The Multiple Description Rate Region for High Resolution Source Coding *

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Abstract

Consider encoding a memoryless source using two descriptions, the first at rate R_1 and distortion d_1, the second at rate R_2 and distortion d_2. Combining the two descriptions the source can be reconstructed with distortion d_0. For a Gaussian source of variance σ^2, Ozarow found an explicit characterization of the region $\mathcal{R}^*(\sigma^2; d_1, d_2, d_0) \subset \mathbb{R}^2$ of achievable rate pairs (R_1, R_2) with given mean squared distortions d_1, d_2, and d_0. This is the only case for which the multiple description rate-distortion region is completely known. We show that for a general real valued source X and a locally quadratic distortion measure of the form $\rho(x, \hat{x}) = w(x)^2(x - \hat{x})^2 + o((x - \hat{x})^2)$, the region of admissible rate pairs is arbitrary well approximated in the limit of small distortions by the region

$$\mathcal{R}^* \left(P_X 2^{2E\{\log w(X)\}}; d_1, d_2, d_0 \right)$$

where $\mathcal{R}^*(\sigma^2; d_1, d_2, d_0)$ denotes the multiple description rate region of a Gaussian source with variance σ^2, and where P_X is the entropy-power of the source. Applications to companding quantization are also considered.

I. Introduction and Main Result

The multiple description problem [2, 5, 11, 12] arises for example in situations where source information (e.g. speech, image, video) is transmitted via lossy packet networks. Transmitted information is broken into packets, some of which may not arrive

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to the destination. The decoder's goal is to achieve a certain basic reproduction quality if a small number of the packets arrive, and to improve the quality if more (or all of the) packets arrive.

The basic formulation of the multiple description problem in the information-theoretic literature involves two (lossless) sub-channels of rates R_1 and R_2, corresponding to two "packets", and three receivers. Each receiver corresponds to a possible case of packet arrival: the first arrived, the second arrived, or both arrived. Given a source block $\boldsymbol{X} = (X_1, \ldots, X_n)$, the encoder generates two code words (indices) $f_i(\boldsymbol{X})$ and $f_2(\boldsymbol{X})$ at rates

$$R_i = \frac{1}{n} \log |f_i| \qquad i = 1, 2$$

where $|f_i|$ denotes the size of code f_i and log denotes base 2 logarithm. Then the codeword $f_i(\boldsymbol{X})$ is transmitted through sub-channel i. The two individual ("marginal") receivers and the combined ("central") receiver then generate the reconstructions $\widehat{\boldsymbol{X}}_1, \widehat{\boldsymbol{X}}_2$, and $\widehat{\boldsymbol{X}}_0$, respectively, using the decoding functions g_1, g_2, and g_0, i.e.,

$$\begin{aligned}
\widehat{\boldsymbol{X}}_1 &= g_1\big(f_1(\boldsymbol{X})\big) \\
\widehat{\boldsymbol{X}}_2 &= g_2\big(f_2(\boldsymbol{X})\big) \\
\widehat{\boldsymbol{X}}_0 &= g_0\big(f_1(\boldsymbol{X}), f_2(\boldsymbol{X})\big).
\end{aligned}$$

Assume that $\{X_j\}_{j=1}^{\infty}$ is a real memoryless source with a generating random variable X. The distortions in the three cases of packet arrivals are given by $\frac{1}{n} \sum_{j=1}^{n} E\rho(X_j, \widehat{X}_{ij})$, where $(\widehat{X}_{i1}, \ldots, \widehat{X}_{in}) = \widehat{\boldsymbol{X}}_i$ for $i = 0, 1, 2$, and where the distortion measure $\rho(x, \hat{x})$ is common to all receivers. The quintuple $(R_1, R_2, d_0, d_1, d_2)$ is said to be *achievable* if for any $\epsilon > 0$ and sufficiently large n there exists a multiple description code for which

$$\frac{1}{n} \log |f_i| \leq R_i + \epsilon \quad \text{for } i = 1, 2$$

and

$$\frac{1}{n} \sum_{j=1}^{n} E\rho(X_j, \widehat{X}_{ij}) \leq d_i + \epsilon \quad \text{for } i = 0, 1, 2.$$

Let

$$\mathcal{R}_X(d_0, d_1, d_2) = \{(R_1, R_2) : (R_1, R_2, d_0, d_1, d_2) \quad \text{is achievable}\} \tag{1}$$

be the set of achievable rate pairs at distortions (d_0, d_1, d_2). Similarly, let

$$\mathcal{D}_X(R_1, R_2) = \{(d_0, d_1, d_2) : (R_1, R_2, d_0, d_1, d_2) \quad \text{is achievable}\}$$

be the set of achievable distortion triplets for a given rate pair (R_0, R_1). Let $R_X(d)$

denote the rate-distortion function of the source. In the multiple description problem it is usually impossible to achieve the Shannon limits $R_1 = R_X(d_1), R_2 = R_X(d_2)$ while at the same time achieving the central receiver distortion d_0 which satisfies the Shannon limit $R_1 + R_2 = R_X(d_0)$.

