High-Resolution Source Coding for Non-difference Distortion Measures: Multidimensional Companding *

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Abstract

Entropy coded vector quantization is studied using high resolution multidimensional companding over a class of non-difference distortion measures. For distortion measures which are "locally quadratic" a rigorous derivation of the asymptotic distortion and entropy coded rate of multidimensional companders is given along with conditions for the optimal choice of the compressor function. This optimum compressor, when it exists, depends on the distortion measure but not on the source distribution. The rate-distortion performance of the companding scheme is studied using a recently obtained asymptotic expression for the rate-distortion function which parallels the Shannon lower bound for difference distortion measures. It is proved that the high resolution performance of the scheme is arbitrarily close to the rate-distortion limit for large quantizer dimensions if the compressor function and the lattice quantizer used in the companding scheme are optimal, extending an analogous statement for entropy coded lattice quantization and MSE distortion. The companding approach is applied to obtain a high resolution quantizing scheme for noisy sources.

Index Terms: non-difference distortion measures, asymptotic quantization theory, multidimensional companding, lattice quantizers, entropy coding, rate-distortion function.

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1 Introduction

The high resolution (asymptotic, low distortion) behavior of vector quantizers is relatively well understood for so-called difference distortion measures where the distortion is measured by a function of the difference between the source and the reproduction vectors. In particular, for the mean squared error, and more generally for "nice" functions of a norm-based distance measure, the asymptotic distortion of optimal quantizers, as well as the asymptotic distortion of sequences of quantizers with a given "point density", have been identified as a function of the codebook size, or as a function of the entropy of the output [1, 2, 3, 4, 5, 6]. These results give insight to the structure of asymptotically optimal quantizers. On the practical side, the expressions for quantizer performance provide useful guidance for quantizer design at even small to moderate rates.

Source coding is less understood when the distortion is not measured by a difference distortion measure. Non-difference distortion measures occur naturally in source coding problems. Prominent examples include the log spectral distortion and the Itakura-Saito distortion which are used in linear predictive speech coding [7], certain perceptual distortion measures in image coding [8], and most distortion measures that arise in noisy (or remote) source coding if the original distortion measure is other than the squared error. The design of vector quantizers for certain classes of non-difference distortion measures is possible using the generalized Lloyd-Max algorithm [9], and the same approach can be extended to modified distortion measures in noisy source coding [10].

Due to the difficulty in analyzing such systems, there exist only a few known results for high resolution quantization with non-difference distortion measures. By assuming the existence of a limiting quantizer point density, a lower bound was calculated in [11] for the high resolution performance of fixed rate optimal vector quantizers for locally quadratic distortion measures. The log spectral distortion and the Itakura-Saito distortion are examples of such measures. A more formal treatment of the same lower bound is given in [8], and a new lower bound on the variable rate (i.e. entropy coded) performance is developed using optimal point densities. It is also pointed out in [8] that some important "perceptual distortion measures" in image coding are locally quadratic. In [12] an asymptotically tight expression for the rate distortion function is derived for locally quadratic distortion measures. As will be shown in this paper, the expression given in [12] plays the same important role in high resolution quantization for these distortion measures as does the Shannon lower bound in quantizing for squared error loss.

To develop the basics of a high resolution quantization theory for locally quadratic distortion measures, we investigate variable rate (entropy coded) companding vector quantization.

Multidimensional companding is a type of structured vector quantization of low complexity where a k-dimensional source vector X is "compressed" by an invertible mapping F (called the compressor function). Then $F(X)$ is quantized by a uniform (or more generally, a lattice) quantizer, and the inverse mapping F^{-1} is applied to obtain the reproduction $\hat{X}$. Thus the scheme is

$$X \to F(\cdot) \to Q_U(\cdot) \to F^{-1}(\cdot) \to \hat{X}$$

where Q_U is a uniform or lattice quantizer. In this paper, the output of the quantizer is entropy coded and the rate of the system is given by the Shannon entropy of Q_U.

In a classical paper [2], entropy coded scalar quantization with *mean squared* distortion was considered. It was found that for smooth sources, uniform scalar quantizers have asymptotically the minimum possible entropy of all scalar quantizers with a given distortion D, and this minimum entropy is given for small D by

$$H(Q_U(X)) \approx h(X) - \frac{1}{2}\log(12D), \tag{1}$$

where H and h denote discrete and differential entropies, respectively, the logarithm is base 2, and $\approx$ means that the difference between the corresponding quantities goes to zero as $D \to 0$. Thus the optimal compressor characteristic for entropy coded scalar quantization is *uniform*. More generally, the entropy of a lattice quantizer Q that encodes a smooth k-dimensional vector source X with squared distortion D is given for small D by [3]

$$H(Q(X)) \approx h(X) - \frac{k}{2}\log(D/(kL(P_0))), \tag{2}$$

where $L(P_0)$ denotes the normalized second moment of the basic cell of the lattice (see also [13] and [14]). The above implies (by means of the Shannon lower bound [15]) that the asymptotic rate redundancy of an entropy coded lattice quantizer above the rate-distortion function is $\frac{1}{2}\log(2\pi e L(P_0))$ bits per dimension [2, 4]. In this paper we will show that analogous results hold for locally quadratic non-difference distortion measures. For example, the optimal compressor function is again independent of the source distribution. However the compressor now depends on the distortion measure through the so-called sensitivity matrix.

We have two main reasons for considering a companding realization of vector quantizers. First, in [12] it has been observed that for a large class of non-difference distortion measures the asymptotically optimal forward test channel which realizes Shannon's rate-distortion function has a certain structure very similar to that of multidimensional companding quantizers. It has also been conjectured that such a companding scheme, together with entropy coding, performs arbitrarily close to the rate-distortion limit. Second, our aim is to develop a rigorous theory. In high resolution quantization theory it is rather common to obtain results via informal reasoning (see, e.g., [16, 2, 3, 4, 11, 8]) and most of the rigorously derived results deal with fixed rate quantization [5, 17, 18, 6, 19] (due primarily to the difficulty of handling the problem of "quantizer point density" in the variable rate case, e.g., [20]). On the other hand, for the purpose of directly relating the quantizer's performance to the rate-distortion function it is more suitable to consider the variable rate performance. For this reason we choose to introduce structure in the coding scheme by using a companding realization which allows us to deal with the point density problem for variable rate quantizers.

As will be seen in the paper, in some cases it is not possible to realize a (heuristically derived) desired optimal point density using a compander. However, the companding approach can serve as a basis for the development of a rigorous theory of quantizers with a given point density.

In this paper we consider entropy coded multidimensional companding quantizers with non-difference distortion measures satisfying rather general regularity conditions. The main requirement is a smoothness condition which implies that the distortion $d(x, y)$ between $x, y \in \mathrm{R}^k$ can be approximated as $d(x, y) \approx (x-y)^T M(x)(x-y)$ for y close to x, where $M(x)$ is an input-dependent positive definite matrix. In Section 3, Theorem 1, we give a rigorous derivation of the asymptotic entropy coded rate as a function of the distortion for sources with densities. A general sufficient condition for the optimal choice of the compressor function is derived in Theorem 2 and examples are shown for the existence of optimal compressors, which are determined by the distortion measure and do not depend on the source distribution. In Section 4 the rate-distortion performance is considered. Using a result from [12] we prove in Theorem 4 that if the compressor function satisfies the sufficient condition for optimality, and if the lattice quantizer used in the companding scheme is optimal, then the high resolution performance of the scheme is arbitrarily close to the rate-distortion limit for large quantizer dimensions. When specialized to mean squared error, this result gives back the well known fact that for large rate and large quantizer dimension, lattice quantizers combined with entropy coding are asymptotically optimal. The above results can be applied to obtain a simple encoding scheme for quantizing sources corrupted by noise. In this problem, the original distortion measure in the "source space" is transformed into a modified distortion measure in the "measurement space". We show in Theorem 5 that if the modified distortion measure satisfies certain regularity conditions, then an estimation-companding quantization scheme gives asymptotically optimal performance. This is the asymptotic analogue of the well known separation principle in the "Wolf-Ziv type" encoding [21] of noisy sources with MSE original distortion measure.

2 Preliminaries

A k-dimensional *vector quantizer* Q is a mapping defined by

$$Q(x) = y_i \quad \text{if} \quad x \in B_i,$$

where $B_1, \ldots, B_n$ form a measurable partition of R^k, and the collection of *codepoints* $y_i \in \mathrm{R}^k$, $1 \le i \le n$ is called the *codebook*. We do not eliminate the possibility that $n = \infty$, i.e., the codebook of Q can contain countably infinite number of codepoints. The distortion between x and $Q(x)$ is measured by $d(x, Q(x))$, where $d : \mathrm{R}^k \times \mathrm{R}^k \to [0, \infty)$ is a Borel measurable function. The expected distortion in quantizing a k-dimensional random vector X is

$$D(Q) = \mathbf{E}[d(X, Q(X))],$$

and we assume the expectation is finite. The *rate* of Q will be measured by the Shannon entropy of $Q(X)$,

$$H(Q) = -\sum_i \mathbf{P}\{Q(X) = y_i\} \log \mathbf{P}\{Q(X) = y_i\},$$

where the logarithm is base two. The per dimension rate of the system can be reach within $\frac{1}{k}$ of the normalized entropy $\frac{1}{k}H(Q)$ by use of entropy coding techniques.

