Shannon-Type Bounds for Multiple Descriptions of a Stationary Source

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Accepted to the special issue of JSPI
devoted to the Maine Conference, July 1997

Abstract

Multiple description source coding is aimed at achieving graceful degradation in reconstruction quality if portions of the code do not arrive at the decoder. Unfortunately, even the simplest model of two descriptions (and three "receivers") does not have a complete information theoretic solution, with the exception of the quadratic-Gaussian case. We show that the set of achievable marginal (d_1, d_2) and central (d_0) mean squared errors in decoding a real-valued, stationary and ergodic source from two descriptions at rates R_1 and R_2 is inner / outer bounded by the multiple description distortion region associated with a memoryless Gaussian source with the same variance / entropy-rate. We further show that like in the single description case, the outer bound is asymptotically tight at high resolution conditions.

Key Words: multiple descriptions, Gaussian codes, Shannon lower bound, high resolution.

Classification No.: 94A24, 94A29, 94A34.

*This work was supported by the ISF, administered by the Israel Academy of Science and Humanities. This work was presented in part in the Maine Conference, Portland, Maine, July 1997, and in the International Symposium on Information Theory, Cambridge, MA, August 1998.

I. Introduction

Transmission of compressed analog data (speech,image,video) over a "lossy" packet network (e.g., the Internet), motivates the problem of multiple description source coding [GC82]. In this digital communication scenario, a source code is broken into a few packets, some of which may not arrive to the destination. The decoder wishes to achieve a certain basic reproduction quality if a small subset of the packets arrives, and an improved quality if more packets or the whole source code arrives. Thus, portions of various size of the code should contain individually good, complementary descriptions of the source.

The basic formulation of the multiple description problem in the information theoretic literature involves two (noiseless) sub-channels of rates R_1 and R_2, corresponding to two "packets", and three receivers. Each receiver corresponds to a possible case of packet arrival, the first arrived, the second arrived or both arrived. In response to a source block $\boldsymbol{x} = (x_1, \ldots x_n)$, the encoder generates two code words (indices) $f_1(\boldsymbol{x})$ and $f_2(\boldsymbol{x})$ at rates

$$R_i = \tfrac{1}{n} \log |f_i(\cdot)| \qquad i = 1, 2$$

where $|f_i|$ denotes the size of code f_i $(\cdot)$, and transmits codeword f_i through sub-channel $i, i = 1, 2$. The two individual ("marginal") receivers and the combined ("central") receiver then generate reconstructions $\widehat{\boldsymbol{x}}_1, \widehat{\boldsymbol{x}}_2$ and $\widehat{\boldsymbol{x}}_0$, respectively, using the decoding functions g_1, g_2 and g_0, i.e.

$$\widehat{\boldsymbol{x}}_1 = g_1\big(f_1(\boldsymbol{x})\big)$$

$$\widehat{\boldsymbol{x}}_2 = g_2\big(f_2(\boldsymbol{x})\big)$$

$$\widehat{\boldsymbol{x}}_0 = g_0\big(f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\big) \ .$$

Assume that $X = \{X_i \in \mathrm{R}^1, i = 1, \ldots, n\}$ is a real-valued stationary and ergodic source, and we wish to satisfy mean squared errors d_1, d_2 and d_0 in the three non void cases of packet arrival, i.e.,

$$\begin{aligned} \tfrac{1}{n} E\|\hat{\boldsymbol{X}}_1 - \boldsymbol{X}\|^2 &\leq d_1 \\ \tfrac{1}{n} E\|\hat{\boldsymbol{X}}_2 - \boldsymbol{X}\|^2 &\leq d_2 \\ \tfrac{1}{n} E\|\hat{\boldsymbol{X}}_0 - \boldsymbol{X}\|^2 &\leq d_0 \end{aligned} \tag{1}$$

where the reconstructions $\hat{\boldsymbol{X}}_0, \hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_2$ are vectors in R^n. Let

$$\mathcal{R}_X(d_0, d_1, d_2) = \text{closure}\,\{(R_1, R_2) : d_0, d_1, d_2 \quad \text{are achievable for some } n\} \tag{2}$$

denote the set of *achievable rate pairs*, i.e., $(R_1, R_2) \in \mathcal{R}_X(d_0, d_1, d_2)$ if and only if for any $\epsilon > 0$ and sufficiently large n there exist descriptions at rates $R_1 + \epsilon$ and $R_2 + \epsilon$ whose reconstructions achieve expected distortions d_0, d_1 and d_2. Inversely, let

$$\mathcal{D}_X(R_1, R_2) = \{(d_0, d_1, d_2) : \quad (R_1, R_2) \in \mathcal{R}_X(d_0, d_1, d_2)\ \}$$

denote the set of *achievable distortions* for a given rate pair (R_1, R_2). The multiple description problem differs from the classical rate-distortion problem since it is usually impossible to satisfy the Shannon limits $R_1 = R_X(d_1)$ and $R_2 = R_X(d_2)$, and at the same time obtain from the central receiver distortion d_0 such that $R_1 + R_2 = R_X(d_0)$, where $R_X(d)$ denotes the rate distortion function of the stationary source X,

$$R_X(d) = lim_{n\to\infty} \frac{1}{n} \inf_{\{\mathrm{X}:\ E\|\mathrm{X}-\mathrm{X}\|^2 \leq d\}} I(\boldsymbol{X}; \hat{\boldsymbol{X}})$$

and $I(\cdot)$ denotes mutual information.

