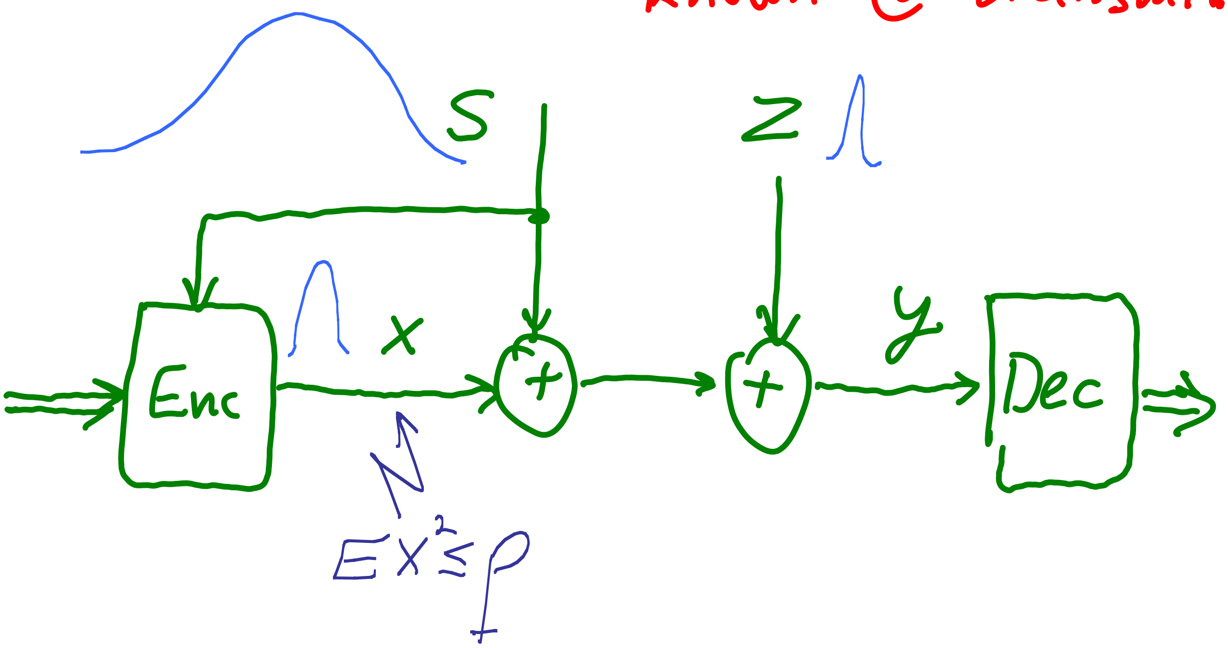
Writing on Dirty Paper with Lattice Codes

Rami Zamir @ ETH

"Writing on Dirty Paper"

(channel coding with Interference known @ transmitter)