The basic building block in a multidimensional companding quantizer is a lattice quantizer, the k-dimensional generalization of a uniform scalar quantizer. Let Λ be a k-dimensional nonsingular lattice, i.e., Λ is the set of all points of the form $\sum_{j=1}^k n_j v_j$, where $v_1, \ldots, v_k$ are linearly independent vectors of R^k (the basis vectors of the lattice) and $(n_1, ..., n_k)$ ranges over all k-tuples of integers. For $\alpha > 0$ let $\alpha\Lambda$ denote the scaled lattice $\alpha\Lambda = \{\alpha z : z \in \Lambda\}$. The lattice quantizer $Q_{\alpha\Lambda}$ is then defined so that its codepoints are the points of $\alpha\Lambda$ and its quantization regions are the corresponding Voronoi regions of $\alpha\Lambda$, i.e.,

$$Q_{\alpha\Lambda}(x) = z \in \alpha\Lambda \quad \text{if} \quad \|x - z\| \le \|x - z'\| \quad \text{for all } z' \in \alpha\Lambda,$$

where $\|\cdot\|$ denotes the Euclidean norm, and ties are broken arbitrarily. The quantization regions of $\alpha\Lambda$ are translated and scaled copies of P_0, the *basic Voronoi cell* of Λ, which is defined by

$$P_0 = \{x \in \mathrm{R}^k : \|x\| \le \|x - z\| \text{ for all } z \in \Lambda\}.$$

An important performance figure of Λ is the (dimensionless) normalized second moment of its basic cell, namely

$$L(P_0) = \frac{\int_{P_0} \|x\|^2 \, dx}{kV(P_0)^{2/k+1}},$$

where $V(B)$ denotes the volume (k-dimensional Lebesgue measure) of any measurable $B \subset \mathrm{R}^k$. For a given k, we call a lattice *optimal* if its normalized second moment is minimum over all k-dimensional lattices [22]. It was proved in [23] that the basic cell of an optimal lattice is *white* in the sense that if $Z = (Z_1, \ldots, Z_k)$ is a random vector uniformly distributed over P_0, then the covariance matrix of Z is

$$\mathbf{E}[ZZ^T] = \sigma^2 I,$$

where I denotes the $k \times k$ identity matrix. In other words, the Z_i are uncorrelated and have equal second moments. We will assume that the lattice Λ used in the companding scheme has a white basic cell P_0.

The concept of a companding realization of a nonuniform quantizer originates from Bennett [16]. The idea is to apply a nonlinear transformation (called the compressor) to the input, followed by a uniform (more generally a lattice) quantizer, and then the inverse of the transformation to obtain the reproduction. Let $F : \mathrm{R}^k \to \mathrm{R}^k$ be a one-to-one continuously differentiable mapping whose derivative matrix $F'(x)$ is nonsingular for all x. Then F has an inverse $F^{-1} = G$ which is continuously differentiable on its domain and whose derivative G' is nonsingular. F and G are called the *compressor* and *expander* functions, respectively.

The *companding vector quantizer* $Q_{\alpha,F}$ realized by the compressor function F and the scaled lattice quantizer $Q_{\alpha\Lambda}$ is defined by

$$Q_{\alpha,F}(x) = G(Q_{\alpha\Lambda}(F(x))), \quad x \in \mathrm{R}^k.$$

Our goal is to analyze the entropy coded rate of $Q_{\alpha,F}$ as a function of its distortion for absolutely continuous source distributions and non-difference distortion measures. In general, an analytical evaluation of the rate is not possible for any given $\alpha > 0$, so we alternatively take a high resolution approach and determine the asymptotic behavior of the rate as the distortion (or equivalently α) tends to zero.

3 Multidimensional Companding

3.1 Asymptotic Performance

Let $x = (x_1, \ldots, x_k) \in \mathrm{R}^k$, $y = (y_1, \ldots, y_k) \in \mathrm{R}^k$ and assume that the distortion measure $d(x, y)$ satisfies the following three conditions.

(a) For all fixed $x \in \mathrm{R}^k$, $d(x,y)$ is three times continuously differentiable in the variable y, and the third order partial derivatives

$$\frac{\partial^3 d(x,y)}{\partial y_i \partial y_j \partial y_n}, \quad i,j,n \in \{1,\ldots,k\} \tag{3}$$

are uniformly bounded.

(b) For all $x \in \mathrm{R}^k$, $d(x,y) \geq 0$ with equality if and only if $y = x$.

(c) Let $M(x) = \{m_{ij}(x)\}$ be the $k \times k$ matrix whose ijth element is given by

$$m_{ij}(x) = \frac{1}{2} \frac{\partial^2 d(x,y)}{\partial y_i \partial y_j}\bigg|_{y=x}. \tag{4}$$

Then $M(x)$ is positive definite for all x and its elements $m_{ij}(x)$ are continuous functions.

$M(x)$ is symmetric by (a). Condition (b) implies that the gradient of $d(x,y)$ with respect to y is zero at $y = x$. Thus for any fixed x, a second order Taylor expansion of $d(x,y)$ in y gives

$$d(x,y) = (x-y)^T M(x)(x-y) + O(\|x-y\|^3). \tag{5}$$

Since $d(x,y) > 0$ if $y \neq x$, (a) and (b) already imply through (5) that $M(x)$ is *nonnegative definite*. $M(x)$ was named the *sensitivity matrix* of d in [11] where the fact that certain useful distortion measures can be represented in the form of (5) was first pointed out.

Remark. (i) The conditions given above are not the weakest under which we can prove our results. For example, it suffices to assume that the elements of $M(x)$ are continuous on an open set of probability 1. Also, if $d(x,y)$ is assumed to be three times continuously differentiable as a function on R^{2k}, then the third order partial derivatives in (3) need not be uniformly bounded. (ii) Consider an *input weighted quadratic* distortion measure given by

$$d(x,y) = \|W(x)(x-y)\|^2,$$

where $W(x)$ is a nonsingular $k \times k$ matrix depending on the input x [24]. Since $d(x,y) = (x-y)^T W^T(x) W(x)(x-y)$, and since $M(x) = W(x)^T W(x)$ is positive definite, it is easy to see that $d(x,y)$ satisfies condition (a)–(c) if the elements of $W(x)$ are continuous functions of x. (iii) Very similar conditions are used in [8] to compute lower bounds on the asymptotic distortion of a sequence of fixed rate quantizers with a given point density. Some important measures of image quality [25, 26] satisfy these regularity conditions, for example.

To study the rate of $Q_{\alpha,F}$ as a function of its distortion, one needs to eliminate the scaling factor α. One reasonable way to do this is to choose for each $D > 0$ an $\alpha(D) > 0$ such that

$$D(Q_{\alpha(D),F}) = D. \tag{6}$$

If X has a density, it is not hard to see that $D(Q_{\alpha,F})$ is a continuous function of $\alpha > 0$ which converges to zero as $\alpha \to 0$. Thus in this case for all $D > 0$ small enough there exists an $\alpha(D)$ satisfying (6). For such values of D we define

$$Q_{D,F} = Q_{\alpha(D),F}.$$

The next theorem determines the asymptotic behavior of the rate of $Q_{D,F}$ as $D \to 0$ for any source whose density has a bounded support.

Theorem 1 *Assume that the source X has a density which is zero outside a bounded subset of R^k and suppose the distortion function $d(x,y)$ satisfies conditions (a)–(c). If X has a finite differential entropy $h(X)$, then the rate $H(Q_{D,F})$ and the distortion D of the multidimensional companding quantizer $Q_{D,F}$ satisfy*

$$\lim_{D\to 0}\left(H(Q_{D,F}) + \frac{k}{2}\log D\right)$$
$$= h(X) + \mathbf{E}[\log|\det F'(X)|] + \frac{k}{2}\log\big(L(P_0)\mathbf{E}[\mathrm{tr}\{\Gamma(X)\}]\big),$$

where F is the compressor function, $L(P_0)$ is the normalized second moment of the basic cell P_0 of the white lattice quantizer, $\mathrm{tr}\{\Gamma(x)\}$ denotes the trace of the matrix

$$\Gamma(x) = F'(x)^{-T}M(x)F'(x)^{-1}, \tag{7}$$

$M(x)$ is the sensitivity matrix of $d(x,y)$, and $F'(x)^{-T}$ is the inverse transpose of the derivative of $F(x)$.

The above theorem is a consequence of the following two results which determine the asymptotics of the distortion and the rate of $Q_{\alpha,F}$ as $\alpha \to 0$. Both results are proved in Section 5.

Proposition 1 *Assume that the source X has a density which is zero outside a bounded subset of R^k and suppose $d(x,y)$ satisfies conditions (a)–(c). Then*

$$\lim_{\alpha\to 0}\alpha^{-2}D(Q_{\alpha,F}) = L(P_0)V(P_0)^{2/k}\mathbf{E}[\mathrm{tr}\{\Gamma(X)\}].$$

Proposition 2 *Assume that the source X has a density and finite differential entropy $h(X)$. If $\mathbf{E}[\log|\det F'(X)|] < \infty$ and there exists an $\alpha > 0$ such that $H(Q_{\alpha,F})$ is finite, then*

$$\lim_{\alpha\to 0} [\, H(Q_{\alpha,F}) + k\log\alpha\,] = h(X) + \mathbf{E}[\log|\det F'(X)|] - \log V(P_0).$$

Theorem 1 is proved by noticing that by Proposition 1,

$$\frac{k}{2}\log D(Q_{\alpha,F}) - k\log\alpha \to \frac{k}{2}\log\big(L(P_0)\mathbf{E}[\mathrm{tr}\{\Gamma(X)\}]\big) + \log V(P_0)$$

as $\alpha \to 0$. Combining this with Proposition 2 and the fact that $\alpha(D) \to 0$ as $D \to 0$ proves the statement. Note that since the source density has a bounded support, $H(Q_{\alpha,F})$ is finite for all $\alpha > 0$ and therefore the conditions of Proposition 2 are satisfied.