A "single letter" characterization of the set of achievable distortions $\mathcal{D}_X(R_1, R_2)$ for the case of a Gaussian source and squared error distortion measure was found by Ozarow [10]. This set, denoted $\mathcal{D}^*(\sigma^2; R_1, R_2)$, and its "inverse" set $\mathcal{R}^*(\sigma^2; d_0, d_1, d_2)$, where σ^2 is the source variance, are described in the next section. For general sources and distortion measures, however, the complete solution for $\mathcal{R}_X(d_0, d_1, d_2)$ is still unknown.

El Gamal and Cover [5] characterized an inner bound for $\mathcal{R}_X(d_0, d_1, d_2)$, i.e., a set of achievable rates (see Section IV). Only the special case of no excess rate sum, i.e., $R_1 + R_2 = R_X(d_0)$, was solved completely, by Ahlswede [1]. For the case of a *discrete* source, Zhang and Berger [14, 15] provided inner and outer bounds which are relatively tight for the case of no excess marginal rate, i.e., $R_i = R_X(d_i)$ for $i = 1, 2$.

Since the general problem is still unsolved, it is of interest to obtain bounds on the multiple description rate region which, at least for high resolution, become tight. In a recent paper [13], inner and outer bounds on $\mathcal{R}_X(d_0, d_1, d_2)$ were found for general real valued sources and a squared error distortion measure, in terms of the Gaussian rate region $\mathcal{R}^*(\cdot)$. These bounds have the form

$$\mathcal{R}^*(\sigma_X^2; d_0, d_1, d_2) \subset \mathcal{R}_X(d_0, d_1, d_2) \subset \mathcal{R}^*(P_X; d_0, d_1, d_2) \ , \tag{2}$$

where σ_X^2 is the variance of X, P_X is its entropy power defined by

$$P_X = \frac{1}{2\pi e} 2^{2h(X)},$$

and $h(X)$ is the differential entropy of X. The bounds in (2) parallel Shannon's lower and upper bounds for the rate-distortion function subject to a mean squared error constraint [3]. It was also shown in [13] that the outer bound (the right-hand side of (2)) becomes tight in the limit of small distortions in the following sense. For any $\epsilon > 0$ and $\mathcal{R} \subset \mathbb{R}^2$, let $\mathcal{R}^{(+\epsilon)}$ and $\mathcal{R}^{(-\epsilon)}$ denote the planar sets

$$\mathcal{R}^{(+\epsilon)} = \{(R_1, R_2) : (R_1 + \epsilon, R_2 + \epsilon) \in \mathcal{R}\} \tag{3}$$

and

$$\mathcal{R}^{(-\epsilon)} = \{(R_1, R_2) : (R_1 - \epsilon, R_2 - \epsilon) \in \mathcal{R}\}. \tag{4}$$

The outer bound of (2) is asymptotically tight in the sense that as $(d_0, d_1, d_2) \to (0, 0, 0)$, for any $\epsilon > 0$ we eventually have

$$\mathcal{R}^*(P_X; d_0, d_1, d_2)^{(-\epsilon)} \subset \mathcal{R}_X(d_0, d_1, d_2) \subset \mathcal{R}^*(P_X; d_0, d_1, d_2)^{(+\epsilon)}. \tag{5}$$

In other words, the rate region $\mathcal{R}^*$ of a Gaussian source of variance $\sigma^2 = P_X$ arbitrarily well approximates $\mathcal{R}_X$ for small mean squared distortions. In the sequel, we will denote such asymptotically tight approximation by $\mathcal{R}_X(\cdot) \cong \mathcal{R}^*(\cdot)$.