A "single letter" characterization of the set of achievable distortions $\mathcal{D}_X(R_1, R_2)$ for the case of a memoryless ("white") Gaussian source of variance σ^2 and squared error distortion

measure was found by Ozarow [Oza80]. This set, denoted $\mathcal{D}^*(\sigma^2, R_1, R_2)$, and its inverse set $\mathcal{R}^*(\sigma^2, d_0, d_1, d_2)$, are described in the next section. For a general source, however, the complete solution for $\mathcal{R}_X(d_0, d_1, d_2)$ is still unknown. El-Gamal and Cover [GC82] characterize an inner bound for $\mathcal{R}_X(d_0, d_1, d_2)$, i.e., a set of achievable rates; see Section IV. We do not know of any general *outer* bound for $\mathcal{R}_X(d_0, d_1, d_2)$. Partial solutions exist, e.g., for the "successive refinement" problem, where there is no constraint on one of the marginal distortions, or for the case of no excess rate sum, i.e., $R_1 + R_2 = R_X(d_0)$ [Ahl85]. Zhang and Berger [ZB95] derive relatively tight inner and outer bounds for the case of no excess marginal rate, i.e., $R_i = R_X(d_i), i = 1, 2$, but these bounds may be hard to compute in general.

In this paper we develop explicit outer and inner bounds on the multiple description rate-distortion region for a general real-valued stationary source and squared-error distortion measure, in terms of Ozarow's solution for the white Gaussian case. The version of these results for memoryless sources, as well as coding schemes induced from it, appear in [Zam99]. This manuscript, prepared for the special issue of JSPI devoted to the Maine Conference, considers the more general case of sources with memory. Our bounds have the form

$$\mathcal{R}^*(\sigma_x^2, d_0, d_1, d_2) \subseteq \mathcal{R}_X(d_0, d_1, d_2) \subseteq \mathcal{R}^*(P_x, d_0, d_1, d_2) \ , \tag{3}$$

and they extend Shannon's lower and upper bounds for the rate-distortion function [Ber71]

$$\frac{1}{2}\log\left(\frac{\sigma_x^2}{d}\right) \geq R_X(d) \geq \frac{1}{2}\log\left(\frac{P_x}{d}\right) \ , \tag{4}$$

where σ_x^2 is the variance of X, $\bar{h}(X) = \lim_{n\to\infty} \frac{1}{n} h(X_1 \ldots X_n)$ is the entropy-rate of X, and

$$P_x = 2^{2\bar{h}(X)}/2\pi e$$

is the entropy-rate power of X. Equality in (3) and (4) holds for a white Gaussian source X^*, in which case $P_x = \sigma_x^2$. The tightness of (3) and (4) for a general source X depends

on the closeness of the source to white Gaussianity, e.g., as measured by the divergence [CT91, ZF96]

$$\mathcal{D}(X;X^*) = \frac{1}{2}\log\left(\frac{\sigma_x^2}{P_x}\right) \ . \tag{5}$$

We state the outer and inner bounds in Theorem 1 and Theorem 2 in Sections III and IV, respectively, after briefly describing Ozarow's solution for a Gaussian source in Section II. In Section V we show that for a "smooth" source at high resolution conditions, i.e., in the limit as d_1 and d_2 go to zero along a straight line, the outer bound above becomes tight, and is realized by the optimum Gaussian "test channel" [GC82]. This result parallels the well known asymptotic tightness of the *Shannon lower bound* (right hand side of (4)) at high resolution; see [Ber71, LZ94].

II. Ozarow's Solution of the White Gaussian Case

Ozarow [Oza80] proved that the set of achievable squared distortions $\mathcal{D}^*(\sigma^2, R_1, R_2)$ for a memoryless (i.i.d.) Gaussian source with variance σ^2 is given by the union of all triplets d_0, d_1, d_2 satisfying

$$d_1 \ \geq \ \sigma^2 \cdot 2^{-2R_1} \tag{6a}$$

$$d_2 \ \geq \ \sigma^2 \cdot 2^{-2R_2} \tag{6b}$$

$$d_0 \ \geq \ \frac{\sigma^2 \cdot 2^{-2(R_1+R_2)}}{1 - \left(\left|\sqrt{\pi} - \sqrt{\triangle}\right|^+\right)^2} \tag{6c}$$

where $\pi = (1 - d_1/\sigma^2)(1 - d_2/\sigma^2)$ and $\triangle = d_1 d_2/\sigma^4 - 2^{-2(R_1+R_2)}$, and where

$$|x|^+ \stackrel{\triangle}{=} \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The $|\cdot|^+$ sign in (6c) does not appear in [Oza80]; it becomes effective only when $d_1 + d_2 > \sigma^2(1 + 2^{-2(R_1+R_2)})$ ("very high marginal distortions"), and can be omitted without affecting the boundary of the distortion region $\mathcal{D}^*(\sigma^2, R_1, R_2)$.

In the sequel we will need also the inverse region of (6). Given a triplet of squared distortions d_0, d_1, d_2, the set $\mathcal{R}^*(\sigma^2, d_0, d_1, d_2)$ of admissible rate pairs is given by the union of all (R_1, R_2) satisfying

$$R_1 \geq \tfrac{1}{2}\log\left(\frac{\sigma^2}{d_1}\right) \tag{7a}$$

$$R_2 \geq \tfrac{1}{2}\log\left(\frac{\sigma^2}{d_2}\right) \tag{7b}$$

$$R_1 + R_2 \geq \tfrac{1}{2}\log\left(\frac{\sigma^4}{d_1 d_2}\right) + \delta \tag{7c}$$

where $\delta = \delta(\sigma^2, d_0, d_1, d_2)$ is defined by

$$\delta = \begin{cases} \frac{1}{2}\log\left(\frac{1}{1-\rho^2}\right), & d_0 \leq d_{0max} \\ 0, & d_0 \geq d_{0max}\ , \end{cases} \tag{7d}$$