Remarks. (i) Note that no smoothness conditions are imposed on the source density in Theorem 1 except the requirement that the differential entropy be finite. The only restrictive condition is the assumption that the source density has bounded support (see Proposition 1). In principle, the distortion formula can be proved for source densities with unbounded support, but in that case extra conditions on the compressor function are needed. These conditions are associated with the tail of the source density, leading to a substantially more complicated proof. (ii) Proposition 1 can be used to obtain the high-rate distortion of the companding quantizer as a function of the number of codepoints (which is finite since the source is bounded). When specialized to mean squared error ($M(x) = I$), we obtain Bucklew's heuristically derived formula [27] for fixed rate multidimensional companding.

3.2 Optimal Compressor Functions

The question of the optimal choice of the compressor F is considered next. Let us define

$$C_1(F) = \mathbf{E}[\mathrm{tr}\{\Gamma(X)\}]$$

and

$$C_2(F) = \mathbf{E}[\log|\det F'(X)|],$$

where $\Gamma(x)$ is defined in (7). Then the statement of Theorem 1 becomes

$$\lim_{D\to 0}\left(H(Q_{D,F}) + \frac{k}{2}\log D\right) = h(X) + \frac{k}{2}\log L(P_0) + C_2(F) + \frac{k}{2}\log C_1(F). \tag{8}$$

It is clear that if F minimizes the right hand side of (8), i.e., if

$$C_2(F) + \frac{k}{2}\log C_1(F) \le C_2(\hat{F}) + \frac{k}{2}\log C_1(\hat{F})$$

for all allowable $\hat{F}$, then $Q_{D,F}$ asymptotically outperforms all other companding quantizers $Q_{D,\hat{F}}$. Thus to find a best compressor one has to minimize the functional

$$C_2(F) + \frac{k}{2}\log C_1(F) \tag{9}$$

over all one-to-one and continuously differentiable F such that $F'(x)$ is nonsingular for all x.

A lower bound on (9) is obtained next. Since $M(x)$ is positive definite and $F'(x)$ is nonsingular, $\Gamma(x) = F'(x)^{-T}M(x)F'(x)^{-1}$ is also positive definite. Thus by the arithmetic-geometric mean inequality we have $\mathrm{tr}\{\Gamma(x)\} \geq k(\det\Gamma(x))^{1/k}$ with equality if and only if the eigenvalues of $\Gamma(x)$ are all equal. Therefore

$$\begin{aligned}
\frac{k}{2}\log\mathbf{E}[\mathrm{tr}\{\Gamma(X)\}] &\geq \frac{k}{2}\log\mathbf{E}\left[k(\det\Gamma(X))^{1/k}\right] && (10)\\
&\geq \frac{k}{2}\mathbf{E}\left[\log(k(\det\Gamma(X))^{1/k})\right] && (11)\\
&= \frac{k}{2}\log k + \frac{1}{2}\mathbf{E}[\log(\det M(X))] - \mathbf{E}[\log|\det F'(X)|],
\end{aligned}$$

where (11) follows from Jensen's inequality. The above is equivalent to

$$\frac{k}{2}\log C_1(F) + C_2(F) \geq \frac{k}{2}\log k + \frac{1}{2}\mathbf{E}[\log(\det M(X))]. \tag{12}$$

Let us examine the conditions for achieving the above lower bound. We have equality in (12) iff both (10) and (11) are equalities. Equality holds in (10) iff the eigenvalues of $\Gamma(x)$ are equal a.e. $[\mu_X]$ (where μ_X denotes the probability measure induced by X on R^k). Since $\Gamma(x)$ is positive definite, this implies that $\Phi(x)^T\Gamma(x)\Phi(x) = \beta(x)I$ a.e. $[\mu_X]$ for some $\beta(x) > 0$ and some orthogonal matrix $\Phi(x)$ (i.e., $\Phi(x)^T\Phi(x) = I$). This in turn implies that $\Gamma(x) = \beta(x)I$ a.e. $[\mu_X]$. The condition of equality in (11) is that the determinant of $\Gamma(x)$ be constant a.e. $[\mu_X]$. Thus equality holds in (12) if and only if $\Gamma(x) = \beta I$ a.e. $[\mu_X]$, where $\beta > 0$ is a constant. This is equivalent to $M(x) = \beta F'(x)^T F'(x)$ a.e. $[\mu_X]$. Since $M(x)$ is positive definite, it has a unique positive definite square root $W(x)$ (i.e., $W(x)$ is symmetric and positive definite and $M(x) = W(x)W(x)$). Therefore another equivalent condition is that $\beta^{-1}W(x)[F'(x)]^{-1}$ be an orthogonal matrix. We have thus proved the following sufficient condition for the optimality of a compressor function in terms of a condition involving the sensitivity matrix of d.

Theorem 2 *Assume the conditions of Theorem 1 hold. Then for any compressor F we have*

$$\lim_{D\to 0}\left(H(Q_{D,F}) + \frac{k}{2}\log D\right) \geq h(X) + \frac{k}{2}\log(kL(P_0)) + \frac{1}{2}\mathbf{E}[\log(\det M(X))] \tag{13}$$

with equality if and only if F satisfies

$$F'(x)^T F'(x) = cM(x) \tag{14}$$

a.e. $[\mu_X]$, *where $c > 0$ is a scalar constant. Thus, if F satisfies (14), then it is an optimal compressor function in the sense that*

$$\lim_{D\to 0}\Big(H(Q_{D,\hat{F}}) - H(Q_{D,F})\Big) \geq 0$$

for all other compressors $\hat{F}$.

Remarks. (i) It is interesting to observe that at high rates and for an optimal compressor satisfying $F'(x)^T F'(x) = M(x)$ the *squared error* at the output of the lattice quantizer is approximately equal to the overall distortion of the companding scheme. This can be proved by making precise the following informal derivation. Assuming that the cells of $Q_{\alpha\Lambda}$ are small enough, we have

$$\begin{aligned} x - G(Q_{\alpha\Lambda}(F(x))) &= G(F(x)) - G(Q_{\alpha\Lambda}(F(x))) \\ &\approx G'(F(x))(F(x) - Q_{\alpha\Lambda}(F(x))) \\ &= F'(x)^{-1}(F(x) - Q_{\alpha\Lambda}(F(x))) \end{aligned}$$

and therefore by (5) we obtain

$$\begin{aligned} d(x, Q_{\alpha,F}(x)) &\approx (x - G(Q_{\alpha\Lambda}(F(x))))^T M(x)(x - G(Q_{\alpha\Lambda}(F(x)))) \\ &= \|F(x) - Q_{\alpha\Lambda}(F(x))\|^2 \end{aligned}$$

since $F'(x)^{-T}M(x)F'(x)^{-1} = I$. It follows that $\mathbf{E}[d(X, Q_{\alpha,F}(X))] \approx \mathbf{E}\|F(X) - Q_{\alpha\Lambda}(F(X))\|^2$, which together with Proposition 2 enables one to guess the optimal rate-distortion characteristics given in (13). By analyzing the proof of Proposition 1 it also becomes clear that the sufficient condition of optimality in the above theorem means the following. The optimal compressor function shapes the lattice quantizer so that for small α the weighted quantization error vector $e = W(X)(X - Q_{\alpha,F}(X))$, where $W(X)$ is the square root of $M(X)$, is approximately white and its conditional power $\mathbf{E}[\|e\|^2 | Q_{\alpha,F}(X) = y_{\alpha,i}]$ does not depend on the codepoint $y_{\alpha,i}$. (ii) Note that the optimality condition of Theorem 2 does not depend on the source density. This observation nicely parallels the fact that for mean squared error the asymptotically optimal entropy coded quantizer is an infinite level uniform quantizer [2]. It is also analogous to a widely cited conjecture made by Gersho [3] that the asymptotically optimal entropy coded quantizer has a so called tessellating structure, i.e., its quantization regions are congruent polytopes which tessellate the whole space. For technical reasons (e.g., lack of the whiteness property), we limit our investigation to lattice quantizers, which are a special but important case of tessellating quantizers.

The condition $F'(x)^T F'(x) = M(x)$ is a system of partial differential equations which might not have a solution for a general $M(x)$. Thus, as in the case of fixed rate multidimensional companding for the squared error [3, 27, 28], in general there may not exist a compressor function $F(x)$ satisfying the above condition. The following example shows that the condition of Theorem 2 can be satisfied by an F consisting of scalar compressors if d is a single letter distortion measure.

Example 1 (*single letter distortion measures*) Assume that $d(x, y)$ can be written as

$$d(x, y) = \sum_{i=1}^{k} d_i(x_i, y_i), \tag{15}$$

where $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$, and the scalar distortion measures $d_i : \mathrm{R} \times \mathrm{R} \to [0, \infty)$, $1 \leq i \leq k$ satisfy conditions (a)–(c). Namely, for each d_i we require that $\frac{\partial^3}{\partial u^3} d_i(t, u)$ be uniformly bounded, $d_i(t, u) \geq 0$ with equality iff $u = t$, and

$$m_i(t) = \frac{1}{2} \frac{\partial^2 d_i(t, u)}{\partial u^2}\Big|_{u=t}$$

be positive and continuous for all t. Since $\frac{\partial^2 d(x,y)}{\partial y_i \partial y_j} = 0$ if $i \neq j$, $M(x)$ becomes the diagonal matrix

$$M(x) = \mathrm{diag}\{m_1(x_1), \ldots, m_k(x_k)\}.$$

Define $F(x) = (F_1(x), \ldots, F_k(x))$ by setting

$$F_i(x) = \int_0^{x_i} m_i(t)^{1/2}\, dt,$$

where we used the convention that $\int_a^b = -\int_b^a$ if $a > b$. Then $F(x)$ is one-to-one and continuously differentiable since each $m_i(t)$ is positive and continuous by assumption. Obviously, $F'(x)$ is diagonal and $[F'(x)]^2 = M(x)$ so the optimality condition of Theorem 2 is satisfied. Note that the ith component of F is a scalar compressor which is optimal for d_i. There is an interesting analogy with fixed rate multidimensional companding for squared error. For this problem it has recently been reported [29] that if the source is stationary and memoryless, then the optimal compressor function compresses each vector component independently, using the scalar compressor optimal for the marginal distribution of the source.