In this paper we determine the high resolution multiple description rate region for more general distortion measures. We will consider distortion measures $\rho(x, \hat{x})$ such that $d(x, \hat{x}) \geq 0$ with equality if and only if $x = \hat{x}$, and furthermore $\frac{\partial^2 \rho(x,\hat{x})}{\partial \hat{x}^2}\Big|_{\hat{x}=x} > 0$ for all x. Distortion measures of this type are called *locally quadratic* since by second order Taylor expansion, if $|x - \hat{x}|$ is small, then

$$\rho(x, \hat{x}) \approx w^2(x)(x - \hat{x})^2,$$

where the positive "weighting function" $w(x)$ is defined by

$$w^2(x) = \frac{1}{2}\frac{\partial^2 \rho(x, \hat{x})}{\partial \hat{x}^2}\Big|_{\hat{x}=x}. \tag{6}$$

We note here that some important *perceptual distortion measures* for speech and image coding are locally quadratic [4, 6]. We will assume that $\rho(x, \hat{x})$ also satisfies some additional mild regularity conditions detailed in [8]. With these assumption, our main result is the following.

Theorem 1 *Let X be a real source with finite entropy-power $P_X > 0$, and assume the distortion measure $\rho(x, \hat{x})$ is locally quadratic satisfying the regularity conditions of [8]. Assume that $(d_0, d_1, d_2) \to 0$ in such a way that the ratios d_1/d_0 and d_2/d_0 remain constant. Then the multiple description rate region is given asymptotically as*

$$\mathcal{R}_X(d_0, d_1, d_2) \cong \mathcal{R}^*\left(P_X 2^{2E\{\log w(X)\}}; d_0, d_1, d_2\right) \tag{7}$$

where $\mathcal{R}^*(\cdot)$ is the quadratic-Gaussian multiple description rate region and $w(x)$ is the weighting function associated with $\rho(x, \hat{x})$ through (6).

The rest of the paper is organized as follows. In Section II, Ozarow's solution $\mathcal{R}^*$ for a Gaussian source is described. The structure of this solution is essential for our result. Then in Section III and Section IV we present the converse and direct results, respectively, which together imply Theorem 1. In Section V a simple realization scheme using companding quantization is discussed briefly.

II. The Quadratic-Gaussian Case

Ozarow [10] proved that for mean squared error, the set of achievable distortions $\mathcal{D}^*(\sigma^2; R_1, R_2)$ for a Gaussian source with variance σ^2 is the union of all triplets (d_0, d_1, d_2) satisfying

$$d_1 \geq \sigma^2 \cdot 2^{-2R_1} \tag{8a}$$

$$d_2 \geq \sigma^2 \cdot 2^{-2R_2} \tag{8b}$$

$$d_0 \geq \frac{\sigma^2 \cdot 2^{-2(R_1+R_2)}}{1 - \left(\sqrt{\Pi} - \sqrt{\Delta}\right)^2}, \tag{8c}$$

where $\Pi = (1 - d_1/\sigma^2)(1 - d_2/\sigma^2)$ and $\triangle = (d_1 d_2/\sigma^4) - 2^{-2(R_1+R_2)}$. In the sequel we will need also the inverse function of (8). Given a triplet of mean square errors (d_0, d_1, d_2), the set $\mathcal{R}^*(\sigma^2; d_0, d_1, d_2)$ of admissible rate pairs is the union of all (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\geq R^*(\sigma^2; d_1) && \text{(9a)} \\ R_2 &\geq R^*(\sigma^2; d_2) && \text{(9b)} \\ R_1 + R_2 &\geq R^*(\sigma^2; d_1) + R^*(\sigma^2; d_2) + \delta, && \text{(9c)} \end{aligned}$$

where $R^*(\cdot)$ is the quadratic-Gaussian rate-distortion function

$$R^*(\sigma^2; d) = \frac{1}{2}\log\Big(\frac{\sigma^2}{d}\Big), \tag{9d}$$

$\delta = \delta(d_0, d_1, d_2)$ is defined by

$$\delta = \begin{cases} \frac{1}{2}\log\left(\frac{1}{1-\rho^2}\right), & d_0 \leq d_{0max} \\ 0, & d_0 \geq d_{0max} \end{cases}, \tag{9e}$$

where

$$\begin{aligned} \rho &= -\frac{\sqrt{\Pi\epsilon_0^2 + \gamma} - \sqrt{\Pi\epsilon_0^2}}{(1-\epsilon_0)\sqrt{\epsilon_1\epsilon_2}}, \quad \gamma = (1-\epsilon_0)[(\epsilon_1 - \epsilon_0)(\epsilon_2 - \epsilon_0) + \epsilon_0\epsilon_1\epsilon_2 - \epsilon_0^2] && \text{(9f)} \\ \Pi &= (1-\epsilon_1)(1-\epsilon_2), \qquad \epsilon_i = d_i/\sigma^2 \quad \text{for} \quad i = 0, 1, 2 && \text{(9g)} \end{aligned}$$

