5. If the source spectrum is bounded away from zero and the $S_{Z0}(e^{j2\pi f})$ is bounded from above, we can always take $\widetilde{\text{SNR}}_0$ high enough such that the source-channel passband $\mathcal{F}_0$ includes all frequencies, and then we have for all $\widetilde{\text{SNR}} \geq \widetilde{\text{SNR}}_0$:

$$D(e^{j2\pi f}) \leq \frac{1}{1-\epsilon(\widetilde{\text{SNR}}_0)} \cdot \frac{\widetilde{\text{SNR}}_0}{\widetilde{\text{SNR}}} \Theta_S$$

resulting in

$$\text{SDR} \geq (1-\epsilon(\widetilde{\text{SNR}}_0)) \frac{\widetilde{\text{SNR}}}{\widetilde{\text{SNR}}_0} \text{SDR}_0 = (1-\epsilon(\widetilde{\text{SNR}}_0)) \frac{\widetilde{\text{SNR}}}{\widetilde{\text{SNR}}_0} \Gamma_S \Gamma_C (1+\widetilde{\text{SNR}}_0)$$

where the second transition is due to Proposition 1. Again due to the same proposition, this is nearly optimal. If the spectra are not bounded, then we artificially set the pre-filters to be 1 outside their respective bands, and in the case of the channel pre-filter we slightly attenuate other frequencies to comply with the power constraint. While this has the effect of worsening the performance at $\widetilde{\text{SNR}}_0$, it allows for linear improvement of the SDR with $\widetilde{\text{SNR}}$ for all frequencies. The effect of worsening at $\widetilde{\text{SNR}}_0$ can be made as small as desired, by taking large enough $\widetilde{\text{SNR}}_0$. ■

Alternatively, we could prove this result using a the zero-forcing scheme of Section II with high lattice dimension. Actually, using such a scheme, an even stronger result can be proven: Not only can the encoder be SNR-independent, but so can the decoder.

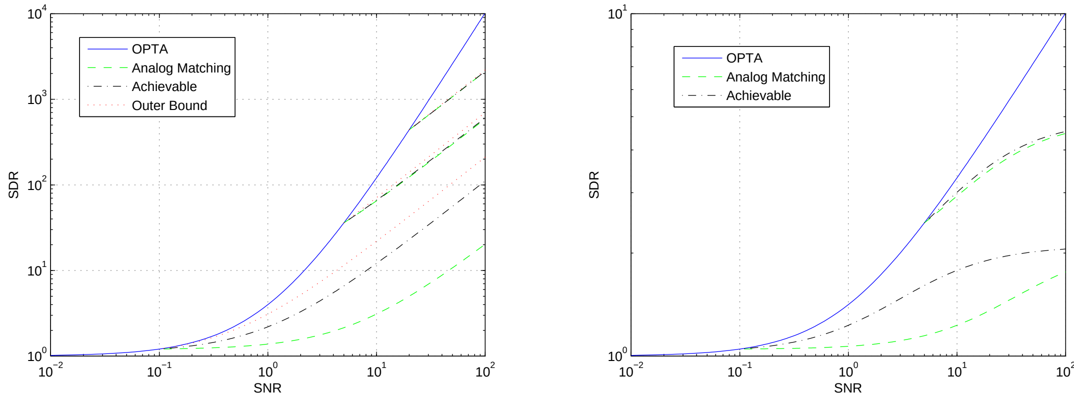
C. BW Expansion and Compression

We go back now to the cases of bandwidth expansion and compression discussed at the end of Section IV. In these cases, we can no longer have a single Analog Matching encoder which is universal for different SNRs, even in the high SNR limit. For bandwidth expansion ($\rho > 1$), the reason is that the source is perfectly predictable, thus at the limit of high SNR we have that

$$\text{Var}\{U_n | V_{n-1}, V_{n-2}, \ldots\} \rightarrow \text{Var}\{U_n | U_{n-1}, U_{n-2}, \ldots\} = 0 \quad ,$$

thus the optimum β goes to infinity. Any β value chosen to ensure correct decoding at some finite SNR, will impose unbounded loss as the SNR further grows. For bandwidth compression, the reason is that using any channel predictor suitable for some finite SNR, we have in the equivalent noise $\tilde{Z}_n$ some component which depends on the channel input. As the SNR further grows, this component does not decrease, inflicting again unbounded loss.