The above discussion of single letter distortion measures can be extended to the case when $d(x, y)$ is given as the sum of n-dimensional distortion measures where n divides k.

Consider now the more general case when $d(x, y)$ is not a single letter distortion measure, but there exists an orthogonal transformation such that $d(x, y)$ becomes a single letter distortion measure in the transformed space. That is,

$$d(x, y) = d^*(Vx, Vy)$$

where V is a fixed orthogonal matrix and d^* is in the form of (15). Then it is easy to see that the optimal compressor F is given by $F(x) = F^*(Vx)$, where F^* is optimal for d^*.

Example 2 (*scalar compressors*) Let us consider now the problem of optimizing the asymptotic performance for a general $d(x, y)$ with sensitivity matrix $M(x) = \{m_{ij}(x)\}$ under the constraint that F compresses the vector coordinates independently. An obvious advantage of such a companding scheme is a dramatic decrease in complexity. Another interesting application of independent companders can be given in lossy multiterminal source coding [30].

The goal is to minimize $C_2(F) + \frac{k}{2}\log C_1(F)$ subject to the constraint that the compressor be of the form

$$F(x) = (F_1(x_1), \ldots, F_k(x_k)),$$

where the $F_i : \mathrm{R} \to \mathrm{R}$ are scalar compressors which are invertible and possess nonzero and continuous derivatives F_i', $1 \le i \le k$. Then

$$F'(x) = \mathrm{diag}\{F_1'(x_1), \ldots, F_k'(x_k)\}$$

so that we obtain

$$\begin{aligned} C_2(F) + \frac{k}{2}\log C_1(F) &= \mathbf{E}[\log|\det F'(X)|] + \frac{k}{2}\log \mathbf{E}[\mathrm{tr}\{F'(X)^{-T}M(X)F'(X)^{-1}\}] \\ &= \sum_{i=1}^{k} \mathbf{E}[\log|F_i'(X_i)|] + \frac{k}{2}\log \mathbf{E}\left(\sum_{i=1}^{k} \frac{m_{ii}(X)}{F_i'(X_i)^2}\right) \\ &= \sum_{i=1}^{k} \mathbf{E}[\log|F_i'(X_i)|] + \frac{k}{2}\log \mathbf{E}\left(\sum_{i=1}^{k} \frac{q_i(X_i)}{F_i'(X_i)^2}\right), \end{aligned} \tag{16}$$

where

$$q_i(x_i) = \mathbf{E}[m_{ii}(X)|X_i = x_i]. \tag{17}$$

In the Appendix we show that $F'_1, \ldots, F'_k$ minimize (16) if and only if

$$F'_i(x_i) = cq_i(x_i)^{1/2} \quad \text{a.e. } [\mu_{X_i}] \quad 1 \le i \le k \tag{18}$$

for some $c > 0$. Now suppose that the density of X is continuous and positive on the closed k-dimensional hypercube $[-B, B]^k$ and vanishes outside this hypercube. Then since $m_{ii}(x)$ is continuous and positive (recall that $M(x)$ is positive definite), it is easy to see that the conditional expectation defining $q_i(x_i)$ has a strictly positive and continuous version on $[-B, B]$. With these q_i, define

$$\tilde{F}_i(x_i) = \int_0^{x_i} q_i(t)^{1/2}\, dt, \quad x_i \in [-B, B], \quad 1 \le i \le k. \tag{19}$$

Then each $\tilde{F}_i$ is a valid scalar compressor function on $[-B, B]$ and $\tilde{F}'_1, \ldots, \tilde{F}'_k$ satisfy (18). Therefore $\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_k)$ is an optimal solution. (The definition of $\tilde{F}$ outside $[-B, B]^k$ is immaterial.) Furthermore, for this optimal $\tilde{F}$ we have

$$C_2(\tilde{F}) + \frac{k}{2} \log C_1(\tilde{F}) = \frac{k}{2} \log k + \frac{1}{2} \sum_{i=1}^{k} \mathbf{E}[\log q_i(X_i)]. \tag{20}$$

The excess rate resulting from using an optimal scalar compressor instead of a globally optimal compressor can be calculated from (13) and (20). Suppose F is a globally optimal compressor given in Theorem 2 (so that F achieves equality in (12)) and let F be a optimal componentwise scalar compressor given in (19). Then

$$\lim_{D \to 0} \left(H(Q_{D,\tilde{F}}) - H(Q_{D,F}) \right) = \frac{1}{2} \mathbf{E} \left[\log \left(\frac{\prod_{i=1}^{k} q_i(X_i)}{\det M(X)} \right) \right]. \tag{21}$$

The expectation on the right hand side is of course nonnegative, and equals zero if and only if, on a set of X-probability one, $M(x)$ is diagonal and each $m_{ii}(x)$ depends only on x_i.

3.3 Source and Distortion Measure Mismatch

Consider first the situation where the source statistics are imperfectly known. In [8] this problem was treated for fixed rate coding of memoryless sources. It was found that if the quantizer's point density is optimized for a model probability density $\tilde{f}$ instead of the true source density f, the resulting excess distortion in decibels is proportional to the relative entropy of f and $\tilde{f}$, namely

$$\mathbf{D}(f \| \hat{f}) = \int f(x) \log \frac{f(x)}{\tilde{f}(x)}\, dx,$$

for large quantizer dimensions and small distortions.

We obtain a similar result for rate redundancy due to source mismatch. In fact, the derivation is straightforward in our case, since the asymptotically optimal compressor (when it exists) and the lattice quantizer are independent of the source density and therefore knowledge of the source statistics is only needed for designing the lossless variable length code. Suppose we model the true source density f by $\tilde{f}$ such that $\mathbf{D}(f\|\tilde{f})$ is finite. Let the random vector $\tilde{X}$ have density $\tilde{f}$. Thus the variable rate lossless code is designed to be optimal for the known model distribution of the quantizer output $Q_{\alpha\Lambda}(F(\tilde{X}))$ instead of the true output distribution of $Q_{\alpha\Lambda}(F(X))$, where $\alpha = \alpha(D)$ (see the definition of $\alpha(D)$ in (6)). For any pair of random vectors Y and $\tilde{Y}$ such that either both have discrete distributions or both have densities, let $\mathbf{D}(Y\|\tilde{Y})$ denote the relative entropy [31] of the corresponding probability distributions of Y and $\tilde{Y}$. It is known [31] that the rate increase due to designing an optimal variable length code for $Q_{\alpha\Lambda}(F(\tilde{X}))$ and then using it for $Q_{\alpha\Lambda}(F(X))$ is within 1 bit of the relative entropy $\mathbf{D}\big(Q_{\alpha\Lambda}(F(X))\|Q_{\alpha\Lambda}(F(\tilde{X}))\big)$. The Voronoi partitions of R^k induced by the family of scaled lattices $\{\alpha\Lambda; \alpha > 0\}$ generate the Borel σ-field in R^k. Since $\alpha(D) \to 0$ as $D \to 0$, we have

$$\lim_{D\to 0} \mathbf{D}\big(Q_{\alpha\Lambda}(F(X))\|Q_{\alpha\Lambda}(F(\tilde{X}))\big) = \mathbf{D}\big(F(X)\|F(\tilde{X})\big)$$

(see [32, Corollary 5.2.4]). Since F is invertible, $\mathbf{D}\big(F(X)\|F(\tilde{X})\big) = \mathbf{D}\big(X\|\tilde{X}\big)$, and we conclude that the asymptotic rate redundancy due to source mismatch is within 1 bit (or $1/k$ bit per dimension) of the relative entropy $\mathbf{D}(X\|\tilde{X}) = \mathbf{D}(f\|\tilde{f})$ of the true and the model source densities.

Consider now the effect of distortion mismatch. Assume that the compressor $\tilde{F}$ satisfies the optimality condition $\tilde{F}'(x)^T\tilde{F}'(x) = \tilde{M}(x)$ for a distortion measure $\tilde{d}(x,y)$ whose sensitivity matrix is $\tilde{M}(x)$. There can be many reasons for optimizing the compressor for $\tilde{d}(x,y)$ instead of the true distortion measure $d(x,y)$. For example, the optimal compressor for $\tilde{d}$ may have simpler structure, or the optimality condition could be satisfied for $\tilde{d}$ (see Example 1 in Section 3.2) but not for d. In this case one is interested in the excess rate resulting from designing the compander for a mismatched distortion measure. This problem was also considered in [11] where an asymptotic expression was heuristically derived for the distortion redundancy due to using quantizers whose point density is optimized for a mismatched distortion measure.