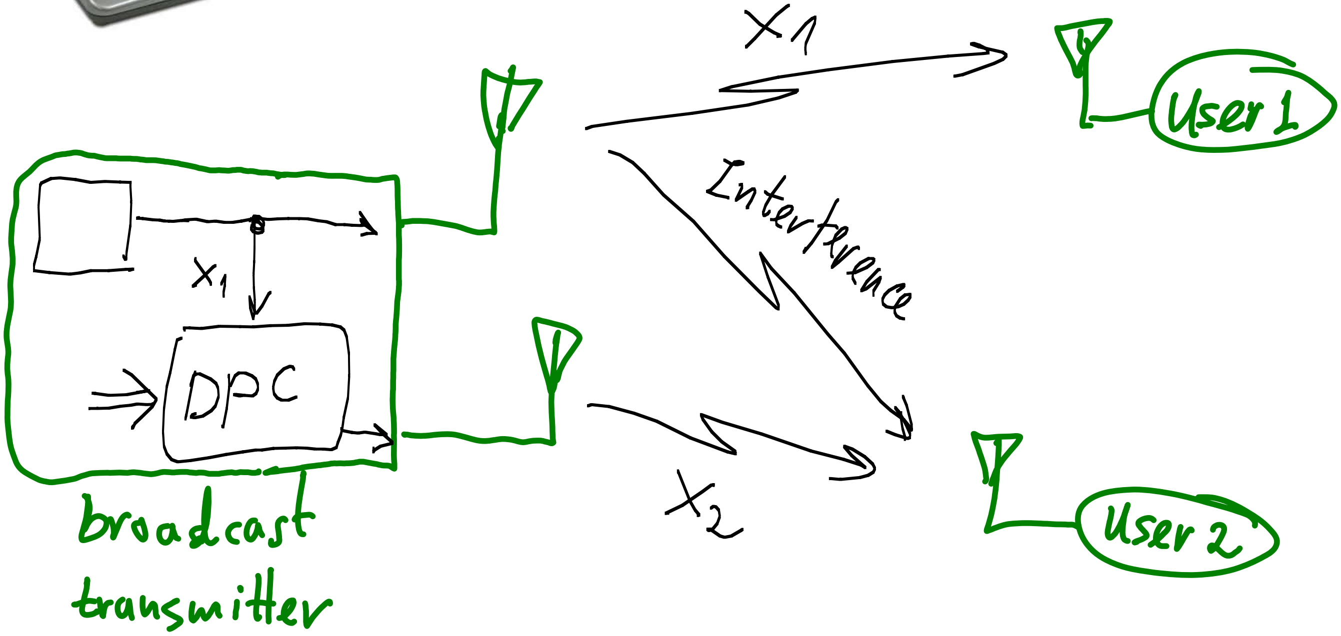
$$C_{S1@Tx} = \frac{1}{2}\log\left(1+\frac{P}{\sigma_z^2}\right) = C_{AWGN}$$

Gelfand-Pinsker 1980

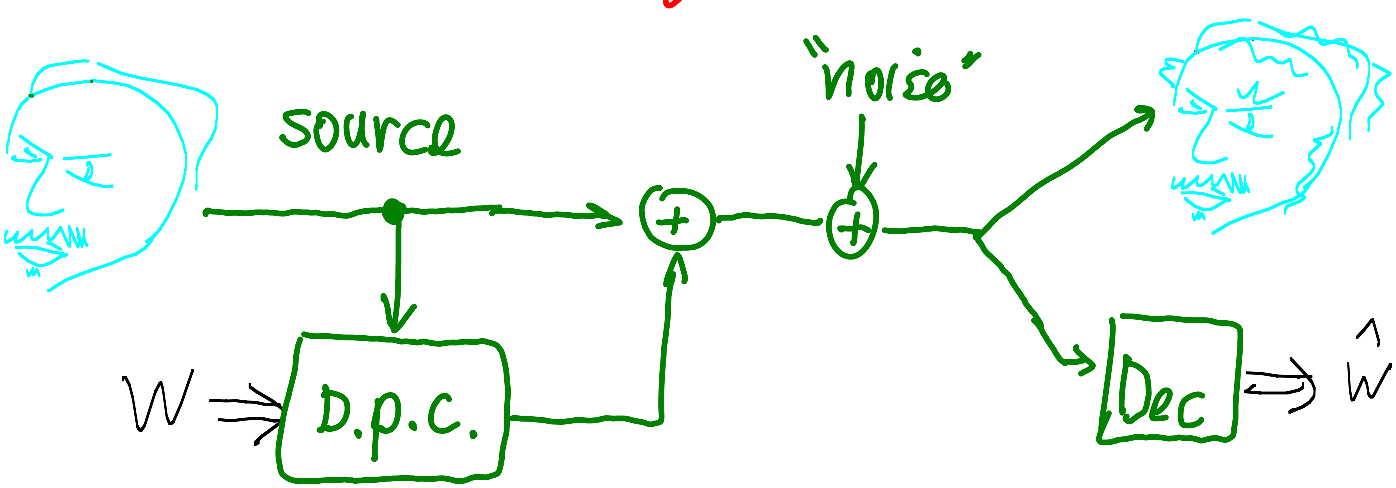
Costa 1983

Surprising: interference cancellation with no power penalty?