By straightforward substitution in Lemma 5, we arrive at the following.



(a) $\rho = 2$ (b) $\rho = \frac{1}{2}$

Figure 13: Unknown SNR Performance: BW Expansion and Compression

Corollary 1: Assume white source and AWGN channel where we are allowed ρ channel uses per source sample. Then using an optimum Analog Matching scheme which assumes signal to noise ratio SNR_0 and a suitable decoder, it is possible to approach:

$$\frac{1}{\mathrm{SDR}} = \frac{1-\min(1,\rho)}{\left(1+\widetilde{\mathrm{SNR}}_0\right)^{\rho}} + \frac{\min(1,\rho)}{1+\Phi_\rho(\widetilde{\mathrm{SNR}},\widetilde{\mathrm{SNR}}_0)} \quad , \tag{53}$$

where

$$\Phi_\rho(\widetilde{\mathrm{SNR}},\widetilde{\mathrm{SNR}}_0) \triangleq \frac{1+\widetilde{\mathrm{SNR}}}{1+\widetilde{\mathrm{SNR}}_0}\left[\left(1+\widetilde{\mathrm{SNR}}_0\right)^{\rho} - 1\right] \tag{54}$$

and $\tilde{\rho} =$, for any $\widetilde{\mathrm{SNR}} \geq \widetilde{\mathrm{SNR}}_0$.

Note that the choice of filters in the SNR-dependent decoder remains simple in this case: For $\rho > 1$ the channel post-filter is flat while the source post-filter is an ideal low-pass filter, while for $\rho < 1$ it is vice versa. The only parameters which change with SNR, are the scalar filter gains.

Comparison of performance: In comparison, the performance reported by different methods in [15], [21] for these cases has, in terms of (53):

$$\Phi_\rho(\widetilde{\mathrm{SNR}},\widetilde{\mathrm{SNR}}_0) = (1+\widetilde{\mathrm{SNR}}) \cdot (1+\widetilde{\mathrm{SNR}}_0)^{\rho-1} - 1 \tag{55}$$

while [21] also proves an outer bound for BW expansion ($\rho > 1$) on any scheme which is optimal at some SNR:

$$\Phi_\rho(\widetilde{\mathrm{SNR}},\widetilde{\mathrm{SNR}}_0) = \frac{\widetilde{\mathrm{SNR}}}{\widetilde{\mathrm{SNR}}_0}\left[(1+\widetilde{\mathrm{SNR}}_0)^{\rho} - 1\right] \quad . \tag{56}$$

In both BW expansion and compression, the Analog Matching scheme does not perform as good as the previously reported schemes, although the difference vanishes for high SNR. The basic drawback of analog matching compared to methods developed specifically for these special cases seems to be, that these methods apply different "zooming" to different source or channel frequency bands, analog matching uses the same "zooming factor" β for all bands. Enhancements to the scheme, such as the combination of analog matching with pure analog transmission, may improve these results. Figure V-3 demonstrates these results, for systems which are optimal at different SNR levels.

At high SNR, the performance of all these methods and of the outer bound converge to:

$$\frac{1}{\text{SDR}} = \frac{1-\min(\rho,1)}{\text{SNR}_0^{\rho}} + \frac{\min(\rho,1)}{\text{SNR}\cdot\text{SNR}_0^{\rho-1}} \quad . \tag{57}$$

Thus the Analog Matching scheme, as well as the schemes of [15], [21], are all asymptotically optimal for high SNR among the schemes which achieve SDR^{opt} at some SNR.

D. Asymptotic Behavior with BW Change

Finally we turn back to the general case of non-white spectra with any ρ, and examine it in the high-SNR regime. As in Section V-2, we assume that the channel ISI filter is known, corresponding with an equivalent noise spectrum $S_Z(e^{j2\pi f})$ known up to a scalar factor.