Since $\tilde{F}'(x)^T\tilde{F}'(x) = \tilde{M}(x)$, we have

$$\mathrm{tr}\{\tilde{F}'(x)^{-T}M(x)\tilde{F}'(x)^{-1}\} = \mathrm{tr}\{\tilde{M}(x)^{-1}M(x)\}$$

and

$$\log|\det \tilde{F}'(x)| = \frac{1}{2}\log(\det \tilde{M}(x)).$$

Suppose first that there exists an optimal compander $F'(x)^T F'(x) = M(x)$, where $M(x)$ is the sensitivity matrix of $d(x, y)$. Then, if the distortion is measured using $d(x, y)$, the asymptotic rate redundancy of $Q_{D,\tilde{F}}$ over $Q_{D,F}$ is given by Theorems 1 and 2 as

$$\begin{aligned}
&\lim_{D\to 0}\left(H(Q_{D,\tilde{F}}) - H(Q_{D,F})\right) \\
&\quad = \frac{1}{2}\mathbf{E}\left[\log\left(\frac{\det \tilde{M}(X)}{\det M(X)}\right)\right] + \frac{k}{2}\log \mathbf{E}\left[\frac{\operatorname{tr}\{\tilde{M}(X)^{-1}M(X)\}}{k}\right]. \qquad (22)
\end{aligned}$$

If the optimal compressor does not exist, then the right hand side of (22) is the rate redundancy of $Q_{D,\tilde{F}}$ over the lower bound of Theorem 2. Note that (22) reduces to (21) of Example 2 if $\tilde{F}$ is the compressor with optimal scalar components. In that case $\tilde{F}$ is the globally optimal compressor for a distortion measure whose sensitivity matrix is $\tilde{M}(x) = \operatorname{diag}\{q_1(x_1), \ldots, q_k(x_k)\}$, where $q_i(x_i) = \mathbf{E}[m_{ii}(X)|X_i = x_i]$.

4 Rate-Distortion Performance

4.1 Asymptotic Optimality

The rate-distortion function of the random vector X is defined by

$$R(D) = \inf\{I(X, Y) : \mathbf{E}[d(X, Y)] \le D\}, \qquad (23)$$

where $I(X; Y)$ denotes the mutual information between the the k-dimensional random vectors X and Y, and the infimum is taken over all joint distributions of the pair (X, Y) such that $\mathbf{E}[d(X, Y)] \le D$. $R(D) = \infty$ by definition if no such Y exists. The rate-distortion function characterizes the lowest rate achievable by any source coding scheme in coding a memoryless vector source with marginal X at distortion level D. In particular, $R(D)$ is a lower bound on the rate of any vector quantizer for X whose distortion is less than or equal to D. In what follows we describe a result given in [12] that makes it possible to relate the asymptotic performance of the compander to the optimal performance given by $R(D)$.

We consider a more general class of distortion measures for reasons that will later be apparent. The principal difference is that now we do not require that for a given x the distortion takes its minimum at $y = x$, but rather that the unique minimum occurs at $y = r(x)$, where $r(x)$ is a smooth function of x. To be more concrete, we require that $d(x, y)$ satisfy the following.

(a') For all fixed $x \in \mathrm{R}^k$, $d(x,y)$ is three times continuously differentiable in the variable y, and the third order partial derivatives

$$\frac{\partial^3 d(x,y)}{\partial y_i \partial y_j \partial y_n} \quad i,j,n \in \{1,\ldots,k\}$$

are uniformly bounded.

(b') There is a function $r : \mathrm{R}^k \to \mathrm{R}^k$ such that for all $x \in \mathrm{R}^k$, $d(x,y) \geq d(x,r(x))$ with equality if and only if $y = r(x)$. Assume $r(x)$ is continuously differentiable and has a continuously differentiable inverse $g = r^{-1}$.

(c') $\liminf_{\|y\|\to\infty} d(x,y) > 0$ for all $x \in \mathrm{R}^k$.

(d') Let $M(x)$ be a $k \times k$ matrix with entries $m_{ij}(x)$, where

$$m_{ij}(x) = \frac{1}{2} \left.\frac{\partial^2 d(x,y)}{\partial y_i \partial y_j}\right|_{y=r(x)}.$$

Assume that for all i,j, the functions $m_{ij}(x)$ are continuously differentiable.

Let us define $d_{\min}(x) = d(x,r(x))$. Conditions (a'), (b'), and a second order Taylor expansion imply that as $\|y - r(x)\| \to 0$,

$$d(x,y) = d_{\min}(x) + (r(x)-y)^T M(x)(r(x)-y) + O(\|r(x)-y\|^3), \tag{24}$$

where $M(x)$ (the sensitivity matrix) is symmetric and nonnegative definite, analogous to (5). Furthermore, define

$$D_{\min} = \mathbf{E}[d_{\min}(X)]$$

and assume that $D_{\min}$ is finite (otherwise all quantizers have infinite distortion). Note that $R(D) = \infty$ by definition if $D < D_{\min}$.

The next theorem is a slightly specialized form of [12, Theorem 1]. It describes the asymptotic behavior of $R(D)$ as $D \to D_{\min}$ from above.

Theorem 3 *Suppose conditions (a')-(d') hold and assume that* $h(r(X))$, $\mathbf{E}[\log(\det M(X))]$, $\mathbf{E}[\|r(X)\|^2]$, *and* $\mathbf{E}\left[(\mathrm{tr}\{M^{-1}(X)\})^{3/2}\right]$ *are finite. Then the low distortion asymptotic behavior of* $R(D)$ *is given by*

$$\lim_{D\to D_{\min}} \left(R(D) + \frac{k}{2}\log(2\pi e(D-D_{\min})/k)\right) = h(r(X)) + \frac{1}{2}\mathbf{E}[\log(\det M(X))],$$

if $D \to D_{\min}$ *from above.*

Observe that if $d(x,y)$ satisfies conditions (a)-(c) of the previous sections, then it also satisfies (a') and (b'), with $r(x) = x$ and $d(x, r(x)) = 0$. In this case $D_{\min} = 0$ and if (c') and (d') also hold, we obtain

$$\lim_{D\to 0}\left(R(D) + \frac{k}{2}\log(2\pi e D/k)\right) = h(X) + \frac{1}{2}\mathbf{E}[\log(\det M(X))]. \tag{25}$$

Assume now that there exists a compressor F which is optimal in the sense of Theorem 2. Then we also have

$$\lim_{D\to 0}\left(H(Q_{D,F}) + \frac{k}{2}\log\left(\frac{D}{kL(P_0)}\right)\right) = h(X) + \frac{1}{2}\mathbf{E}[\log(\det M(X))]. \tag{26}$$

Combining (25) and (26) gives the following result.

Theorem 4 *Suppose X has a density which vanishes outside a bounded set, $h(X)$ is finite, and assume $d(x,y)$ satisfies the conditions (a')-(d') above with $r(x) = x$ and $\min_y d(x,y) = 0$. If $M(x)$ is positive definite for all x, and the compressor F is optimal, then the low distortion asymptotic behavior of the multidimensional companding quantizer relative to $R(D)$ is given by*

$$\lim_{D\to 0}\big(H(Q_{D,F}) - R(D)\big) = \frac{k}{2}\log(2\pi e L(P_0)). \tag{27}$$

Thus, for low distortions, the per dimension rate of $Q_{F,D}$ is about $\frac{1}{2}\log(2\pi e L(P_0))$ bits above the rate-distortion function.

Proof. We only have to check the validity of (25) and (26). The conditions given are clearly strong enough to imply (25) by Theorem 1. On the other hand, since the elements of $M(x)$ are continuous, it follows that $\mathrm{tr}\{M^{-1}(x)\}$ and $|\log(\det M(x))|$ are continuous functions, and since $M(x)$ is positive definite, they are also bounded on the compact support of the density of X. Similarly, $\|r(x)\|^2$ is bounded on the support of X by condition (b'). Thus the conditions of Theorem 3 are satisfied and (26) holds. Λ

Remark. This statement has a well known analogue for mean squared error and entropy coded lattice (or tessellating) quantizers [2, 4]. In fact, the same upper bound applies there, but the result is conceptually much simpler since the well known Shannon lower bound for the squared error can be used in place of Theorem 3.

Let G_k be the minimum value of $L(P_0)$ for any k-dimensional lattice. Based on a result of Poltyrev it was proved in [23, Lemma 1] that as $k \to \infty$, $G_k \to (2\pi e)^{-1}$ at a rate $\log(2\pi e G_k) = O(k^{-1}\log k)$. Thus, for optimal lattices and compressors,

$$\lim_{D\to 0}\frac{1}{k}\big(H(Q_{D,F}) - R(D)\big) = O\left(\frac{\log k}{k}\right)$$

which indicates that for high dimensions and low distortions, an entropy constrained companding lattice vector quantizer with an optimal compressor function can arbitrarily approach the rate-distortion performance limit.

4.2 Noisy Source Quantization

Let U be a k-dimensional random vector, called the *clean source* and let X be a k-dimensional random vector obtained by passing U through a noisy channel. The encoder has access only to the *noisy source* X, but the quantized signal $Q(X)$ has to approximate the clean source so that the distortion is measured by

$$D(Q) = \mathbf{E}[\rho(U, Q(X))],$$

where ρ is a given distortion measure, called the *original distortion measure.* This problem can be reformulated by introducing the *modified distortion measure* given by the conditional expectation

$$d(x, y) = \mathbf{E}[\rho(U, y)|X = x].$$

Then

$$D(Q) = \mathbf{E}[d(X, Q(X))]$$

and the noisy source problem is reduced to an ordinary quantization problem relative to the modified distortion measure $d(x, y)$ (which, however, depends on the joint distribution of U and X). In general, the modified distortion measure d is not a difference distortion measure and $d(x, y)$ is not minimized in y at $y = x$. Moreover, $\mathbf{E}[\inf_y d(X, y)] = D_{\min} > 0$ in all nontrivial cases. If $\mathbf{E}[\rho(U, y)|X = x]$ has a unique minimum $y = r(x)$, then $r(x)$ can be viewed as the optimal estimator (in the ρ distortion sense) of U given $X = x$.

By a classical result [15] the ordinary rate distortion function $R(D)$ of X relative to the modified distortion measure d is equal to the operational rate-distortion function for the noisy source quantization problem relative to the original distortion measure ρ. Let us assume that $d(x, y)$ satisfies the regularity conditions (a')-(d') and consider the companding quantizer scheme where the input to the compressor is $r(X)$. This companding scheme can be visualized as

$$X \to r(\cdot) \to F(\cdot) \to Q_{\alpha\Lambda}(\cdot) \to \left| \begin{array}{c} \text{entropy} \\ \text{coding} \end{array} \right| \to F^{-1}(\cdot) \to \hat{X} \tag{28}$$

If $M(x)$ is positive definite for all x we have the following result for companding quantization of noisy sources.