MIMO - Broadcast using D.P.C

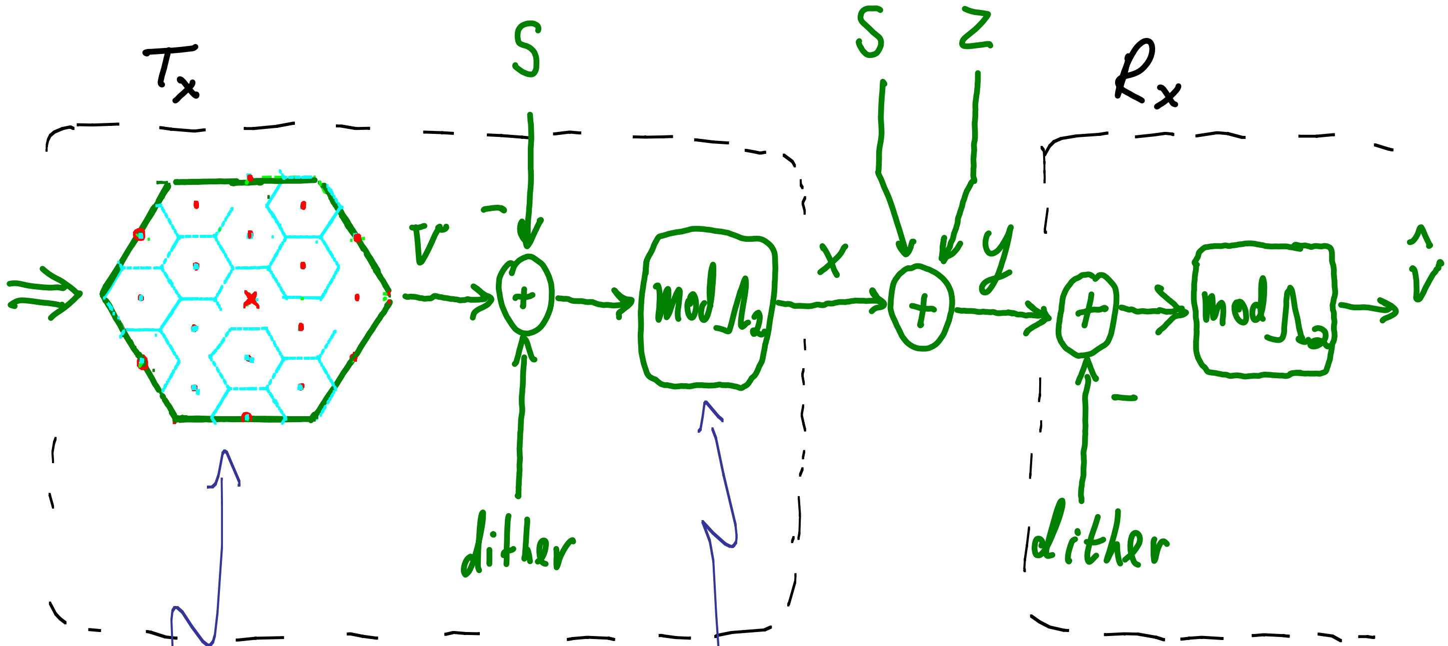


Information Embedding ("Watermarking")



Lattice Dirty Paper Coding

[Erez Shamai & Z]