$$\rho = \begin{cases} -\frac{\sqrt{\pi\epsilon_0^2+\gamma}-\sqrt{\pi\epsilon_0^2}}{(1-\epsilon_0)\sqrt{\epsilon_1\epsilon_2}}, & \epsilon_1 + \epsilon_2 \leq 1 + \epsilon_0 \\ -\sqrt{\frac{\pi}{\epsilon_1\epsilon_2}}, & \text{otherwise} \end{cases} \tag{7e}$$

$$\gamma = (1-\epsilon_0)[(\epsilon_1 - \epsilon_0)(\epsilon_2 - \epsilon_0) + \epsilon_0\epsilon_1\epsilon_2 - \epsilon_0^2] \tag{7f}$$

$$\pi = (1-\epsilon_1)(1-\epsilon_2)\ , \qquad \epsilon_i = d_i/\sigma^2 \quad \text{for} \quad i = 0, 1, 2 \tag{7g}$$

and

$$d_{0max} = \frac{1}{\frac{1}{d_1} + \frac{1}{d_2} - \frac{1}{\sigma^2}} = \frac{d_1 d_2}{d_1 + d_2 - d_1 d_2/\sigma^2}\ .$$

Note that the second case of (7e) corresponds to "very high marginal distortions", as discussed regarding (6c).

Note that $\delta, \gamma \geq 0$ and $-1 \leq \rho \leq 0$, for all $d_1, d_2 \leq \sigma^2$ and $d_0 \leq d_{0max}$. Note also that δ depends on d_0, d_1, d_2 and σ^2 only through the ratios d_i/σ^2. As we will see in Section IV, ρ has the meaning of a correlation coefficient in the Gaussian optimal test channel. The right hand sides of (7a)-(7b) are the rate-distortion functions of the Gaussian source $X^* \sim \mathcal{N}(0, \sigma^2)$ at distortion levels d_1 and d_2 respectively. Thus, the quantity $\delta = \delta_1 + \delta_2 \geq 0$, where

$$\delta_i \stackrel{\Delta}{=} R_i - \frac{1}{2}\log\left(\frac{\sigma^2}{d_i}\right), \quad i = 1, 2$$

represents the Total Excess Marginal Rate (TEMR) in the Gaussian case, where we borrow the term "excess marginal rate" from [ZB95]. If d_1 and d_2 are held fixed, then the maximum central distortion d_{0max} can be achieved without excess marginal rate (i.e., with $\delta = \triangle = 0$). As δ increases from zero to infinity (ρ varies from zero to -1), the central distortion d_0 decreases from d_{0max} to zero at a rate $O(2^{-2\delta})$.

The *no excess rate sum* case, i.e., $R_1 + R_2 = R_x(d_0) = \frac{1}{2}\log(\sigma^2/d_0)$ ($\pi \leq \Delta$ in (6)), whose solution was a breakthrough in the research of multiple descriptions [Ahl85], has little practical significance in the Gaussian case. The reason is that the excess rate sum vanishes only when $d_1 + d_2 \geq \sigma^2(1 + 2^{-2(R_1+R_2)}) = \sigma^2 + d_0$ (see [Oza80]), i.e., when at least one of the two marginal receivers suffers from distortion higher than $\sigma^2/2$, corresponding to a signal-to-noise ratio that is worse than 3dB. Having both types of excess rate (marginal-rate and rate-sum) zero is possible only in the trivial case where one of the sub-channels is disabled (e.g. $R_2 = 0$ and $d_2 = \sigma^2$).

Another special case in which we will be interested in the sequel is that of high resolution coding, i.e., the limit as d_1/σ^2 and d_2/σ^2 go to zero along a straight line. In this limit the TEMR and the maximum central distortion do not depend on the variance of the source.

Specifically we have

$$\begin{aligned}\delta_{HR}(d_0,d_1,d_2) &\stackrel{\Delta}{=} \lim_{\sigma\to\infty}\delta(\sigma^2,d_0,d_1,d_2) \stackrel{\Delta}{=} \lim_{\lambda\to 0}\delta(\sigma^2,\lambda d_0,\lambda d_1,\lambda d_2)\\ &= \frac{1}{2}\log\left(\frac{1}{1-\rho_{HR}^2}\right) \end{aligned} \tag{8}$$

where

$$\rho_{HR} = -\frac{\sqrt{d_1/d_0-1}\sqrt{d_2/d_0-1}-1}{\sqrt{d_1d_2/d_0^2}} \tag{9}$$

and

$$\lim_{\sigma\to\infty} d_{0max} = \frac{d_1d_2}{d_1+d_2}\,. \tag{10}$$

III. A "Shannon Lower Bound" for Multiple Descriptions

In this section we prove an outer bound for the set of achievable mean squared errors in multiple description encoding of a general real-valued source.

Theorem 1 *The distortion region of any real-valued stationary source X with finite differential entropy-rate is bounded by*

$$\mathcal{D}_X(R_1,R_2) \subseteq \mathcal{D}^*(P_x,R_1,R_2) \tag{11}$$

where $P_x = (2\pi e)^{-1}2^{2\bar{h}(x)}$ and $\bar{h}(X)$ are the entropy-rate power and the differential entropy-rate of the source, respectively, and $\mathcal{D}^*(P_x,R_1,R_2)$ is the set defined in (6) (here at $\sigma^2 = P_x$)

of all triplets (d_0, d_1, d_2) *satisfying*

$$d_1 \geq P_x \cdot 2^{-2R_1} \tag{12a}$$

$$d_2 \geq P_x \cdot 2^{-2R_2} \tag{12b}$$

$$d_0 \geq \frac{P_x \cdot 2^{-2(R_1+R_2)}}{1 - (\left|\sqrt{\tilde{\pi}} - \sqrt{\tilde{\triangle}}\right|^+)^2} \tag{12c}$$

where $\tilde{\pi} = (1 - d_1/P_x)(1 - d_2/P_x)$ *and* $\tilde{\triangle} = d_1 d_2/P_x^2 - 2^{-2(R_1+R_2)}$.