and

$$d_{0max} = \frac{d_1 d_2}{d_1 + d_2 - (d_1 d_2/\sigma^2)}.$$

Note that $\delta, \gamma \geq 0$, and $-1 \leq \rho \leq 0$ for all $d_1, d_2 \leq \sigma^2$. Note also that δ depends on d_0, d_1, d_2, and σ^2 only through the ratios d_i/σ^2. As we will see in Section IV, ρ has the meaning of a correlation coefficient in the Gaussian optimal test channel.

A case of special interest for us is that of high resolution coding, i.e., the limit as (d_0, d_1, d_2) converges to zero along a straight line. In this limit the so-called total excess marginal rate δ and the maximum central distortion d_{0max} do not depend on the variance of the source. Specifically, we have

$$\begin{aligned} \delta_{HR}(d_0, d_1, d_2) &\triangleq \lim_{\sigma\to\infty} \delta(\sigma^2, d_0, d_1, d_2) \triangleq \lim_{\lambda\to 0} \delta(\sigma^2, \lambda d_0, \lambda d_1, \lambda d_2) \\ &= \frac{1}{2}\log\left(\frac{1}{1-\rho_{HR}^2}\right), \end{aligned} \tag{10}$$

where

$$\rho_{HR} = -\frac{\sqrt{d_1/d_0 - 1}\sqrt{d_2/d_0 - 1} - 1}{\sqrt{d_1 d_2/d_0^2}} \tag{11}$$

and

$$\lim_{\sigma\to\infty} d_{0max} = \frac{d_1 d_2}{d_1 + d_2} \ . \tag{12}$$

Using these definitions, our Theorem 1 can be rewritten in the following more explicit form. For a general real source X with entropy power P_X and for a locally quadratic distortion measure $\rho(x, \hat{x})$ with associated weighting function $w(x)$, the multiple description rate region for small (d_0, d_1, d_2) is approximately given by

$$R_1 \ \geq \ R^*(\sigma^2; d_1) + E\{\log w(X)\} \tag{13a}$$

$$R_2 \ \geq \ R^*(\sigma^2; d_2) + E\{\log w(X)\} \tag{13b}$$

$$R_1 + R_2 \ \geq \ R^*(\sigma^2; d_1) + R^*(\sigma^2; d_2) + 2E\{\log w(X)\} + \delta_{HR} \ . \tag{13c}$$

III. Converse Theorem

In this section an asymptotic outer bound is given for the set of achievable distortions, which establishes the converse part (the asymptotic outer bound) of Theorem 1.

Theorem 2 *Under the conditions of Theorem 1, the set of achievable distortion triplets* (d_0, d_1, d_2) *satisfies*

$$\mathcal{D}_X(R_1, R_2) \ \tilde{\subset} \ \mathcal{D}^*\left(P_X 2^{2E\{\log w(X)\}}; R_1, R_2\right), \tag{14}$$

where $\mathcal{D}^*(\cdot)$ *is the quadratic-Gaussian distortion region (8), and the asymptotic inclusion* $\tilde{\subset}$ *means that for any* $\epsilon > 0$*, if* R_1 *and* R_2 *are large enough, then*

$$\mathcal{D}_X(R_1, R_2) \subset \mathcal{D}^*\left(P_X 2^{2E\{\log w(X)\}}; R_1, R_2\right)^{(+\epsilon)} .$$

The proof of this result follows the line of the proof of the converse in [10], and its generalization to a mean squared distortion and non-Gaussian source in [13]. However, while in [13] the proof was facilitated by the Shannon lower bound, we are forced to carry out a more involved analysis for non-difference distortion measures since the Shannon lower bound works for difference distortion measures only. Our approach is to use a recently derived result [8] stating that for locally quadratic distortion measures the rate-distortion function $R_X(d)$ of X is asymptotically lower bounded by $R^*(P_X; d) + E\{\log w(X)\}$, i.e.,

$$\limsup_{d\to 0}\Big(R_X(d) - R^*\left(P_X 2^{2E\{\log w(X)\}}; d\right)\Big) \geq 0.$$

This result allows us to replace the use of the Shannon lower bound, but due to its asymptotic nature it introduces difficulties not encountered in the mean squared distortion case.

