Lattice Strategies for the Dirty Multiple Access Channel

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***Abstract*— We consider a generalization of the Gaussian dirty-paper problem to a multiple access setup. There are two additive interferences, one known to each transmitter but none to the receiver. We derive the rate region achievable using lattice strategies and also derive an outer bound for the capacity region. We find conditions under which the rate region achieved using lattice strategies meets the outer bound. We observe that lattice strategies offer an advantage over standard random binning techniques for this problem. In fact, standard random binning schemes fail to achieve any positive rate in this problem. We also derive a lattice transmission scheme for the asymmetric case, where there is only one interference which is known to one of the users. In particular, when there is one user which is ignorant of the interference and that wishes to send information (the *helper problem*), we find conditions under which lattice strategies are optimal. Furthermore, we show that lattice strategies asymptotically achieve the capacity region in the single interference problem in the limit of high SNR.**

I. Introduction

We consider a two-user Gaussian multiple access channel (MAC) with two known interferences as shown in Figure 1. The interference S_1 and S_2 can be *arbitrary* and are known non-causally to the transmitters of user 1 and user 2, respectively. Specifically, we consider the following *dirty MAC* model

$$Y = X_1 + X_2 + S_1 + S_2 + Z, \tag{1}$$

where $Z \sim \mathcal{N}(0, N)$ is independent of X_1, X_2, S_1, S_2, and where user 1 and user 2 must satisfy the power constraints, $E[X_1^2] \leq P_1$ and $E[X_2]^2 \leq P_2$, respectively. We define the signal to noise ratio for each user as $SNR_1 = \frac{P_1}{N}$ and $SNR_2 = \frac{P_2}{N}$.

This channel model generalizes Costa's dirty-paper channel [1] to a multiple access setup. In [1], Costa considered the single-user case, $Y = X + S + Z$, where the interference is assumed to be i.i.d. Gaussian. It was shown in [1] that in this case, the capacity is $\frac{1}{2}\log_2(1 + SNR)$, where $SNR = P/N$.

The proof of Costa uses the general capacity formula derived by Gelfand and Pinsker [4] for channels with (non-causal) side information at the transmitter. The technique of Gelfand and Pinsker falls in the framework of random binning which is widely used in the analysis of multi-terminal source and channel coding problems. They obtained a general capacity expression (originally derived for the DMC case) which involves an auxiliary random variable U:

$$C = \max_{P(u,x|s)} \{H(U|S) - H(U|Y)\} \tag{2}$$

where the maximization is over all the joint distributions of the form $p(u, s, y, x) = p(s)p(u, x|s)p(y|x, s)$. Selecting the auxiliary random variable u to be

$$U = X + \alpha S, \tag{3}$$

where $X \sim \mathcal{N}(0, P)$, $S \sim \mathcal{N}(0, Q)$, and taking $\alpha = \frac{P}{P+N}$, the random binning scheme is capacity achieving.

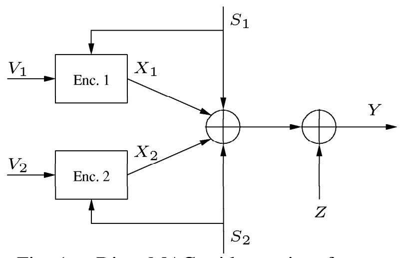

Fig. 1. Dirty MAC with two interferences

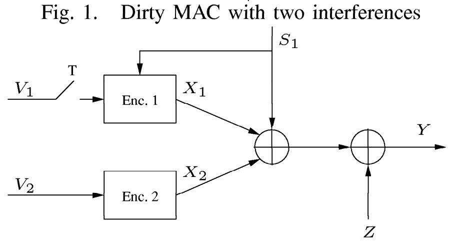

Fig. 2. Dirty MAC with one interference (when T is off: the helper problem)

Another special case of the channel model (1) was considered by Gel'fand and Pinsker in [3]. They showed that in the noiseless case ($N = 0$), arbitrary large rate pairs (R_1, R_2) are achievable. For the general case ($N > 0$) and Gaussian interferences, i.e., $S_1 \sim \mathcal{N}(0, Q_1)$ and $S_2 \sim \mathcal{N}(0, Q_2)$ are independent, they conjectured that the capacity region is the same as that of the MAC with no interference. The outer bound in Section II below shows that their conjecture is incorrect. Moreover, in the limit of strong interferences ($Q_1, Q_2 \to \infty$) "standard" random binning is not able to achieve *any positive rates*.