Proof: Assume that some multiple description scheme $(f_1, f_2, g_0, g_1, g_2)$ encodes the source $\mathbf{X} \in \mathrm{R}^n$ with rates R_1, R_2, and reconstructs $\mathbf{X}$ with central and marginal distortions

$$d_i = \tfrac{1}{n} E\|\hat{\boldsymbol{X}}_i - \boldsymbol{X}\|^2 \quad i = 0, 1, 2.$$

We will show that d_1, d_2 and d_0 satisfy (12).

The first two constraints (12a) and (12b) follow straightforwardly by applying the conventional (single description) Rate-Distortion Theorem, and the Shannon lower bound $R_X(d) \geq \frac{1}{2}\log(\frac{P_x}{d})$ (right side of (4)), to the marginal receivers, i.e.,

$$R_i \geq R_X(d_i) \geq \frac{1}{2}\log\left(\frac{P_x}{d_i}\right) \quad i = 1, 2.$$

The proof of the constraint (12c) on the central distortion d_0 below follows the line of the proof of the converse in [Oza80], except that at few points it uses the Shannon lower bound instead of the explicit rate distortion function, and at one point it uses the entropy-power inequality instead of the exact value of the entropy.

Since the encoder transmits the codewords $f_1(\boldsymbol{X})$ and $f_2(\boldsymbol{X})$, we have

$$nR_1 \geq H(f_1(\boldsymbol{X})) = I(\mathbf{X}; f_1(\boldsymbol{X})) \tag{13}$$

$$nR_2 \geq H(f_2(\boldsymbol{X})) = I(\mathbf{X}; f_2(\boldsymbol{X})) \tag{14}$$

where $H()$ and $I(\ ;\ )$ denote entropy and mutual information, respectively. In the sequel we use the following identity which follows by applying twice the chain rule

$$I(A;BC) = I(A;B) + I(A;C) + I(B;C|A) - I(B;C)\ . \tag{15}$$

Adding (13) and (14) and applying (15), we obtain

$$\begin{aligned} n(R_1+R_2) &\geq I(\boldsymbol{X}; f_1(\boldsymbol{X})) + I(\boldsymbol{X}; f_2(\boldsymbol{X})) \\ &= I(\boldsymbol{X}; f_1(\boldsymbol{X}), f_2(\boldsymbol{X})) + I(f_1(\boldsymbol{X}); f_2(\boldsymbol{X})) - I(f_1(\boldsymbol{X}); f_2(\boldsymbol{X})|\boldsymbol{X}) \\ &= I(\boldsymbol{X}; f_1(\boldsymbol{X}), f_2(\boldsymbol{X})) + I(f_1(\boldsymbol{X}); f_2(\boldsymbol{X}))\ , \end{aligned} \tag{16}$$

where the last equality follows since f_1 and f_2 are deterministic functions of $\mathbf{X}$. Since $\widehat{\boldsymbol{X}}_1 = g_1(f_1(\boldsymbol{X})), \hat{\boldsymbol{X}}_2 = g_2(f_2(\boldsymbol{X}))$ and $\widehat{\boldsymbol{X}}_0 = g_0(f_1(\boldsymbol{X}), f_2(\boldsymbol{X}))$, the data processing theorem implies from (16)

$$R_1 + R_2 \geq \tfrac{1}{n} I(\boldsymbol{X}; \hat{\boldsymbol{X}}_0) + \tfrac{1}{n} I(\hat{\boldsymbol{X}}_1; \hat{\boldsymbol{X}}_2) \tag{17}$$

$$R_1 + R_2 \geq \tfrac{1}{n} I(\boldsymbol{X}; \hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_2) + \tfrac{1}{n} I(\hat{\boldsymbol{X}}_1; \hat{\boldsymbol{X}}_2)\ . \tag{18}$$

By the definition of the rate-distortion function we have $\frac{1}{n} I(\boldsymbol{X}; \hat{\boldsymbol{X}}_0) \geq R_X(d_0)$, so from (17)

$$R_1 + R_2 \ \ \geq \ \ R_X(d_0) + \tfrac{1}{n} I(\hat{\boldsymbol{X}}_1; \hat{\boldsymbol{X}}_2)\ . \tag{19}$$

Using the Shannon lower bound (4) we thus have

$$d_0 \geq P_x \cdot 2^{-2(R_1+R_2)} \cdot 2^{\frac{2}{n} I(\mathrm{X}_1;\mathrm{X}_2)}\ . \tag{20}$$

To lower bound $2^{\frac{2}{n} I(\mathrm{X}_1;\mathrm{X}_2)}$, we introduce, like in [Oza80], an auxiliary random vector $\boldsymbol{Y} \in \mathrm{R}^n$, where

$$Y_j = X_j + Z_j \qquad j = 1, \ldots, n$$

and where the Z_j's are independent zero mean Gaussian variables, jointly independent of $(\boldsymbol{X}, \hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_2)$, and having variance σ_z^2. (Later in the proof we pick a specific value for σ_z^2.) Using identity (15) with $A = \boldsymbol{Y}, B = \hat{\boldsymbol{X}}_1$ and $C = \hat{\boldsymbol{X}}_2$, we have