IV. Direct Theorem

In this section we use the El Gamal – Cover inner bound [5] to prove the direct part (asymptotic inner bound) in Theorem 1. We recall from [5] that a distortion triplet (d_0, d_1, d_2) is achievable from two descriptions of a source X with rates R_1 and R_2 if (but not necessarily only if) there exist random variables U and V, jointly distributed with X, and three real valued functions functions $\widehat{X}_1 = \widehat{X}_1(U), \widehat{X}_2 = \widehat{X}_2(V)$, and $\widehat{X}_0 = \widehat{X}_0(U,V)$, such that

$$\begin{aligned} R_1 &\geq I(X;U) && (15a)\\ R_2 &\geq I(X;V) && (15b)\\ R_1 + R_2 &\geq I(X;UV) + I(U;V) && (15c)\\ d_i &\geq E\rho(X, \widehat{X}_i) \qquad\qquad i = 0,1,2\ . && (15d)\end{aligned}$$

The El Gamal – Cover solution is optimal for a Gaussian source and squared error distortion measure. More specifically, the entire rate region $R^*(\sigma_x^2; d_0, d_1, d_2)$ characterized in (9) is realized in this case by choosing

$$U = X + N_1 \qquad V = X + N_2 \tag{16a}$$

$$\widehat{X}_1 = \alpha_1 U \qquad \widehat{X}_2 = \alpha_2 V \qquad \widehat{X}_0 = \beta_1 U + \beta_2 V, \tag{16b}$$

where (N_1, N_2) is zero mean bivariate Gaussian, independent of X, and with covariance matrix

$$\mathrm{Cov}(N_1, N_2) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}. \tag{17}$$

The parameters ρ, σ_1, and σ_2 are chosen so that $\mathrm{Var}(X|U) = d_1$, $\mathrm{Var}(X|V) = d_2$, and $\mathrm{Var}(X|UV) = d_0$, where $\mathrm{Var}(A|B) = E[A - E(A|B)]^2$ denotes conditional variance. The scaling factors $\alpha_1, \alpha_2, \beta_1$, and β_2 are chosen to yield the conditional means $\widehat{X}_1 = E(X|U), \widehat{X}_2 = E(X|V)$, and $\widehat{X}_0 = E(X|UV)$, which are linear functions since (X, U, V) is jointly Gaussian. The above implies that $\sigma_i^2 = \sigma_x^2 d_i/(\sigma_x^2 - d_i)$, $i = 1, 2$, and that ρ is given by the expression in (9f). See [10] for an explicit characterization of $\alpha_1, \alpha_2, \beta_1, \beta_2$ as a function of $\sigma_x^2, d_0, d_1, d_2$.

At high resolution, i.e., as $(d_0, d_1, d_2) \to 0$ along a straight line, we have the following limiting values for the parameters of the quadratic-Gaussian solution given by (16)-(17):

$$\begin{aligned}&\alpha_1 = \alpha_2 = 1 \\ &\beta_1 = 1 - \beta_2 \triangleq \beta_{HR}\end{aligned} \qquad \mathrm{Cov}(N_1, N_2) = \begin{pmatrix} d_1 & \rho_{HR}\sqrt{d_1 d_2} \\ \rho_{HR}\sqrt{d_1 d_2} & d_2 \end{pmatrix}, \tag{18}$$

where ρ_{HR} is given in (11), and $\beta_{HR} = (d_2 + |\rho_{HR}|\sqrt{d_1 d_2})/(d_1 + 2|\rho_{HR}|\sqrt{d_1 d_2} + d_2)$ (see [10, 5]). We use this set of parameters in proving the asymptotic inner bound below.

Theorem 3 *Under the conditions of Theorem 1, the set of achievable rate pairs (R_1, R_2) satisfies*

$$\mathcal{R}_X(d_0, d_1, d_2) \mathrel{\tilde{\supset}} \mathcal{R}^* \left(P_X 2^{2E\{\log w(X)\}}; d_0, d_1, d_2\right) . \tag{19}$$