To see that, consider for simplicity the limit of high SNR where $SNR_1, SNR_2 \gg 1$, and high power independent Gaussian interferences, i.e., $S_1 \sim \mathcal{N}(0, Q_1)$ and $S_2 \sim$

$\mathcal{N}(0, Q_2)$ where $Q_1, Q_2 \gg \max\{P_1, P_2\}$. From (3), the auxiliary random variables are $U_1^h = X_1 + S_1$ and $U_2^h = X_2 + S_2$, since $\frac{P_1}{P_1+N}, \frac{P_2}{P_2+N} \approx 1$ at high SNR, and $X_1 \sim \mathcal{N}(0, P_1)$, $X_2 \sim \mathcal{N}(0, P_2)$ are independent. By letting the transmitters to cooperate, from (2) the achievable sum rate using the auxiliary random variables U_1^h, U_2^h is upper bounded by $h(U_1^h, U_2^h|S_1, S_2) - h(U_1^h, U_2^h|Y)$. Therefore, we have that

$$
\begin{aligned}
R_1 + R_2 &\leq h(U_1^h, U_2^h|S_1, S_2) - h(U_1^h, U_2^h|Y) \\
&= h(X_1, X_2) - h(Y|U_1^h, U_2^h) - h(U_1^h, U_2^h) + h(Y) \\
&= h(X_1, X_2) - h(Z) - h(U_1^h) - h(U_2^h) + h(Y) \\
&\approx h(S_1 + S_2) - h(S_1) - h(S_2).
\end{aligned}
$$

Since $h(S_1) + h(S_2) > h(S_1 + S_2)$ for independent Gaussian random variables S_1 and S_2 with $Q_1, Q_2 \to \infty$, we can not achieve any positive sum rate using such random binning scheme. Generally, it is not clear how to generalize random binning in such a way that may achieve a positive rate in this problem.

In contrast, lattice strategies can achieve positive rate. Furthermore, under sufficient conditions we show that lattice strategies are optimal. Thus, this coding problem is an instance where linear codes are superior to any known random binning technique. A similar situation was observed by Korner and Marton [5] in a problem involving distributed lossless source coding, where they showed that the achievable rate region using linear codes is optimal.

We also consider the case where $S_2 = 0$, i.e., the Gaussian dirty MAC problem with one interference known non causally to the transmitter of user 1

$$Y = X_1 + X_2 + S_1 + Z \tag{4}$$

as shown in Figure 2. For this case, random binning can achieve positive rates as shown recently in [6].

In Section II we present an outer bound on the achievable rates in this problem. Section III provides an optimal transmission scheme using lattice strategies for dirty MAC with two interferences at high SNR. In Section IV we present lattice strategies for any SNR and provides conditions for optimality.

II. Outer Bounds

We first establish outer bounds on the capacity region of the dirty MAC with one and two interferences.

Theorem 1: In the limit of strong interference, the outer bound for the capacity region of the dirty MAC with one interference (4) is:

$$
\begin{aligned}
R_1 + R_2 &\leq \frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right) \\
R_2 &\leq \frac{1}{2}\log_2\left(1 + \frac{P_2}{N}\right).
\end{aligned}
\tag{5}
$$

Strong interference can be either infinite power Gaussian interference or an arbitrary interference similarly to [2].

Proof: Assume a genie reveals the message of user 1 to user 2 and vice versa. Both users intend to transmit a common message W. An upper bound on the rate of this message clearly upper bounds $R_1 + R_2$. Applying Fano's inequality to the common message rate R we have,

$$nR \leq H(W) = H(W|Y^n) + I(W; Y^n) \leq n\epsilon_n + I(W; Y^n),$$

where $\epsilon_n \to 0$ as the error probability ($P_e^{(n)}$) goes to zero. The following chain of inequalities can be easily verified.