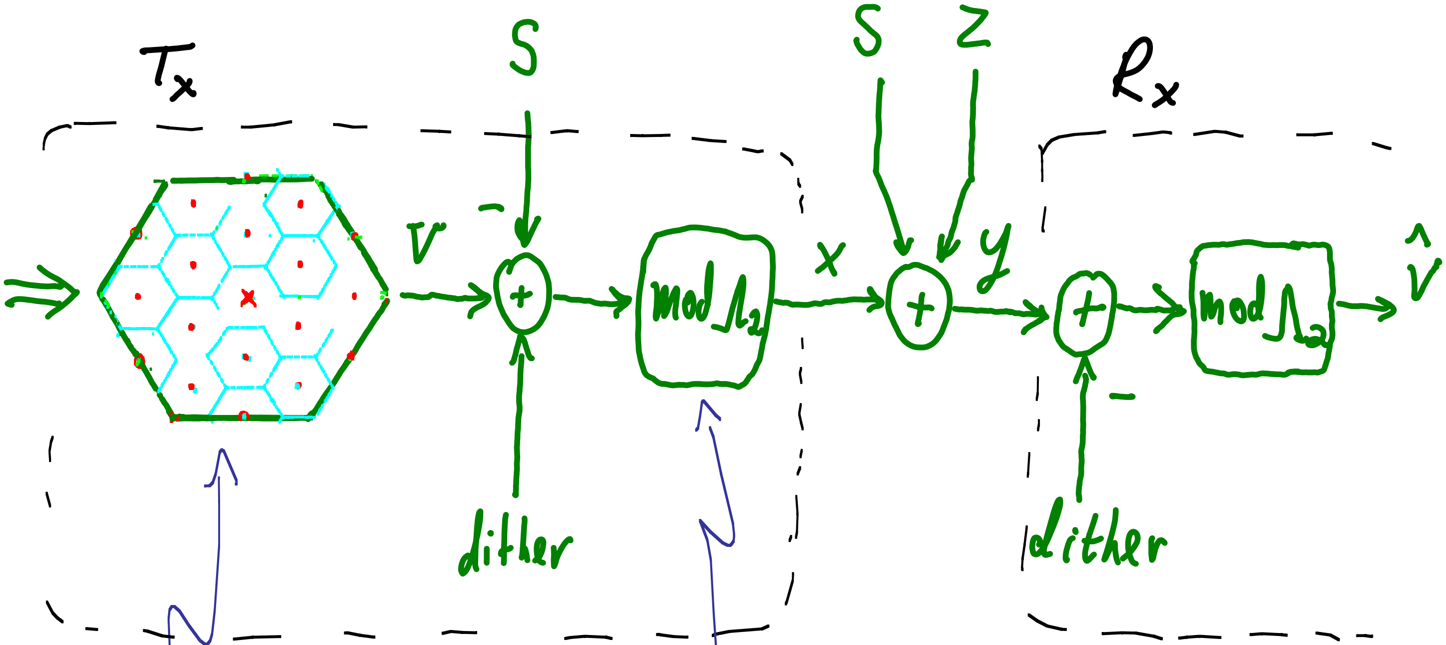


Λ_1/Λ_2
Voronoi
Constellation

Λ_2 = Good quantizer
$\sigma^2(\Lambda_2) = P$

Λ_1 = good channel
code for $N(0, \sigma_Z^2)$

Lattice Dirty Paper Coding



Λ_1 / Λ_2
Voronoi