Proof To make the proof more transparent we assume the simplified form of an input weighted distortion measure $\rho(x, y) = w^2(x)(y - x)^2$. Let us substitute the following auxiliary random variables in the El Gamal – Cover inner bound (15):

$$\begin{aligned}
\widehat{X}_1 &= U = X + \frac{N_1}{w(X)} \\
\widehat{X}_2 &= V = X + \frac{N_2}{w(X)}, \\
\widehat{X}_0 &= \beta_{HR} U + (1 - \beta_{HR})V = X + \frac{\beta_{HR} N_1 + (1 - \beta_{HR}) N_2}{w(X)}
\end{aligned} \tag{20}$$

where (N_1, N_2) is bivariate Gaussian, independent of X, and has covariance matrix given in (18). It is easy to verify that these choices of $\widehat{X}_0$, $\widehat{X}_1$, and $\widehat{X}_2$ satisfy the distortion constraints in (15d). Thus, (15a)-(15c) describe an inner bound for $\mathcal{R}_X(d_0, d_1, d_2)$, which includes all rate pairs (R_1, R_2) satisfying

$$R_1 \geq I\left(X; X + \frac{N_1}{w(X)}\right) \tag{21a}$$

$$R_2 \geq I\left(X; X + \frac{N_2}{w(X)}\right) \tag{21b}$$

$$R_1 + R_2 \geq I\left(X; X + \frac{N_1}{w(X)}, X + \frac{N_2}{w(X)}\right) + I\left(X + \frac{N_1}{w(X)}; X + \frac{N_2}{w(X)}\right) \tag{21c}$$

$$= I\left(X; X + \frac{N_1}{w(X)}\right) + I\left(X; X + \frac{N_2}{w(X)}\right) + I\left(\frac{N_1}{w(X)}; \frac{N_2}{w(X)} \middle| X\right) \tag{21d}$$

$$= I\left(X; X + \frac{N_1}{w(X)}\right) + I\left(X; X + \frac{N_2}{w(X)}\right) + \frac{1}{2}\log\left(\frac{1}{1 - \rho_{HR}^2}\right). \tag{21e}$$

Here ρ_{HR} is given in (11), equation (21d) follows from the identity

$$I(A; BC) = I(A; B) + I(A; C) + I(B; C|A) - I(B; C),$$

and in (21e) we used the fact that $I\left(\frac{N_1}{w(X)}; \frac{N_2}{w(X)} \middle| X\right) = I(N_1; N_2)$ (since $w(x)$ is positive for all x), and then applied the formula for the mutual information between correlated Gaussian random variables. It can be shown (see [8]) that as $d_i \to 0$,

$$I\left(X; X + \frac{N_i}{w(X)}\right) = \frac{1}{2}\log(P_X/d_i) + E\{\log w(X)\} + o(1).$$

Since $\frac{1}{2}\log(P_X/d_i) = R^*(P_X; d_i)$ and $\delta_{HR} = -\frac{1}{2}\log(1-\rho_{HR}^2)$, substitution into (21) gives approximately the high resolution Gaussian rate region of (13) and the theorem follows. □

V. Companding Model for Multiple Description

The proof of the direct theorem in the previous section has the additional benefit of suggesting how to realize an asymptotically optimal multiple description coding scheme. We note first that as $d_i \to 0$, the right hand sides in (21a) and (21b) can be shown to satisfy

$$I\left(X; X + \frac{N_i}{w(X)}\right) \approx I(q(X); q(X) + N_i),$$

where $q(x)$ is defined by $q(x) = \int_0^x w(t)\,dt$ (see [8, 9]). Furthermore, as $d_i \to 0$,

$$q^{-1}\Big(q(X) + N_i\Big) \approx X + N_i/w(X)\ .$$

Thus, the realization of $\mathcal{R}^*\left(P_X 2^{2E\{\log w(X)\}}; d_0, d_1, d_2\right)$ in (20) is equivalent to passing the source X through the mapping $q(\cdot)$, then applying the quadratic-Gaussian multiple description test channel of (16), and finally passing the reconstructions through the inverse mapping $q^{-1}(\cdot)$.

This information-theoretic realization suggests that in practice, a combination of a compressor $q(\cdot)$, a multiple description quantizer for the squared error, and an expander $q^{-1}(\cdot)$, can approach the optimal performance for locally quadratic distortion measures $\rho(x,\hat{x}) \approx w^2(x)(\hat{x}-x)^2$, for high quantizer dimension and small distortion. The work of the authors [9] on entropy-coded companding quantization for non-difference distortion measures further supports the feasibility of such a simple, modular solution.

From linder@code.ucsd.edu Fri Jan 9 15:34:44 1998 Date: Thu, 8 Jan 1998 13:59:55 -0800 (PST) From: Tamas Linder ¡linder@code.ucsd.edu¿ To: zamir@eng.tau.ac.il Subject: dcc98.bbl

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