$$
\begin{aligned}
&I(W; Y^n) = h(Y^n) - h(Y^n|W) \\
&\leq h(Y^n) - h(Y^n|W, X_2^n,) \\
&= h(Y^n) - h(Y^n|W, X_2^n, S_1^n) - I(S_1^n; Y^n|W, X_2^n) \\
&\leq h(Y^n) - h(Z^n) - I(S_1^n; Y^n|W, X_2^n) \\
&= h(Y^n) - h(Z^n) - h(S_1^n) + h(S_1^n|W, X_2^n, Y^n) \qquad (6) \\
&= h(Y^n) - h(Z^n) - h(S_1^n) + h(X_1^n + Z^n|W, X_2^n, Y^n) \\
&\leq h(Y^n) - h(Z^n) - h(S_1^n) + h(X_1^n + Z^n), \qquad (7)
\end{aligned}
$$

where the equality in (6) follows from the fact that S_1^n is independent of (X_2^n, W) and the two inequalities are a consequence of the fact that conditioning reduces differential entropy. Now observe that for $Q_1 \to \infty$ (from Cauchy-Schwartz inequality) $h(Y^n) \leq \frac{n}{2}\log_2 2\pi e(N + (\sqrt{P_1} + \sqrt{P_2} + \sqrt{Q_1})^2) = \frac{n}{2}\log_2 Q_1 + no(1)$, and $h(S_1^n) = \frac{n}{2}\log_2 2\pi e Q_1$. Substituting in (7), and setting $\epsilon'_n = \epsilon_n + o(1)$ we have

$$nR \leq n\epsilon' + h(X_1^n + Z^n) - h(Z^n) \leq \frac{n}{2}\log_2\left(1 + \frac{P_1}{N}\right) + n\epsilon'_n,$$

as stated in the sum-rate bound in (5). The bound on R_2 trivially follows by revealing S_1^n to the decoder. ■

The outer bound in Theorem 1 is specialized to the *helper* problem below.

Corollary 1: Suppose that in the single interference model (4), only user 2 intends to send the message i.e., $R_1 = 0$. In the limit of strong interference, an upper bound on the rate R_2 is given by

$$R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{\min\{P_1, P_2\}}{N}\right). \tag{8}$$

The outer bound for the capacity region of dirty MAC with one interference is also an outer bound for the capacity region of the dirty MAC with two interferences, provided the two interferences are independent. One can show a tighter outer bound by taking the intersection of the outer bounds for dirty MAC with one interference S_1 and dirty MAC with one interference S_2.

Corollary 2: In the limit of strong interferences, the outer bound for the capacity region of the dirty MAC with two interferences (1) is given by

$$R_1 + R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{\min\{P_1, P_2\}}{N}\right). \tag{9}$$

Remark: The proof for Theorem 1, develops an upper bound on the common message capacity of the MAC channel. After developing our upper bound, we learned that the common message MAC channel has been studied in an independent parallel work in [7]. In particular, the authors provide (without proof) an expression for the common message capacity in dirty MAC with one interference. The problem of independent messages as well as the helper problem in dirty MAC, and the problem of dirty MAC with two-interferences have not been considered in [7].

III. Lattice Strategies for High SNR

In this section, we show an achievable rate using lattice strategies for the Dirty MAC with two interferences (1). We consider the high SNR case, i.e., $SNR_1, SNR_2 \gg 1$. The transmission schemes in this Section and in Section IV are based on lattice strategies which shown to be optimal for "writing on dirty paper" problem [2].

In the following scheme we use k-dimensional lattice Λ. We denote the normalized second moment and the basic cell of Λ by $G_k(\Lambda)$ and $\mathcal{V}$, respectively. We consider that Λ has second moment $\min\{P_1, P_2\}$. Assume that the transmitters send

$$\begin{aligned}
\mathbf{x}_1 &= [\mathbf{v}_1 - \mathbf{s}_1] \bmod \Lambda \\
\mathbf{x}_2 &= [\mathbf{v}_2 - \mathbf{s}_2] \bmod \Lambda,
\end{aligned}$$

where $\mathbf{v}_1, \mathbf{v}_2 \sim U(\mathcal{V})$ carry the information for user 1 and user 2, respectively. The received signal $\mathbf{y}$ is reduced modulo-Λ which results the additive modulo MAC equivalent channel

$$\mathbf{y}' = \mathbf{y} \bmod \Lambda = [\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{z}] \bmod \Lambda.$$