Constellation

$\Lambda_1 =$ good channel code for $N(0, \sigma_Z^2)$

Good quantizer
$\sigma^2(\Lambda_2) = P$ + dither

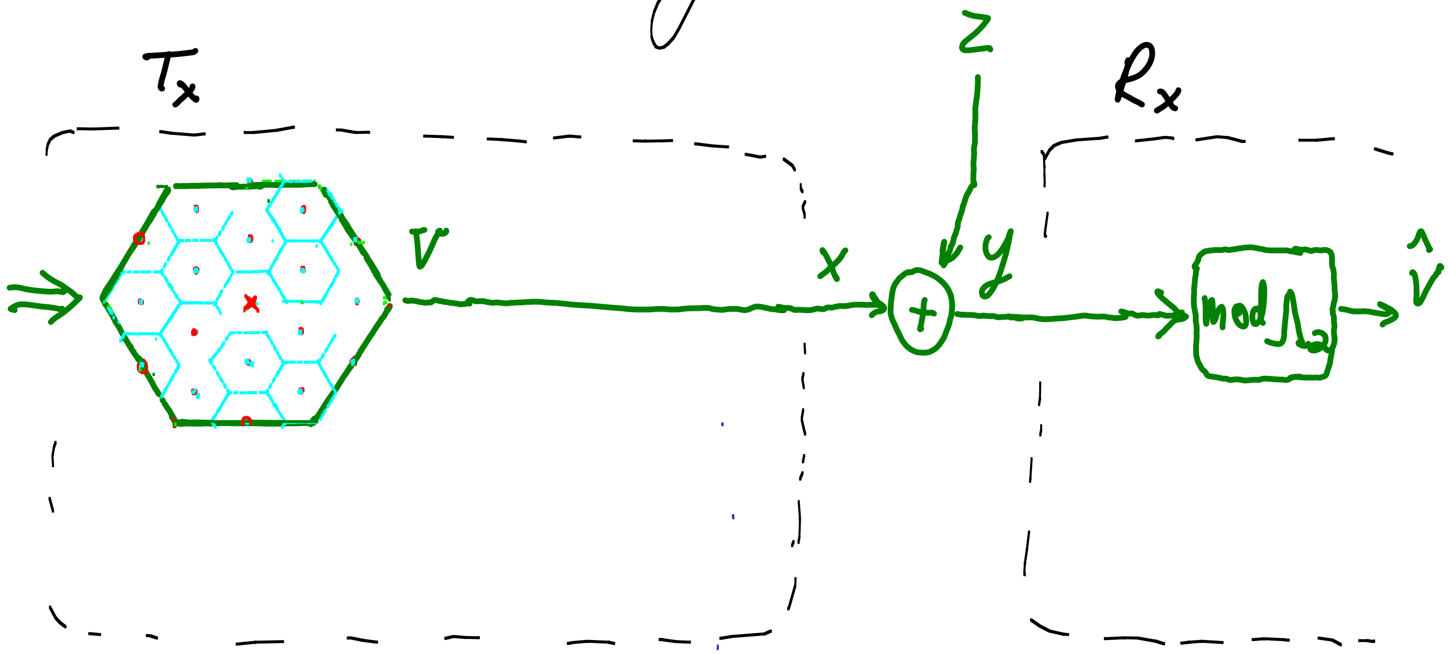
$\Downarrow$

$E \, \frac{1}{k} \|X\|^2 = P$

for any codeword!