$$\begin{aligned} I(\hat{\boldsymbol{X}}_1; \hat{\boldsymbol{X}}_2) &= I(\boldsymbol{Y}; \hat{\boldsymbol{X}}_1) + I(\boldsymbol{Y}; \hat{\boldsymbol{X}}_2) + I(\hat{\boldsymbol{X}}_1; \hat{\boldsymbol{X}}_2 | \boldsymbol{Y}) - I(\boldsymbol{Y}; \hat{\boldsymbol{X}}_1 \hat{\boldsymbol{X}}_2) \\ &\geq I(\boldsymbol{Y}; \hat{\boldsymbol{X}}_1) + I(\boldsymbol{Y}; \hat{\boldsymbol{X}}_2) - I(\boldsymbol{Y}; \hat{\boldsymbol{X}}_1 \hat{\boldsymbol{X}}_2) \end{aligned} \tag{21}$$

where the inequality follows from the nonnegativity of the mutual information. To lower bound the first two terms in (21) we use the definition of the rate-distortion function of the vector $\boldsymbol{Y}$, denoted $R_{\mathrm{Y}}(d)$, and the fact that

$$\tfrac{1}{n} E\|\hat{\boldsymbol{X}}_i - \boldsymbol{Y}\|^2 = \tfrac{1}{n} E\|\hat{\boldsymbol{X}}_i - \boldsymbol{X}\|^2 + \tfrac{1}{n} E\|\mathbf{Z}\|^2 = d_i + \sigma_z^2 \quad i = 1, 2$$

and obtain

$$\tfrac{1}{n} I(\boldsymbol{Y}; \hat{\boldsymbol{X}}_i) \geq R_{\mathrm{Y}}(d_i + \sigma_z^2) \geq \tfrac{1}{2} \log \left(\frac{P_y^{(n)}}{d_i + \sigma_z^2} \right) \qquad i = 1, 2 \tag{22}$$

where $P_y^{(n)} = (2\pi e)^{-1} 2^{2h(\mathrm{Y})/n}$ is the entropy power of the vector $\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{Z}$ (where $Z \sim \mathcal{N}(0, \sigma_z^2)$) and the second inequality follows from the Shannon lower bound (4) for the rate distortion function of the *vector* $\boldsymbol{Y}$. We upper bound the third term in (21) as follows

$$\begin{aligned} I(\boldsymbol{Y}; \hat{\boldsymbol{X}}_1 \hat{\boldsymbol{X}}_2) &= h(\boldsymbol{Y}) - h(\boldsymbol{Y} | \hat{\boldsymbol{X}}_1 \hat{\boldsymbol{X}}_2) && \text{(23a)} \\ &= \tfrac{n}{2} \log(2\pi e P_y^{(n)}) - h(\boldsymbol{X} + \boldsymbol{Z} | \hat{\boldsymbol{X}}_1 \hat{\boldsymbol{X}}_2) && \text{(23b)} \\ &\leq \tfrac{n}{2} \log(2\pi e P_y^{(n)}) - \tfrac{n}{2} \log 2\pi e \left(\frac{2^{\frac{2}{n} h(\mathrm{X} | \hat{\mathrm{X}}_1 \hat{\mathrm{X}}_2)}}{2\pi e} + \sigma_z^2 \right) && \text{(23c)} \end{aligned}$$

where in the last step we used the fact that $\boldsymbol{Z}$ is independent of $(\boldsymbol{X}, \hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_2)$ and applied the conditional version of the entropy-power inequality [Oza80, Bla65] to lower bound the conditional entropy of the sum of $\boldsymbol{X}$ and $\boldsymbol{Z}$ given $(\hat{\boldsymbol{X}}_1 \hat{\boldsymbol{X}}_2)$. Finally we lower bound the

conditional entropy $h(\boldsymbol{X}|\hat{\boldsymbol{X}}_1\hat{\boldsymbol{X}}_2)$ in (23c) as follows. From (18) we have

$$\begin{aligned} n(R_1+R_2) &\geq I(\boldsymbol{X};\hat{\boldsymbol{X}}_1\hat{\boldsymbol{X}}_2) + I(\hat{\boldsymbol{X}}_1;\hat{\boldsymbol{X}}_2) \\ &= h(\boldsymbol{X}) - h(\boldsymbol{X}|\hat{\boldsymbol{X}}_1\hat{\boldsymbol{X}}_2) + I(\hat{\boldsymbol{X}}_1;\hat{\boldsymbol{X}}_2) \end{aligned}$$

Thus (using $h(\boldsymbol{X}) \geq \frac{n}{2}\log(2\pi e P_x)$)

$$h(\boldsymbol{X}|\hat{\boldsymbol{X}}_1\hat{\boldsymbol{X}}_2) \geq \tfrac{n}{2}\log(2\pi e P_x) - n(R_1+R_2) + I(\hat{\boldsymbol{X}}_1;\hat{\boldsymbol{X}}_2)\ . \tag{24}$$

Combining (23c) and (24) we upper bound $I(\boldsymbol{Y};\hat{\boldsymbol{X}}_1\hat{\boldsymbol{X}}_2)$ by

$$I(\boldsymbol{Y};\hat{\boldsymbol{X}}_1\hat{\boldsymbol{X}}_2) \leq \tfrac{n}{2}\log(2\pi e P_y^{(n)}) - \tfrac{n}{2}\log 2\pi e\big[P_x \cdot 2^{-2(R_1+R_2)} \cdot 2^{\frac{1}{n}I(\mathrm{X}_1;\mathrm{X}_2)} + \sigma_z^2\big] \tag{25}$$