In this MAC, the achievable sum rate is given by

$$\begin{aligned}
R_1 + R_2 &= \frac{1}{k} I(\mathbf{v}_1, \mathbf{v}_2; \mathbf{y}') \\
&= \frac{1}{k} \left\{ h(\mathbf{y}') - h(\mathbf{y}' | \mathbf{v}_1, \mathbf{v}_2) \right\} \\
&= \frac{1}{2} \log_2 \left(\frac{\min\{P_1, P_2\}}{G_k(\Lambda)} \right) - \frac{1}{k} h(\mathbf{z} \bmod \Lambda) \\
&\geq \frac{1}{2} \log_2 \left(\frac{\min\{P_1, P_2\}}{G_k(\Lambda)} \right) - \frac{1}{2} \log_2 (2\pi e N) \\
&= \frac{1}{2} \log_2 \left(\frac{\min\{P_1, P_2\}}{N} \right) - \frac{1}{2} \log_2 (2\pi e G_k(\Lambda))
\end{aligned}$$

For optimal lattice with $G_k(\Lambda) \to 1/2\pi e$ as $k \to \infty$, we can approach the outer bound (9) at high SNR. The above result is stated in the following Theorem.

Theorem 2: In the limit of strong interferences, the capacity region of the dirty MAC with two interferences (1) at high SNR, is given by

$$R_1 + R_2 \leq \frac{1}{2} \log_2 \left(1 + \frac{\min\{P_1, P_2\}}{N} \right) - o(1),$$

where $o(1) \to 0$ as $P_1, P_2 \to \infty$.

The reason that lattice strategies can achieve significant rate is due to the linear structure of lattice codes as well as the additive model of the dirty MAC problem. This structure enables the users to use decentralized encoders which effectively (at the receiver) have the same structure as in single user problem with full side information $S_1 + S_2$ at the transmitter.

IV. Lattice Strategies for General SNR

In this Section we show a transmission scheme based on lattice strategies for the dirty MAC with two interferences (1) for any SNR. We provide conditions that lattice strategies are optimal. We also show a transmission scheme based on lattice strategies for dirty MAC with one interference (4) and condition for optimality as well.

A. Dirty MAC with Two Interferences

First, we show achievable rate for the point $(0, R_2)$. We denote the nearest neighbor quantizer associated with the lattice Λ by $Q_\Lambda(\cdot)$, that is $[\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x})$.

Consider the case that $P_1 \geq P_2$. User 1 and user 2 use the lattices Λ_1 and Λ_2 with second moments P_1 and P_2, respectively. Specifically, the transmitters send

$$\begin{aligned}
\mathbf{x}_1 &= [-\mathbf{s}_1 + \mathbf{d}_1] \bmod \Lambda_1 \\
\mathbf{x}_2 &= [\mathbf{v}_2 - \alpha_2 \mathbf{s}_2 + \mathbf{d}_2] \bmod \Lambda_2,
\end{aligned}$$

where $\mathbf{v}_2 \sim U(\mathcal{V}_2)$ carries the information of user 2. The dither signals $\mathbf{d}_1$ and $\mathbf{d}_2$ are common randomness, where $\mathbf{d}_1 \sim U(\mathcal{V}_1)$ is known at the encoder of user 1 and to the decoder, and $\mathbf{d}_2 \sim U(\mathcal{V}_2)$ is known at the encoder of user 2 and to decoder as well. The receiver calculates $\mathbf{y}' = [\alpha_2(\mathbf{y} - \mathbf{d}_1) - \mathbf{d}_2] \bmod \Lambda_2$. It can be shown that the resulting channel is given by

$$\mathbf{y}' = [\mathbf{v}_2 - (1 - \alpha_2)\mathbf{u}_2 + \alpha_2 \mathbf{z} - \alpha_2 Q_{\Lambda_1}(\mathbf{d}_1 - \mathbf{s}_1)] \bmod \Lambda_2,$$

where $\mathbf{u}_2 \sim U(\mathcal{V}_2)$ is independent of $\mathbf{v}_2$ due to the dither quantization property [8]. User 1 tries to eliminate the interference $\mathbf{s}_1$, therefore it effectively uses $\alpha_1 = 1$. However, a residual interference remains at the receiver input which consists of Λ_1-lattice point. User 2 send $\mathbf{v}_2$ in presence of the interference $\mathbf{s}_2$. In order to achieve the maximal rate the optimal MMSE factor is used, i.e., $\alpha_2 = \frac{P_2}{P_2 + N}$. For $\Lambda_2 = \alpha_2 \Lambda_1$, we have that $\alpha_2 Q_{\Lambda_1}(\mathbf{d}_1 - \mathbf{s}_1) \in \Lambda_2$. Such a selection of lattices causes the element $\alpha_2 Q_{\Lambda_1}(\mathbf{d}_1 - \mathbf{s}_1)$ to disappear after the modulo Λ_2 operation. However, it restricts the user powers to the ratio $P_2 = \alpha_2^2 P_1$. As a consequence, the equivalent channel is given by

$$\mathbf{y}' = [\mathbf{v}_2 - (1 - \alpha_2)\mathbf{u}_2 + \alpha_2 \mathbf{z}] \bmod \Lambda_2, \tag{10}$$