Theorem 5 *Suppose the density of X vanishes outside a bounded set and $h(X)$ is finite. Assume the modified distortion measure satisfies conditions (a')-(d') and that $M(x)$ is positive definite for all x. If there exists a compressor F such that $F'(x)^T F'(x) = M(r^{-1}(x))$, then the low distortion asymptotic behavior of the noisy source companding quantizer is given by*

$$\lim_{D \to D_{\min}} \left(H(Q_{D,F}) - R(D)\right) = \frac{k}{2}\log(2\pi e L(P_0)).$$

Proof. Let $Z = r(X)$. Then, by condition (*b*'), Z has a density which is zero outside a bounded set. Define the new distortion measure $\widehat{d}$ by

$$\widehat{d}(z, y) = d(r^{-1}(z), y) - d_{\min}(r^{-1}(z)).$$

Then $\widehat{d}(z, y) \geq 0$ with equality if and only if $y = z$ and it is easy to check that $\widehat{d}(z, y)$ satisfies all of the conditions of Theorem 4. Note that for all $\alpha > 0$ we have

$$\mathbf{E}[d(X, Q_{\alpha,F}(Z))] = \mathbf{E}[\widehat{d}(Z, Q_{\alpha,F}(Z))] + D_{\min}, \tag{29}$$

where $D_{\min} = \mathbf{E}[d_{\min}(X)]$. Also, the second derivative matrix associated with $\widehat{d}(z, y)$ is given by $\widehat{M}(z) = M(r^{-1}(z))$. Therefore by (29) the companding scheme is optimal with respect to the distortion measure $\widehat{d}$, and (29) and Theorem 4 imply

$$\lim_{D \to 0} \left(H(Q_{D+D_{\min},F}) - \widehat{R}(D)\right) = \frac{k}{2}\log(2\pi e L(P_0)), \tag{30}$$

where $\widehat{R}(D)$ is the rate-distortion function of Z relative to $\widehat{d}$. Since $Z = r(X)$ and r is invertible, (29) also implies that for all $D > 0$,

$$\widehat{R}(D) = R(D + D_{\min}).$$

Substitution into (30) competes the proof. Λ

Note that the optimal compander exists if the original distortion measure is additive, i.e., $\rho(u, y) = \sum_{i=1}^{k} \rho_i(u_i, y_i)$, where the ρ_i are appropriate scalar distortion measures, and if each U_i is conditionally independent of $\{X_j; j = 1, \ldots, k, j \neq i\}$ given X_i. In this case d will be in the same form with $d_i(x_i, y_i) = \mathbf{E}[\rho_i(U_i, y_i)|X_i = x_i]$, and the existence of an optimal compander follows by the Example after Theorem 2.

A discussion on when the modified distortion function satisfies the regularity conditions is given in [12]. An example of a family of original non-difference distortion measures is given which, if $X = U + \nu$, where U and ν are independent and Gaussian, induce modified distortion measures that satisfy our conditions. In general, the smoothness and integrability conditions are satisfied for "nice" original distortion measures and for "nice" noisy channels such as an additive noise channel where the noise density is sufficiently restricted. In fact, the condition that the sensitivity matrix of d be positive definite for (almost) all x is less restrictive than the same condition for the original distortion measure. This follows because (assuming we can exchange the order of differentiation and integration) we have $M(x) = \mathbf{E}[\widehat{M}(U)|X = x]$, where $\widehat{M}(u)$ is the sensitivity matrix of ρ. Then, for any y, we have $y^T M(x)y = \mathbf{E}[y^T\widehat{M}(U)y|X = x]$. Thus for $M(x)$ to be positive definite it suffices that $\widehat{M}(u)$ be positive definite on a set of nonzero $P_{U|X=x}$ probability.

As pointed out in [12], the primary restriction is that $r(x)$ should be invertible. In this respect we note that an alternative condition is that the sensitivity matrix depend on x only through the optimal estimator $r(x)$, i.e., $M(x) = \tilde{M}(r(x))$ for some positive definite $\tilde{M}(\cdot)$. Then the asymptotic expansion of $d(x, y)$ depends on x only through $r(x)$ and Theorem 5 holds. In [10, Theorem 2] it is proved that the modified distortion measure obtained from noisy source quantization with the Itakura-Saito distortion satisfies this condition. In this case $r(x)$ represents the parameters of the autoregressive model of U optimally estimated from X.

The asymptotic expression of Theorem 3 for the rate distortion function also holds when $r(x)$ is only "piecewise invertible", i.e., there are a finite number of open sets $A_1, \ldots, A_n$ such that r is one-to-one on each A_i and the complement of the union of these sets has zero Lebesgue measure. The companding scheme of (28) can also be extended to this case. Let $C(x) = i$ iff $x \in A_i$, and let F_i, (for $i = 1, \ldots, n$) be valid compressor functions. The companding quantizer is then defined as follows. If $C(x) = i$, the compressor function F_i is used to obtain the quantizer $Q_{D_i,F_i}(x)$, where the lattice scaling is adjusted so that

$$\mathbf{E}[d(X, Q_{D_i,F_i}(X)|X \in A_i] = D_i,$$

subject to the constraint that $D = \sum_{i=1}^{n} p_i D_i$, where $p_i = \mathbf{P}(C(X) = i)$. Since the value of $C(X)$ is needed for decoding (i.e., for choosing the expander G_i), the overall rate of the scheme is

$$R_F(D) = H(C(X)) + H(Q_{D,F}|C(X)).$$

Let $R_i(D)$ denote the rate-distortion function of the conditional distribution of X given $C(X) = i$. It is not hard to see that the compressors are optimal if for some $c > 0$, $F_i'(x)^T F(x) = cM(x)$ for all $x \in A_i$. Then Theorem 5 shows that for D_i close to $D_{m,i} = \mathbf{E}[d_{\min}(X)|C(X) = i]$, we have

$$H(Q_{F,D}|C(X) = i) \approx R_i(D_i) + \frac{k}{2}\log(2\pi e L(P_0)).$$

Thus, as $D \to D_{\min}$,

$$R_F(D) - R(D) \approx H(C(X)) + \sum_{i=1}^{n} p_i R(D_i) - R(D) + \frac{k}{2}\log(2\pi e L(P_0)). \tag{31}$$

From Theorem 3 (applied to each $R_i(D_i)$ separately) we obtain

$$\begin{aligned}\sum_{i=1}^{n} p_i R(D_i) - R(D) \;\;&\approx\;\; h(r(X)|C(X)) - h(r(X)) \\ &\qquad + \frac{k}{2}\log(D - D_{\min}) - \frac{k}{2}\sum_{i=1}^{n} p_i \log(D_i - D_{m,i}).\end{aligned}$$

By Jensen's inequality, $R_F(D)$ is asymptotically minimized if $D_i - D_{m,i} = D - D_{\min}$ for all i. Thus the asymptotically optimal choice in (4.2) is $D_i = D_{m,i} + (D - D_{\min})$. Since $h(r(X)) - h(r(X)|C(X)) = I(r(X); C(X))$, it follows from (31) that the asymptotic rate of the scheme is given by

$$\lim_{D \to D_{\min}} \big(R_F(D) - R(D)\big) = H(C(X)|r(X)) + \frac{k}{2}\log(2\pi e L(P_0)).$$

In general, if $r(x)$ is not invertible, the companding scheme may not be asymptotically optimal in the rate-distortion sense. For example, if $X = (X_1, \ldots, X_k)$ consists of the first k samples of an i.i.d. source and d is a single letter distortion measure, then $\frac{1}{k}H(C(X)|r(X))$ is the same positive constant for all k and thus the companding scheme is *not asymptotically* optimal, contrary to the case of an invertible $r(x)$.

5 Proofs

Proof of Proposition 2 The entropy of $Q_{\alpha,F}(X) = G(Q_{\alpha\Lambda}(F(X)))$ is equal to the entropy of the lattice quantizer output $Q_{\alpha\Lambda}(F(X))$ since G is invertible. It was proved in [14] using a result of Csiszár [33] that if a random vector Y is lattice quantized by the scaled lattice quantizer $Q_{\alpha\Lambda}$, and Y has a density and finite differential entropy $h(Y)$, then the quantizer's entropy is given asymptotically by

$$\lim_{\alpha \to 0} [\, H(Q_{\alpha\Lambda}(Y)) + k \log \alpha \,] = h(Y) - \log V(P_0),$$

provided $H(Q_{\alpha\Lambda}(X))$ is finite for some $\alpha > 0$. Thus by setting $Y = F(X)$ and using the identity $h(F(X)) = h(X) + \mathbf{E}[\log|\det F'(X)|]$, valid for all one-to-one and continuously differentiable F, the proposition is proved. Λ

Proof of Proposition 1 For any mapping $T : \mathrm{R}^k \to \mathrm{R}^k$ and for any set $B \in \mathrm{R}^k$, let $T(B)$ denote the image of B under T, i.e., $T(B) = \{T(x) : x \in B\}$. If T is linear, we will use the notation $TB = T(B)$. Specifically, if a is a real number, then we write $aB = \{ax : x \in B\}$. Also, for $y \in \mathrm{R}^k$, $B + y$ denotes the set $\{x + y : x \in B\}$.