Combining (21), (22) and (25), we thus have

$$\begin{aligned} \tfrac{1}{n}I(\hat{\boldsymbol{X}}_1;\hat{\boldsymbol{X}}_2) &\geq \tfrac{1}{2}\log\left(\frac{P_y^{(n)}}{d_1+\sigma_z^2}\right) + \tfrac{1}{2}\log\left(\frac{P_y^{(n)}}{d_2+\sigma_z^2}\right) - \tfrac{1}{2}\log(2\pi e P_y^{(n)}) + \\ &\quad + \tfrac{1}{2}\log 2\pi e\left[P_x \cdot 2^{-2(R_1+R_2)} \cdot 2^{\frac{2}{n}I(\mathrm{X}_1;\mathrm{X}_2)} + \sigma_z^2\right] \\ &= \tfrac{1}{2}\log\left[\left(\frac{P_y^{(n)}}{(d_1+\sigma_z^2)(d_2+\sigma_z^2)}\right)\left(P_x \cdot 2^{-2(R_1+R_2)} \cdot 2^{\frac{2}{n}I(\mathrm{X}_1;\mathrm{X}_2)} + \sigma_z^2\right)\right] \end{aligned} \tag{26}$$

Define $t = 2^{\frac{2}{n}I(\mathrm{X}_1;\mathrm{X}_2)}$. We have from (26)

$$t \geq \frac{(P_x+\sigma_z^2)(P_x \cdot 2^{-2(R_1+R_2)}t + \sigma_z^2)}{(d_1+\sigma_z^2)(d_2+\sigma_z^2)} \tag{27}$$

where we have also used the entropy-power inequality for the independent sum $Y = X + Z$

$$P_y^{(n)} \geq P_x^{(n)} + P_z \geq P_x + \sigma_z^2$$

where as before, superscript $^{(n)}$ denotes entropy-power of n-vector, while no superscript means entropy-rate power (of process). Define $\widetilde{\epsilon} = \sigma_z^2/P_x$, $\widetilde{\pi} = (1-d_1/P_x)(1-d_2/P_x)$ and $\widetilde{\Delta} = d_1 d_2/P_x^2 - 2^{-2(R_1+R_2)}$. Isolating t in (27), we get

$$t \geq \frac{\widetilde{\epsilon}(1+\widetilde{\epsilon})}{\widetilde{\epsilon}^2 + \widetilde{\epsilon}(1+\widetilde{\Delta}-\widetilde{\pi}) + \widetilde{\Delta}} \tag{28}$$

which holds for any choice of $\widetilde{\epsilon} \geq 0$. To get the tightest bound, we take the $\widetilde{\epsilon}$ which maximizes the bound,

$$\widetilde{\epsilon} = \begin{cases} \frac{\sqrt{\tilde{\triangle}}}{\sqrt{\tilde{\pi}}-\sqrt{\tilde{\triangle}}}, & \text{if } \tilde{\pi} \geq \tilde{\Delta} \\ \infty, & \text{otherwise,} \end{cases}$$

and obtain

$$t \stackrel{\Delta}{=} 2^{\frac{2}{n}I(\mathrm{X}_1;\mathrm{X}_2)} \geq \begin{cases} \frac{1}{1-(\sqrt{\tilde{\pi}}-\sqrt{\tilde{\triangle}})^2}, & \text{if } \tilde{\pi} \geq \tilde{\Delta} \\ 1, & \text{otherwise.} \end{cases}$$

Substituting in (20) we get (12c), and the theorem is proved. Ξ

Note that slightly tighter, though less explicit, outer bound on (d_0, d_1, d_2) follows from the derivation above: $R_1 \geq R_X(d_1),\ R_2 \geq R_X(d_2),\ R_1+R_2 \geq R_X(d_0)-\frac{1}{2}\log(1-(|\sqrt{\tilde{\pi}}-\sqrt{\tilde{\triangle}}|^+)^2$.

IV. An Inner Bound via a Gaussian Test Channel

In this section we show that the El-Gamal-Cover construction for a memoryless Gaussian source (see [Oza80, GC82]) can provide an inner bound for the multiple description rate-region of a general (not necessarily Gaussian or memoryless) source.

We start by stating an achievability result for multiple descriptions under a mean square error constraint which follows from [GC82]. A triplet of mean square errors (d_0, d_1, d_2) is achievable from two descriptions of a memoryless source X with rates R_1 and R_2 if (but not necessarily only if!) there exist random variables U and V jointly distributed with X, and

three functions $\widehat{X}_1 = \widehat{X}_1(U), \widehat{X}_2 = \widehat{X}_2(V)$ and $\widehat{X}_0 = \widehat{X}_0(U,V)$, such that

$$R_1 \geq I(X;U) \tag{29a}$$

$$R_2 \geq I(X;V) \tag{29b}$$

$$R_1 + R_2 \geq I(X;UV) + I(U;V) \tag{29c}$$

$$d_i \geq E(X - \widehat{X}_i)^2 \qquad i = 0,1,2\ . \tag{29d}$$

It seems reasonable that for a source *with* memory, a process form of this characterization (obtained as a limit of vector extensions of this characterization), should provide an achievable rate-region, assuming the source is stationary and ergodic. Specifically, in the process form of (29) U and V are stationary processes, and the mutual information (I) is replaced by mutual-information rate ($\bar{I}$), e.g.

$$\bar{I}(X;U) = \lim_{n\to\infty} \frac{1}{n} I(X_1 \ldots X_n; U_1 \ldots U_n).$$

However, we do not treat this possible extension of the El-Gamal-Cover result here, but rather, state our inner bound below for those sources for which the extension holds. In particular, our inner bound holds for memoryless non-Gaussian sources.