The rate that user 2 can achieve is given by

$$\begin{aligned}
R_2 &= \frac{1}{k} I(\mathbf{v}_2; \mathbf{y}') \\
&= \frac{1}{k} \left\{ h(\mathbf{y}') - h(\mathbf{y}' | \mathbf{v}_2)) \right\} \\
&= \frac{1}{k} \left\{ h(\mathbf{y}') - h([(1 - \alpha_2)\mathbf{u}_2 + \alpha_2 \mathbf{z}] \bmod \Lambda_2) \right\} \\
&= \frac{1}{2} \log_2 \left(\frac{P_2}{G_k(\Lambda_2)} \right) - h([(1 - \alpha_2)\mathbf{u}_2 + \alpha_2 \mathbf{z}] \bmod \Lambda_2) \\
&\geq \frac{1}{2} \log_2 \left(\frac{P_2}{G_k(\Lambda_2)} \right) - \frac{1}{2} \log_2 \left(2\pi e \left((1 - \alpha_2)^2 P_2 + \alpha_2^2 N \right) \right)
\end{aligned}$$

Using optimal lattices with $G_k(\Lambda_1), G_k(\Lambda_2) \to 1/2\pi e$ as $k \to \infty$, and for $\alpha_2 = \frac{P_2}{P_2+N}$, we have that

$$R_2 \geq \frac{1}{2}\log_2\left(1+\frac{P_2}{N}\right).$$

Therefore, for $P_1 = P_2(\frac{P_2+N}{P_2})^2$ the inner bound meets the outer bound (9). For $P_1 \geq P_2(\frac{P_2+N}{P_2})^2$, the outer bound (9) remains $\frac{1}{2}\log_2(1+P_2/N)$ which is also achievable.

Now we consider the case that $P_1 < P_2$. The encoders send

$$\begin{aligned}\mathbf{x}_1 &= [-\alpha_1\mathbf{s}_1+\mathbf{d}_1] \bmod \Lambda_1\\ \mathbf{x}_2 &= [\mathbf{v}_2-\mathbf{s}_2] \bmod \Lambda_2,\end{aligned}$$

where $\mathbf{v}_2 \sim U(\mathcal{V}_2)$ carries the information of user 2, the dither signal $\mathbf{d}_1 \sim U(\mathcal{V}_1)$ is known at the encoder of user 1 and to the decoder. The receiver calculates $\mathbf{y}' = [\alpha_1\mathbf{y}-\mathbf{d}_1] \bmod \Lambda_1$. It can be shown that the resulting channel is given by

$$\mathbf{y}' = [\alpha_1\mathbf{v}_2 - (1-\alpha_1)\mathbf{u}_1 + \alpha_1\mathbf{z} - \alpha_1 Q_{\Lambda_2}(\mathbf{v}_2-\mathbf{s}_2)] \bmod \Lambda_1,$$

where $\mathbf{u}_1 \sim U(\mathcal{V}_1)$ is independent of $\mathbf{v}_2$ due to the dither. User 2 send $\mathbf{v}_2$ and eliminates the interference $\mathbf{s}_1$ as well, therefore it effectively uses $\alpha_2 = 1$. However, a residual interference remains at the receiver input which consists of Λ_2-lattice point. User 1 tries to minimizes the *effective noise* ($(1-\alpha_1)\mathbf{u}_1 + \alpha_1\mathbf{z}$) variance, therefore it uses the optimal MMSE factor, i.e., $\alpha_1 = \frac{P_1}{P_1+N}$. For $\Lambda_1 = \alpha_1\Lambda_2$, we have that $\alpha_1 Q_{\Lambda_2}(\mathbf{v}_2-\mathbf{s}_2) \in \Lambda_1$. Such a selection of lattices causes the element $\alpha_1 Q_{\Lambda_2}(\mathbf{v}_2-\mathbf{s}_2)$ to disappear after the modulo Λ_1 operation. However, it restricts the user powers to the ratio $P_1 = \alpha_1^2 P_2$. As a consequence, we have that

$$\mathbf{y}' = [\alpha_1\mathbf{v}_2 - (1-\alpha_1)\mathbf{u}_1 + \alpha_1\mathbf{z}] \bmod \Lambda_1, \tag{11}$$