The distortion of $Q_{\alpha,F}$ is given by

$$\begin{aligned} D(Q_{\alpha,F}) &= \int_{\mathrm{R}^k} d(x, Q_{\alpha,F}(x)) f(x)\, dx \\ &= \sum_{i=0}^{\infty} \int_{G(\alpha P_i)} d(x, G(\alpha y_i)) f(x)\, dx, \end{aligned}$$

where f is the source density and $\{y_i, P_i\}$ is an enumeration of the lattice points and their corresponding Voronoi cells such that $y_0 = 0$ and P_0 is the basic cell of Λ. Note that only finitely many of the terms in the sum above are nonzero since f is zero outside a compact set $K \subset \mathrm{R}^k$.

First we show that the asymptotics of the distortion are unchanged if $d(x, y)$ is replaced by its second order Taylor polynomial

$$\widehat{d}(x, y) \stackrel{\mathrm{def}}{=} (x - y)^T M(x)(x - y).$$

Indeed, since the remainder term is $O(\|x - y\|^3)$ by (5), we have for all $x \in G(\alpha P_i)$,

$$\left| d(x, G(\alpha y_i)) - \widehat{d}(x, G(\alpha y_i)) \right| = O((\operatorname{diam} G(\alpha P_i))^3), \tag{32}$$

where $\operatorname{diam}(B) = \sup\{\|x - y\| : x, y \in B\}$ denotes the diameter for any $B \subset \mathrm{R}^k$. But

$$\begin{aligned} \operatorname{diam}(G(\alpha P_i)) = \sup_{y,z \in \alpha P_i} \|G(y) - G(z)\| &\le \operatorname{diam}(\alpha P_i) \sup_{y \in \alpha P_i} \|G'(y)\| \\ &= \alpha \operatorname{diam}(P_i) \sup_{y \in \alpha P_i} \|G'(y)\|, \end{aligned}$$

where $\|G'(y)\|$ denotes the norm of the matrix $G'(y)$, defined by

$$\|G'(y)\| = \max_{\|z\|=1} \|G'(x)z\|.$$

Let $P_\alpha(x)$ denote the lattice quantizer cell αP_i in which $F(x)$ falls. Since $F(K)$ is a compact set (K is the support of f) , it follows by the continuity of G' that there exists a constant $c > 0$ such that if α is small enough, then for all $x \in K$ we have $\sup_{y \in P_\alpha(x)} \|G'(y))\| \le c$. Thus

$$\operatorname{diam}(G(P_\alpha(x)) \le \hat{c}\alpha, \quad x \in K \tag{33}$$

for a constant $\hat{c}$ and for all α small enough. Thus by (32) we obtain

$$\frac{1}{\alpha^2}\left|\int_{\mathrm{R}^k} d(x, Q_{\alpha,F}(x))f(x)\,dx - \int_{\mathrm{R}^k} \widehat{d}(x, Q_{\alpha,F}(x))f(x)\,dx\right| = O(\alpha) \tag{34}$$

as $\alpha \to 0$, so that it suffices to consider

$$\hat{D}(Q_{\alpha,F}) = \int_{\mathrm{R}^k} (x - Q_{\alpha,F}(x))^T M(x)(x - Q_{\alpha,F}(x))f(x)\,dx.$$

Let $S_\alpha(x) = G(P_\alpha(x))$ be the image under G of the lattice cell in which $F(x)$ falls. Define the piecewise constant probability density f_α by

$$f_\alpha(x) \stackrel{\text{def}}{=} \frac{1}{V(S_\alpha(x))}\int_{S_\alpha(x)} f(y)\,dy.$$

Then by change of variables,

$$f_\alpha(x) = \frac{1}{V(S_\alpha(x))}\int_{P_\alpha(x)} f(G(y))|\det G'(y)|\,dy.$$

As $\alpha \to 0$, the cells $P_\alpha(x)$ of the scaled lattice $\alpha\Lambda$ shrink to the point $F(x)$ in such a way that:

(i) the diameter of $P_\alpha(x)$ tends to zero;

(ii) there is a constant c such that for the smallest hypercube $C_\alpha(x)$ which has edges parallel to the coordinate axes and that is centered at $F(x)$ and contains $P_\alpha(x)$, we have $V(C_\alpha(x)) \le cV(P_\alpha(x))$, for all α small enough.

Thus by the differentiation theorem of Lebesgue integrals (see, e.g., [34, Theorem 7.16]), we have

$$\begin{aligned}\lim_{\alpha\to 0}\frac{1}{V(P_\alpha(x))}\int_{P_\alpha(x)} f(G(y))|\det G'(y)|\,dy &= f(G(F(x)))|\det G'(F(x))| \\ &= f(x)|\det G'(F(x))|\end{aligned}$$

for all x except possibly on a set of Lebesgue measure zero. On the other hand,

$$\frac{V(S_\alpha(x))}{V(P_\alpha(x))} = \frac{1}{V(P_\alpha(x))}\int_{P_\alpha(x)} |\det G'(y)|\,dy \to |\det G'(F(x))|, \quad \text{as } \alpha \to 0$$

for all x, by the continuity of G'. This gives

$$\lim_{\alpha\to 0}\frac{V(S_\alpha(x))}{V(P_\alpha(x))} = |\det G'(F(x))|, \tag{35}$$

and therefore

$$\lim_{\alpha \to 0} f_\alpha(x) = f(x)$$

almost everywhere. Then by Scheffe's theorem (see, e.g., Billingsley [35])

$$\lim_{\alpha \to 0} \int_{\mathrm{R}^k} |f_\alpha(x) - f(x)| \, dx = 0. \tag{36}$$

Define the normalized local distortion $e_\alpha(x)$ by

$$e_\alpha(x) \stackrel{\text{def}}{=} \alpha^{-2}(x - Q_{\alpha,F}(x))^T M(x)(x - Q_{\alpha,F}(x)).$$

If $W(x)$ is the positive definite square root of $M(x)$, by (33) we obtain

$$\begin{aligned}
e_\alpha(x) &= \alpha^{-2}\|W(x)(x - Q_{\alpha,F}(x))\|^2 \\
&\leq \alpha^{-2}\|x - Q_{\alpha,F}(x)\|^2\|W(x)\|^2 \\
&\leq \alpha^{-2}(\mathrm{diam}(S_\alpha(x)))^2\|W(x)\|^2 \\
&\leq \hat{c}\|W(x)\|
\end{aligned}$$

if α is small enough. Since the matrix norm $\|M(x)\|$ is bounded in K by the continuity of $M(x)$, so is $\|W(x)\|$. It follows that $e_\alpha(x)$ is bounded in K. Thus (36) gives

$$\lim_{\alpha \to 0} \int_{\mathrm{R}^k} e_\alpha(x) f(x) \, dx = \lim_{\alpha \to 0} \int_{\mathrm{R}^k} e_\alpha(x) f_\alpha(x) \, dx \tag{37}$$

if the limit on the right hand side exists. Since $f_\alpha(x)$ is constant over each $G(\alpha P_i)$, we have

$$\begin{aligned}
\int_{\mathrm{R}^k} e_\alpha(x) f_\alpha(x) \, dx &= \sum_{i=0}^{\infty} \frac{1}{V(G(\alpha P_i))} \int_{G(\alpha P_i)} f(y) \, dy \int_{G(\alpha P_i)} e_\alpha(x) \, dx \\
&= \sum_{i=0}^{\infty} \int_{G(\alpha P_i)} f(x) \hat{e}_\alpha(x) \, dx \\
&= \int_{\mathrm{R}^k} \hat{e}_\alpha(x) f(x) \, dx,
\end{aligned} \tag{38}$$

where $\hat{e}_\alpha(x)$ is defined by

$$\hat{e}_\alpha(x) \stackrel{\text{def}}{=} \frac{1}{V(S_\alpha(x))} \int_{S_\alpha(x)} e_\alpha(y) \, dy.$$

Suppose we can prove that

$$\lim_{\alpha \to 0} \hat{e}_\alpha(x) = L(P_0) V(P_0)^{2/k} \operatorname{tr}\{\Gamma(x)\} \tag{39}$$

for all x. Since $e_\alpha(x)$ is bounded on the support of f, so is $\hat{e}_\alpha(x)$, and therefore (37) and (38) and the dominated convergence theorem implies

$$\lim_{\alpha\to 0}\int_{\mathrm{R}^k} e_\alpha(x)f(x)\,dx = L(P_0)V(P_0)^{2/k}\int_{\mathrm{R}^k}\mathrm{tr}\{\Gamma(x)\}f(x)\,dx.$$

This and (34) proves the theorem.