The El-Gamal-Cover characterization is optimal for a memoryless Gaussian source; the entire rate region $R^*(\sigma_x^2, d_0, d_1, d_2)$ characterized in (7) is realized in this case by choosing

$$U = X + N_1 \qquad\qquad V = X + N_2 \tag{30a}$$

$$\widehat{X}_1 = \alpha_1 U \qquad\qquad \widehat{X}_2 = \alpha_2 V \qquad \widehat{X}_0 = \beta_1 U + \beta_2 V \tag{30b}$$

where (N_1, N_2) are correlated Gaussians, independent of X, with covariance matrix

$$\mathrm{COV}(N_1, N_2) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{30c}$$

whose parameters ρ, σ_1 and σ_2 are chosen so that $\mathrm{VAR}(X|U) \le d_1$, $\mathrm{VAR}(X|V) \le d_2$ and $\mathrm{VAR}(X|UV) \le d_0$, where $\mathrm{VAR}(A|B) \stackrel{\triangle}{=} E[A - E(A|B)]^2$ denotes conditional variance. In fact, except from the case of "very high marginal distortions" ($d_1 + d_2 > \sigma^2 + d_0$), the optimum test channel satisfies these three inequalities with equality. The scaling factors $\alpha_1, \alpha_2, \beta_1$ and β_2 are chosen to yield the conditional means $\widehat{X}_1 = E(X|U), \widehat{X}_2 = E(X|V)$ and $\widehat{X}_0 = E(X|UV)$, which are linear functions since (X, U, V) are jointly Gaussian. The above implies that $\sigma_i^2 = \sigma_x^2 d_i/(\sigma_x^2 - d_i)$, $i = 1, 2$, and that ρ is given by the expression in (7e). See [Oza80] for explicit characterization of $\alpha_1, \alpha_2, \beta_1, \beta_2$ as a function of $\sigma_x^2, d_0, d_1, d_2$.

For an *arbitrary* real source X with variance σ_x^2, since the relations in (30) are linear, this test channel yields the *same* distortion triplet (d_0, d_1, d_2) in (29d) as in the white Gaussian source case. Substituting (30) in (29) we obtain a rate region, denoted $\mathcal{R}_X^G(d_0, d_1, d_2)$, which contains all rate pairs (R_1, R_2) satisfying

$$
\begin{aligned}
R_1 &\ge \bar{I}(X; X + N_1) && \text{(31a)}\\
R_2 &\ge \bar{I}(X; X + N_2) && \text{(31b)}\\
R_1 + R_2 &\ge \bar{I}(X; X + N_1, X + N_2) + \bar{I}(X + N_1; X + N_2) && \text{(31c)}\\
&= \bar{I}(X; X + N_1) + \bar{I}(X; X + N_2) + I(N_1; N_2) && \text{(31d)}\\
&= \bar{I}(X; X + N_1) + \bar{I}(X; X + N_2) + \tfrac{1}{2}\log\left(\frac{1}{1-\rho^2}\right) && \text{(31e)}
\end{aligned}
$$

where ρ is given in (7e), (31d) follows using identity (15), and in (31e) we used the formula for the mutual information between correlated Gaussians. Note that $\mathcal{R}_X^G$ has the familiar "broken corner" shape of $\mathcal{R}^*$, with $\frac{1}{2}\log(\sigma^2/d_i)$ replaced by $\bar{I}(X, X + N_i)$.

For a white Gaussian source $X^* \sim \mathcal{N}(0, \sigma_x^2)$ we have

$$
I(X^*; X^* + N_i) = \tfrac{1}{2}\log\left(1 + \frac{\sigma_x^2}{\sigma_i^2}\right) = \tfrac{1}{2}\log\left(\frac{\sigma_x^2}{d_i}\right),
$$

so $\mathcal{R}_X^G(d_0, d_1, d_2)$ coincides with the Gaussian rate region $\mathcal{R}^*(\sigma_x^2, d_0, d_1, d_2)$ defined in (7). Furthermore, in this case the total excess marginal rate (TEMR) is given directly by

$$\delta = I(N_1; N_2) = \tfrac{1}{2} \log\left(\frac{1}{1-\rho^2}\right) , \tag{32}$$

in accordance with our notation in (7d).

For a general real source (not necessarily memoryless or Gaussian) X, it is easy to derive an inner bound for $\mathcal{R}_X^G(d_0, d_1, d_2)$, which therefore is also an inner bound for the multiple description rate region $\mathcal{R}_X(d_0, d_1, d_2)$. This bound is stated in the following theorem.

Theorem 2 *Let X be a real-valued stationary source with variance σ_x^2, and assume that the process form of the El-Gamal-Cover rate region of X, given in (29), is achievable. Then*

$$\mathcal{R}_X(d_0, d_1, d_2) \supseteq R_X^G(d_0, d_1, d_2) \supseteq \mathcal{R}^*(\sigma_x^2, d_0, d_1, d_2) \tag{33}$$

Proof: Since $\widehat{X}_0, \widehat{X}_1, \widehat{X}_2$ of (30b) satisfy the distortion constraints for any X with variance σ_x^2, the process form of the El-Gamal-Cover Theorem [GC82], assuming it holds for X, implies that the rate region $\mathcal{R}_X^G(d_0, d_1, d_2)$ inner bounds $\mathcal{R}_X(d_0, d_1, d_2)$. Now, since N_1 and N_2 are white Gaussian, the quantities $\bar{I}(X; X + N_1)$ and $\bar{I}(X; X + N_2)$ in (31) are upper bounded by their respective values for a white Gaussian source, (see e.g. [CT91, pp. 263]), i.e.

$$\bar{I}(X; X + N_i) \leq I(X^*; X^* + N_i) = \tfrac{1}{2} \log\left(\frac{\sigma_x^2}{d_i}\right) . \tag{34}$$

In view of (29), (31) and (34), and by the tightness of (31) for X^*, we thus obtain (33), and the theorem is proved. Ξ

V. Multiple Descriptions with High Resolution

In this section we show that the outer bound of Theorem 1 is asymptotically tight at high resolution, i.e., when d_0, d_1 and d_2 are small compared to the variance of the source. The only requirement is that the source satisfies a simple "smoothness" condition. Furthermore, we show that this outer bound is asymptotically attainable by the Gaussian test channel described in the previous section.