The rate that user 2 can achieve is given by

$$\begin{aligned}R_2 &= \frac{1}{k}I(\mathbf{v}_2;\mathbf{y}')\\ &= \frac{1}{k}\{h(\mathbf{y}') - h(\mathbf{y}'|\mathbf{v}_2))\}\\ &= \frac{1}{k}\{h(\mathbf{y}') - h([(1-\alpha_1)\mathbf{u}_1+\alpha_1\mathbf{z}] \bmod \Lambda_1)\}\\ &= \frac{1}{2}\log_2\left(\frac{\min\{P_1, \alpha_1^2P_2+(1-\alpha_1)^2P_1+\alpha_1^2N\}}{G_k(\Lambda_1)}\right)\\ &\quad -h([(1-\alpha_1)\mathbf{u}_1+\alpha_1\mathbf{z}] \bmod \Lambda_1)\\ &\geq \frac{1}{2}\log_2\left(\frac{\min\{P_1, \alpha_1^2P_2+(1-\alpha_1)^2P_1+\alpha_1^2N\}}{G_k(\Lambda_1)}\right)\\ &\quad -\frac{1}{2}\log_2\left(2\pi e\left((1-\alpha_1)^2P_1+\alpha_1^2N\right)\right).\end{aligned} \tag{12}$$

Using optimal lattices with $G_k(\Lambda_1), G_k(\Lambda_2) \to 1/2\pi e$ as $k \to \infty$, and for $\alpha_1 = \frac{P_1}{P_1+N}$, we have that

$$R_2 = \frac{1}{2}\log_2\left(\min\left\{1+\frac{P_1}{N}, 1+\frac{P_2}{N}\cdot\frac{P_1}{P_1+N}\right\}\right). \tag{13}$$

For $P_2 = P_1(\frac{P_1+N}{P_1})^2$ we have that $\frac{P_1}{N} < \frac{P_2}{N}\cdot\frac{P_1}{P_1+N}$, thus

$$R_2 \geq \frac{1}{2}\log_2\left(1+\frac{P_1}{N}\right).$$

Therefore, for $P_2 = P_1(\frac{P_1+N}{P_1})^2$ the inner bound meets the outer bound (9). For $P_2 \geq P_1(\frac{P_1+N}{P_1})^2$, the outer bound (9) remains $\frac{1}{2}\log_2(1+P_1/N)$, therefore it is also achievable.

Until now, we show that the achievable rate for the point $(0, R_2)$ is given by

$$R_2 = \begin{cases}\frac{1}{2}\log_2\left(1+\frac{P_1}{N}\right), & P_2 \geq P_1\left(\frac{P_1+N}{P_1}\right)^2\\ \frac{1}{2}\log_2\left(1+\frac{P_2}{N}\right), & P_1 \geq P_2\left(\frac{P_2+N}{P_2}\right)^2\end{cases} \tag{14}$$

Due to the symmetry between users in the dirty MAC with two interferences, the same arguments can be used to show the achievable rate for the point $(R_1, 0)$, which is

$$R_1 = \begin{cases}\frac{1}{2}\log_2\left(1+\frac{P_2}{N}\right), & P_1 \geq P_2\left(\frac{P_2+N}{P_2}\right)^2\\ \frac{1}{2}\log_2\left(1+\frac{P_1}{N}\right), & P_2 \geq P_1\left(\frac{P_1+N}{P_1}\right)^2\end{cases} \tag{15}$$

As a consequence, any rate pair on the line $R_1 + R_2 = \frac{1}{2}\log_2(1+\min\{P_1,P_2\}/N)$ is achievable by using time sharing between (14) and (15), which meets the outer bound (9).

Theorem 3: Suppose that $N \leq \sqrt{P_1P_2} - \min\{P_1,P_2\}$. In the limit of strong interferences, the capacity region of the dirty MAC with two interferences (1) is given by

$$R_1 + R_2 \leq \frac{1}{2}\log_2\left(1+\frac{\min\{P_1,P_2\}}{N}\right).$$

B. Dirty MAC with One Interference

In the dirty MAC with one interference (4), it is easy to verify that the point $(R_1 = \frac{1}{2}\log_2(1+P_1/N), 0)$ can be achieved when user 2 is silent, $X_2 = 0$, while user 1 performs point to point DPC scheme.