The remainder of the proof is devoted to proving (39). Let us introduce the notation $y_\alpha(x) = Q_{\alpha,F}(x)$. We have to find the limit as $\alpha \to 0$ of

$$\hat{e}_\alpha(x) = \frac{\alpha^{-2}}{V(S_\alpha(x))}\int_{V(S_\alpha(x))}(y-y_\alpha(x))^T M(x)(y-y_\alpha(x))\,dy.$$

Define

$$\begin{aligned} b_\alpha(x) &\stackrel{\text{def}}{=} \frac{\alpha^{-2}}{V(G'(F(x))\alpha P_0)}\int_{G'(F(x))\alpha P_0+x}(y-x)^T M(x)(y-x)\,dy \\ &= \frac{\alpha^{-2}}{V(G'(F(x))\alpha P_0)}\int_{G'(F(x))\alpha P_0} y^T M(x) y\,dy \\ &= \frac{1}{V(G'(F(x))P_0)}\int_{G'(F(x))P_0} y^T M(x) y\,dy. \end{aligned}$$

To prove (39) it suffices to show that for all $\alpha > 0$,

$$b_\alpha(x) = L(P_0)V(P_0)^{2/k}\,\mathrm{tr}\{\Gamma(x)\} \tag{40}$$

and that

$$\lim_{\alpha\to 0}\hat{e}_\alpha(x) = \lim_{\alpha\to 0} b_\alpha(x). \tag{41}$$

To prove (40), let us simplify the notation by setting $G'(F(x)) = G'$, $M(x) = M$, and $W(x) = M^{1/2}(x) = W$. Then

$$\begin{aligned} \int_{G'P_0} y^T M y\,dy &= \int_{G'P_0}\|Wy\|^2\,dy \\ &= (\det W)^{-1}\int_{WG'P_0}\|y\|^2\,dy \\ &= \frac{V(WG'P_0)}{\det W}\mathbf{E}[\|Z\|^2] \\ &= \frac{V(WG'P_0)}{\det W}\mathbf{E}[\|WG'Y\|^2], \end{aligned}$$

where Z and Y are random vectors which are uniformly distributed over $WG'P_0$ and P_0, respectively. It is easy to see that

$$\mathbf{E}[\|WG'Y\|^2] = \mathrm{tr}\{WG'R_Y(WG')^T\},$$

where $R_Y = \mathbf{E}[YY^T]$ is the covariance matrix of Y. But the basic cell P_0 is white, so that $R_Y = \epsilon I$ for some $\epsilon > 0$, and therefore

$$\begin{aligned} \operatorname{tr}\{(WG')R_Y(WG')^T\} &= \epsilon \operatorname{tr}\{(WG')^T WG'\} \\ &= \epsilon \operatorname{tr}\{(G')^T MG'\}. \end{aligned}$$

Clearly,

$$\epsilon = \frac{1}{kV(P_0)} \int_{P_0} \|y\|^2 \, dy = L(P_0)V(P_0)^{2/k},$$

so we conclude that

$$\begin{aligned} \frac{1}{V(G'P_0)} \int_{G'P_0} y^T My \, dy &= \frac{V(WG'P_0)}{(\det W)V(G'P_0)} L(P_0)V(P_0)^{2/k} \operatorname{tr}\{(G')^T MG'\} \\ &= L(P_0)V(P_0)^{2/k} \operatorname{tr}\{\Gamma(x)\}, \end{aligned}$$

which proves (40).

In the last step of the proof we show (41). Let $z_\alpha(x) = Q_{\alpha\Lambda}(F(x))$, so that $y_\alpha(x) = G(z_\alpha(x))$, and rewrite $\hat{e}_\alpha(x)$ by a change of variables as

$$\begin{aligned} \frac{\alpha^{-2}}{V(S_\alpha(x))} \int_{S_\alpha(x)} (y - y_\alpha(x))^T M(x)(y - y_\alpha(x)) \, dy &= \frac{\alpha^{-2}}{V(S_\alpha(x))} \int_{S_\alpha(x) - y_\alpha(x)} y^T M(x) y \, dy \\ &= \frac{\alpha^k}{V(S_\alpha(x))} \int_{A_\alpha(x)} y^T M(x) y \, dy, \end{aligned}$$

where

$$\begin{aligned} A_\alpha(x) &= \alpha^{-1} S_\alpha(x) - \alpha^{-1} y_\alpha(x) \\ &= \alpha^{-1} G(\alpha P_0 + z_\alpha(x)) - \alpha^{-1} G(z_\alpha(x)), \end{aligned}$$

since $P_\alpha(x) = z_\alpha(x) + \alpha P_0$. By (35) we have

$$\lim_{\alpha \to 0} \frac{V(S_\alpha(x))}{\alpha^k} = |\det G'(F(x))| \cdot V(P_0) = V(G'(F(x))P_0), \tag{42}$$

so that to obtain (41) we have to prove

$$\lim_{\alpha \to 0} \int_{A_\alpha(x)} y^T M(x) y \, dy = \int_{G'(F(x))P_0} y^T M(x) y \, dy.$$

In fact, since there exists a bounded set which contains $A_\alpha(x)$ for all α small enough, it suffices to prove that

$$\lim_{\alpha \to 0} V(A_\alpha(x) \bigtriangleup G'(F(x))P_0) = 0, \tag{43}$$

where $\bigtriangleup$ denotes symmetric difference of sets: $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$. The proof of (43) is given in the Appendix. Thus (41) holds and the proof is complete. Λ

Appendix

Proof of the optimality of (18)

For all componentwise scalar compressors, we have

$$\frac{k}{2}\log \mathbf{E}\left(\sum_{i=1}^{k}\frac{q_i(X_i)}{F_i'(X_i)^2}\right) \geq \frac{k}{2}\mathbf{E}\log\left(\sum_{i=1}^{k}\frac{q_i(X_i)}{F_i'(X_i)^2}\right) \tag{44}$$

$$= \frac{k}{2}\log k + \frac{k}{2}\mathbf{E}\log\left(\frac{1}{k}\sum_{i=1}^{k}\frac{q_i(X_i)}{F_i'(X_i)^2}\right)$$

$$\geq \frac{k}{2}\log k + \frac{k}{2}\mathbf{E}\left(\frac{1}{k}\sum_{i=1}^{k}\log\frac{q_i(X_i)}{F_i'(X_i)^2}\right) \tag{45}$$

$$= \frac{k}{2}\log k + \frac{1}{2}\sum_{i=1}^{k}\mathbf{E}\log q_i(X_i) - \sum_{i=1}^{k}\mathbf{E}\log|F_i'(X_i)|,$$

where we have used Jensen's inequality in (44) and (45). We obtain

$$C_2(F) + \frac{k}{2}\log C_1(F) \geq \frac{k}{2}\log k + \frac{1}{2}\sum_{i=1}^{k}\mathbf{E}\log q_i(X_i),$$

where equality holds if and only if

$$\mathbf{P}\left(\frac{q_i(X_i)}{F_i'(X_i)^2} = \hat{c}, i = 1, \ldots, k\right) = 1$$

for some $\hat{c} > 0$. This condition is equivalent to (18). Λ

Proof that $\lim_{\alpha\to 0} V(A_\alpha(x) \triangle G'(F(x))P_0) = 0$ *in (43)*

We can assume without loss of generality that P_0 is an open set since P_0 is a convex polytope so that replacing it by its interior will not change the value of the integrals. Then $G'(F(x))P_0$ is also open. For any $z \in P_0$ we have

$$\begin{aligned}
&\|G(\alpha z + z_\alpha(x)) - G(z_\alpha(x)) - G'(F(x))\alpha z\| \\
&\quad = \|G(\alpha z + z_\alpha(x)) - G(F(x)) + G(F(x)) - G(z_\alpha(x)) - G'(F(x))\alpha z\| \\
&\quad \leq \|G'(F(x))(\alpha z + z_\alpha(x) - F(x)) - G'(F(x))(z_\alpha(x) - F(x)) - G'(F(x))\alpha z\| \\
&\qquad + o(\|\alpha z + z_\alpha(x) - F(x)\|) + o(\|z_\alpha(x) - F(x)\|) \\
&\quad = o(\|\alpha z + z_\alpha(x) - F(x)\|) + o(\|z_\alpha(x) - F(x)\|) \\
&\quad = o(\alpha),
\end{aligned}$$

as $\alpha \to 0$, since $\|F(x) - z_\alpha(x)\| \leq \operatorname{diam}(P_\alpha(x)) = O(\alpha)$. Thus

$$\alpha^{-1}G(\alpha z + z_\alpha(x)) - \alpha^{-1}G(z_\alpha(x)) \to G'(F(x))z \quad \text{as} \quad \alpha \to 0,$$

implying (since $G'(F(x))P_0$ is open) that

$$\alpha^{-1}G(\alpha z + z_\alpha(x)) - \alpha^{-1}G(z_\alpha(x)) \in G'(F(x))P_0 \tag{46}$$

if $\alpha > 0$ is small enough. Let

$$E_\alpha(x) = \{z \in P_0 : \alpha^{-1}G(\alpha z + z_\alpha(x)) - \alpha^{-1}G(z_\alpha(x)) \notin G'(F(x))P_0\}.$$

If χ_{E_α} is the indicator function of $E_\alpha(x)$, then (46) implies that $\lim_{\alpha\to 0} \chi_{E_\alpha}(z) = 0$ for all z. Thus $\lim_{\alpha\to 0} V(E_\alpha(x)) = \lim_{\alpha\to 0} \int_{P_0} \chi_{E_\alpha}(z)\, dz = 0$ by bounded convergence. Therefore

$$\begin{aligned} V(A_\alpha(x) \setminus G'(F(x))P_0) &= V\left(\alpha^{-1}G(\alpha E_\alpha(x) + z_\alpha(x)) - \alpha^{-1}G(z_\alpha(x))\right) \\ &= \alpha^{-k} \int_{G(\alpha E_\alpha(x) + z_\alpha(x))} dy \\ &= \int_{E_\alpha(x))} |\det G'(\alpha z + z_\alpha(x))|\, dy \to 0 \quad \text{as} \quad \alpha \to 0, \end{aligned}$$

since G' is continuous and $z_\alpha(x) \to F(x)$. Since we know by (42) that $\lim_{\alpha\to 0} V(A_\alpha(x)) = V(G'(F(x))P_0)$, we obtain

$$\begin{aligned} V(G'(F(x))P_0) &= \lim_{\alpha\to 0} \left[V(A_\alpha(x) \cap G'(F(x))P_0) + V(A_\alpha(x) \setminus G'(F(x))P_0)\right] \\ &= \lim_{\alpha\to 0} V(A_\alpha(x) \cap G'(F(x))P_0). \end{aligned}$$

Thus

$$\lim_{\alpha\to 0} V(G'(F(x))P_0 \setminus A_\alpha(x)) = 0,$$

implying

$$\lim_{\alpha\to 0} V(A_\alpha(x) \triangle G'(F(x))P_0) = 0.$$

as was to be shown. Λ

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