If X has a density, and the distortions go to zero, then the rates must go to infinity. We say that two rate regions $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide asymptotically, and denote that $\mathcal{R}_1 \stackrel{\sim}{=} \mathcal{R}_2$, if for any $\epsilon > 0$ we have asymptotically

$$\mathcal{R}_1^{(-\epsilon)} \subset \mathcal{R}_2 \subset \mathcal{R}_1^{(+\epsilon)}$$

where

$$\mathcal{R}^{(+\epsilon)} \stackrel{\Delta}{=} \{(R_1, R_2) : \ (R_1 + \epsilon, R_2 + \epsilon) \in \mathcal{R}\}$$

$$\mathcal{R}^{(-\epsilon)} \stackrel{\Delta}{=} \{(R_1, R_2) : \ (R_1 - \epsilon, R_2 - \epsilon) \in \mathcal{R}\} \ .$$

Theorem 3 *Let the source X be like in Theorem 2, with finite entropy-rate power $P_x > 0$, and finite variance. Suppose that d_0, d_1 and d_2 go to zero such the ratios d_1/d_0 and d_2/d_0 are held fixed. Then the optimum rate region $\mathcal{R}_X(d_0, d_1, d_2)$, the Gaussian rate region $R_X^G(d_0, d_1, d_2)$ defined in (31), and the outer bound $\mathcal{R}^*(P_x, d_0, d_1, d_2)$ coincide asymptotically, i.e.,*

$$\mathcal{R}_X(d_0, d_1, d_2) \stackrel{\sim}{=} \mathcal{R}_X^G(d_0, d_1, d_2) \stackrel{\sim}{=} \mathcal{R}^*(P_x, d_0, d_1, d_2) \ . \tag{35}$$

In other words, at high resolution conditions $R_X^G(d_0, d_1, d_2)$ and $\mathcal{R}_X(d_0, d_1, d_2)$ are composed of all rate pairs satisfying

$$R_1 \geq \tfrac{1}{2}\log\left(\frac{P_x}{d_1}\right) + \epsilon \tag{36a}$$

$$R_2 \geq \tfrac{1}{2}\log\left(\frac{P_x}{d_2}\right) + \epsilon \tag{36b}$$

$$R_1 + R_2 \geq \tfrac{1}{2}\log\left(\frac{P_x^2}{d_1 d_2}\right) + \delta_{HR}(d_0, d_1, d_2) + \epsilon \tag{36c}$$

where $\delta_{HR}(d_0, d_1, d_2)$ is the TEMR at high resolution defined in (8), and $\epsilon > 0$ is arbitrarily small. An equivalent statement in terms of the distortion region is

$$\mathcal{D}^*(P_x, R_1, R_2) \supseteq \mathcal{D}_X(R_1, R_2) \supseteq \mathcal{D}^*(P_x, R_1 - \epsilon, R_2 - \epsilon)$$

for sufficiently large rates. Note that the "smoothness" condition $P_x > 0$ is equivalent to having finite differential entropy-rate.

Before proving Theorem 3 we show the following lemma.

Lemma 1 *Let $(N_1^\lambda, N_2^\lambda)$ denote the noise pair in (30) associated with distortions $\lambda d_0, \lambda d_1$ and λd_2 and source variance σ_x^2. If $P_x > 0$ and $\sigma_x^2 < \infty$ then for $i = 1, 2$*

$$\lim_{\lambda \to 0}\left[\bar{I}(X; X + N_i^\lambda) - \tfrac{1}{2}\log\left(\frac{P_x}{\lambda d_i}\right)\right] = 0 \ . \tag{37}$$

Proof: Decompose the mutual information into a difference between differential entropies. Since $\bar{h}(X) = \frac{1}{2}\log(2\pi e P_x)$ and $\bar{h}(N_i^\lambda) = \frac{1}{2}\log 2\pi e \mathrm{VAR}(N_i^\lambda)$, we have for $i = 1, 2$

$$\bar{I}(X; X + N_i^\lambda) - \frac{1}{2}\log\left(\frac{P_x}{\lambda d_i}\right) = \left[\bar{h}(X + N_i^\lambda) - \bar{h}(X)\right] - \frac{1}{2}\log\left(\frac{VAR(N_i^\lambda)}{\lambda d_i}\right) . \tag{38}$$

From (30) we have $\frac{1}{\lambda}\mathrm{VAR}(N_i^\lambda) = \frac{1}{\lambda}\frac{\sigma_x^2 \lambda d_i}{\sigma_x^2 - \lambda d_i} \longrightarrow d_i$ as $\lambda \to 0$. Thus the second term in (38) is asymptotically zero. Furthermore, since $\sigma_x^2 < \infty$ and $P_x > 0$, and since N_i^λ is Gaussian whose variance $\to 0$, it follows from [LZ94] that $\bar{h}(X + N_i^\lambda) \to \bar{h}(X)$, and (37) follows. Ξ

Proof of Theorem 3 Substituting the limits (37) and (8) in (31), we see that $\mathcal{R}_X^G(d_0, d_1, d_2)$ satisfies (36) for small enough distortions, i.e., $\mathcal{R}_X^G \supset \mathcal{R}^*(P_x, d_0, d_1, d_2)^{(-\epsilon)}$. On the other hand, for any distortions $\mathcal{R}_X^G \subset \mathcal{R}_X \subset \mathcal{R}^*(P_x, d_0, d_1, d_2)$ by Theorem 2 and (33). Thus the regions coincide asymptotically, and the proof follows. Ξ

Acknowledgments

We thank Meir Feder, Tamas Linder, Vinay Vaishampayan and an anonymous reviewer for helpful comments.

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