We first consider the point $(0, R_2)$ the helper problem. The upper bound for this case is given in (8). Assume that the transmitters send

$$\begin{aligned}\mathbf{x}_1 &= [-\alpha_1\mathbf{s}_1+\mathbf{d}_1] \bmod \Lambda_1\\ \mathbf{x}_2 &= \mathbf{v}_2,\end{aligned} \tag{16}$$

where $\mathbf{v}_2 \sim U(\mathcal{V}_2)$ carries the information and $\mathbf{d}_1 \sim U(\mathcal{V}_1)$ is the dither. The receiver calculates $\mathbf{y}' = [\alpha_1\mathbf{y}-\mathbf{d}_1] \bmod \Lambda_1$. It can be shown that the resulting channel is given by

$$\mathbf{y}' = [\alpha_1\mathbf{v}_2 - (1-\alpha_1)\mathbf{u}_1 + \alpha_1\mathbf{z})] \bmod \Lambda_1,$$

where $\mathbf{u}_1 \sim U(\mathcal{V}_1)$ and $\mathbf{v}_2$ are independent due to the dither. In fact, the resulting channel is like we have in dirty MAC problem with two interferences (11). Therefore, using (13), for $\alpha = \frac{P_1}{P_1+N}$ and $G_k(\Lambda_1) \to 1/2\pi e$ as $k \to \infty$, the achievable rate is given by

$$R_2 \geq \frac{1}{2}\log_2\left(\min\left\{1+\frac{P_1}{N}, 1+\frac{P_2}{N}\cdot\frac{P_1}{P_1+N}\right\}\right).$$

Unlike the dirty MAC with two interferences, user 2 does not perform modulo operation. Therefore, we do not have any power restrictions like we had in the two interferences case. For $P_2 \geq P_1 + N$, we have that

$$R_2 \geq \frac{1}{2}\log_2\left(1+\frac{P_1}{N}\right).$$

In this case the inner bound meets the outer bound (8).

Now, we use the same transmission scheme as in (16), but with $\alpha_1 = 1$. Using (12), for $G_k(\Lambda_1) \to 1/2\pi e$ as $k \to \infty$, the achievable rate is given by

$$R_2 \geq \frac{1}{2}\log_2\left(\frac{\min\{P_1, P_2 + N\}}{N}\right).$$

For $P_1 \geq P_2 + N$, we have that

$$R_2 \geq \frac{1}{2}\log_2\left(1 + \frac{P_2}{N}\right),$$

In this case the inner bound is tight (8).

We conclude the above results with formal statement on the capacity for the helper problem.

Theorem 4: Suppose that $N \leq |P_1 - P_2|$ in the dirty MAC with one interference (4). In the limit of strong interference, the capacity of the *helper problem* is given by

$$R_2 = \frac{1}{2}\log_2\left(1 + \frac{\min\{P_1, P_2\}}{N}\right).$$

Unfortunately, for the case that $|P_1 - P_2| < N$, the inner bound for the helper problem stated (without proof) in the following Lemma is not tight.

Lemma 1: The achievable rate for the *helper problem* for $|P_1 - P_2| < N$ is given by

$$R_2 = \frac{1}{2}\log_2\left(1 + \frac{4P_1P_2}{(P_2 - P_1 + N)^2 + 4P_1N}\right).$$

For $P_1 = P_2 = P$ and $SNR = P/N$, the bound becomes

$$R_2 = \frac{1}{2}\log_2\left(1 + SNR\left(\frac{4SNR}{4SNR + 1}\right)\right).$$

For high SNR where $P_1, P_2 \gg N$ and in case that $|P_1 - P_2| < N$, the achievable rate for user 2 is $R_2 = \frac{1}{2}\log_2(1 + \min\{P_1, P_2\}/N) - o(1)$ where $o(1) \to 0$ as $P_1, P_2 \to \infty$, which meets asymptotically the helper problem outer bound (8).

Now, we consider the achievability for the capacity region using lattice strategies. We focus on the high SNR case, i.e., $SNR_1, SNR_2 \gg 1$. We first introduce the following Lemma without proof.