In the high-SNR limit, Lemma 5 implies:

$$\frac{1}{\text{SDR}} = \left[\frac{1-\min(\rho,1)}{\text{SNR}_0^{\rho}} + \frac{\min(\rho,1)}{\text{SNR}\cdot\text{SNR}_0^{\rho-1}}\right]\Gamma_C\Gamma_S \quad . \tag{58}$$

Comparing with (57), we see that the color of the source and of the noise determines a constant factor by which the SDR is multiplied, but the dependence upon the SNR remains similar to the white BW expansion/compression case. The following definition formalizes this behavior (see [13]).

Definition 2: The *distortion slope* of a continuum of SNR-dependent schemes is :

$$\lambda \triangleq \lim_{\widetilde{\text{SNR}}\to\infty} \frac{\log \text{SDR}}{\log \widetilde{\text{SNR}}} \tag{59}$$

where SDR is the signal to distortion attained at signal to noise ratio $\widetilde{\text{SNR}}$, where the limit is taken for a fixed channel filter with noise variance approaching 0

We use the notation $\lambda = \lambda(\rho)$ in order to emphasize the dependance of the asymptotic slope upon the bandwidth expansion factor. The following follows directly from Proposition 1.

Proposition 4: For any source and channel spectra with BW ratio ρ, and for a continuum of schemes achieving the OPTA performance (31),

$$\lambda(\rho) = \rho$$

As for an analog matching scheme which is optimal for a single SNR, (58) implies:

Corollary 2: For any source and channel spectra and for a single analog-matching encoder,

$$\lambda(\rho) = \left\{ \begin{array}{ll} 1, & \text{if } \rho \geq 1 \\ 0, & \text{otherwise} \end{array} \right\}$$

is achievable.

This asymptotic slope agrees with the outer bound of [21] for the (white) bandwidth expansion problem. For the bandwidth compression problem, no outer bound is known, but we are not aware of any proposed scheme with a non-zero asymptotic slope. We believe this to be true for all spectra:

Conjucture 1: For any source and channel spectra of BW ratio ρ, no single encoder which satisfies (31) at some $\widetilde{\mathrm{SNR}}_0$ can have a better slope than that of Corollary 2.

By this conjecture, the analog matching encoder is asymptotically optimal among all encoders ideally matched to one SNR. It should be noted, that schemes which do not satisfy optimality at one SNR *can* in fact approach the ideal slope $\lambda(\rho) = \rho$, see e.g. approaches for integer ρ such as bit interleaving [23].

VI. Conclusion: Implementation and Applications

We presented the Analog Matching scheme, which optimally transmits a Gaussian source of any spectrum over a Gaussian channel of any spectrum, without resorting to any data-bearing code. We showed the advantage of such a scheme over a separation-based solution, in the sense of robustness for unknown channel SNR.

The analysis we provided was asymptotic, in the sense that a high-dimensional lattice is needed. However, unlike digital transmission where reduction of the code block length has a severe impact on performance, the modulo-lattice framework allows in practice reduction to low-dimensional, even *scalar* lattices, with bounded loss.

One approach for scalar implementation of the Analog Matching scheme, uses *companding* [14]. In this approach, the scalar zooming factor β is replaced by a non-linear function which compresses the unbounded Gaussian source into a finite range, an operation which is reverted at the decoder. There is a problem here, since the entity which needs to be compressed is actually the innovations process $\tilde{Q}_n$, unknown at the encoder since it depends on the channel noise. This can be solved by compressing Q_n, the innovations of the source itself; The effect of this "companding encoder-decoder mismatch" vanishes in the high-SNR limit. An altogether different approach, is to avoid instantaneous decoding of the lattice; Instead, the decoder may at each instance calculate the source prediction using several hypothesis in parallel. The ambiguity will be solved in the future, possibly by a trellis-like algorithm.

Finally, we remark that the robustness analysis made in this paper is by no means the only application of Analog Matching. The scheme has the basic property, that it converts any colored channel to an equivalent additive white noise channel of the same capacity as the original channel, but of the source bandwidth. In the limit of high-SNR, this equivalent noise becomes Gaussian and independent of any encoder signal. This property is plausible in multi-user source, channel and joint source/channel problems, in the presence of bandwidth mismatch. Applications include computation over MACs [18], multi-sensor detection [17] and transmission over the parallel relay network [11].

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