Lemma 2: Let

$$R_1^o = \tfrac{1}{2}\log_2\left(\frac{P_1+N}{N+\frac{P_1P_2}{P_1+N}}\right)$$
$$R_2^o = \tfrac{1}{2}\log_2\left(1 + \frac{P_2}{N}\cdot\frac{P_1}{P_1+N}\right).$$

In the limit of strong interference, for any P_1, P_2, N the point (R_1^o, R_2^o) lies on the boundary of the capacity region.

It is easy to verify that the point (R_1^o, R_2^o) lies on the outer bound (5), since $R_1^o + R_2^o = \frac{1}{2}\log_2(1 + P_1/N)$ where $R_2^o < \frac{1}{2}\log_2(1 + P_2/N)$ for any P_1, P_2, N. This point is achievable using lattice strategies.

For $P_1 \leq P_2$, the outer bound (5) becomes

$$R_1 + R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right).$$

On the other hand, the point $(0, \frac{1}{2}\log_2(1 + P_1/N))$ is achievable at high SNR as shown in the helper problem. Since the point $(\frac{1}{2}\log_2(1+P_1/N), 0)$ is achievable using DPC, therefore at high SNR the inner bound meets the outer bound (5).

For $P_1 > P_2$, and high SNR we have that

$$R_1^o = \frac{1}{2}\log_2\left(\frac{P_1 + N}{P_2 + N}\right) - o(1)$$
$$R_2^o = \frac{1}{2}\log_2\left(1 + \frac{P_2}{N}\right) - o(1),$$

where $o(1) \to 0$ as $P_1, P_2 \to \infty$. This point is exactly the corner point of the outer bound (5), since $R_1^o + R_2^o = \frac{1}{2}\log_2(1 + P_1/N) - o(1)$. The point $(0, \frac{1}{2}\log_2(1 + P_2/N))$ is achievable at high SNR as it was shown in the helper problem. The point $(\frac{1}{2}\log_2(1 + P_1/N), 0)$ is achieved using DPC. Therefore, at high SNR the inner bound meets the outer bound (5). As a consequence for $SNR_1, SNR_2 \gg 1$, the lattice strategies are optimal.

Theorem 5: In the limit of strong interference, the capacity region of dirty MAC with one interference (4) at high SNR, is given by

$$R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{P_2}{N}\right) - o(1)$$
$$R_1 + R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right) - o(1),$$

where $o(1) \to 0$ as $P_1, P_2 \to \infty$.

V. Summary

In this work we studied the dirty Gaussian MAC with two interferences, one known to each transmitter. We derived sufficient conditions under which lattice strategies meet the capacity region outer bound.

We also studied the asymmetric case, i.e., the dirty Gaussian MAC with one interference known only at one transmitter. In particular, for the helper problem we found sufficient conditions under which lattice strategies are optimal.

In both setups, at high SNR lattice strategies asymptotically achieve the capacity region.

References

[1] M.H.M. Costa, *Writing on dirty paper*, IEEE Trans. Information Theory **IT-29** (May 1983), 439–441.

[2] U. Erez, S. Shamai (Shitz), and R. Zamir, *Capacity and lattice strategies for canceling known interference*, IEEE Trans. Information Theory **IT-51** (Nov. 2005), 3820–3833.

[3] S. I. Gel'fand and M. S. Pinsker, *On Gaussian channels with random parameters*, Abstracts of Sixth International Symposium on Information Theory, Tashkent, U.S.S.R, Sep. 1984, pp. 247–250.

[4] S.I. Gelfand and M. S. Pinsker, *Coding for channel with random parameters*, Problemy Pered. Inform. (Problems of Inform. Trans.) **9, No. 1** (1980), 19–31.

[5] J. Korner and K. Marton, *How to encode the modulo-two sum of binary sources*, IEEE Trans. Information Theory **IT-25** (March 1979), 219–221.

[6] S. Kotagiri and J. N. Laneman, *Multiple access channels with state information known at some encoders*, IEEE Trans. Information Theory (July 2006, submitted for publication).

[7] A. Somekh-Baruch, S. Shamai, and S. Verdu, *Cooperative encoding with asymmetric state information at the transmitters*, Proceedings 44th Annual Allerton Conference on Communication, Control, and Computing, Univ. of Illinois, Urbana, IL, USA, Sep. 2006.

[8] R. Zamir, S. Shamai (Shitz), and U. Erez, *Nested linear/lattice codes for structured multiterminal binning*, IEEE Trans. Information Theory **IT-48** (June 2002), 1250–1276.