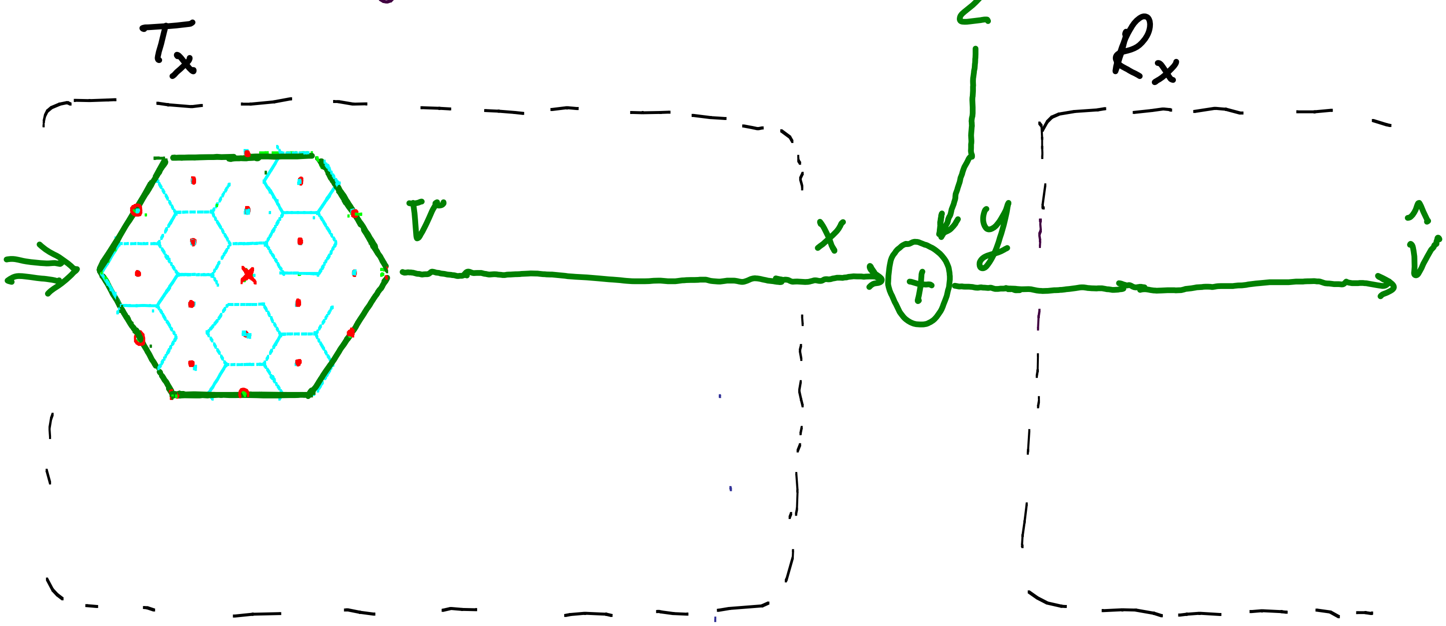
Lattice Dirty Paper Coding

Modulo property $\Rightarrow$



Lattice Dirty Paper Coding

Λ_1 = good for $N(0, \sigma_z^2)$ $\Rightarrow$ $P_e < \varepsilon \ \forall v$



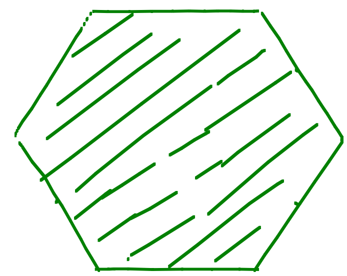
$$\text{Rate} = \frac{1}{k} \log\left(\frac{V_2}{V_1}\right)$$

$$= \text{function}\{\Lambda_1, \Lambda_2, \sigma_z^2\} = ?\ldots$$

P_e P

Lattice Figures of Merit for Coding

- **Quantization efficiency:**



$\underline{X} \sim \text{Uniform}(V_0)$

$\sigma^2(\Lambda) \triangleq \frac{1}{k} E\|X\|^2$

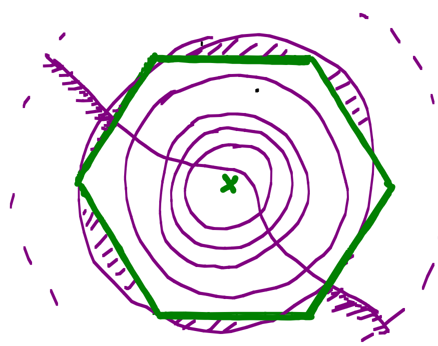
$$G(\Lambda) \triangleq \frac{\sigma^2(\Lambda)}{V^{2/k}}$$ = normalized second moment

good lattices $\Rightarrow$ $G(\Lambda_k) \xrightarrow[k\to\infty]{} \frac{1}{2\pi e}$

- **AWGN coding efficiency:** $\underline{Z} \sim \text{AWGN } N(0, \sigma^2)$

$P_e \triangleq \Pr\{\underline{Z} \notin V_0\}$

= "Poltyrev's error prob."

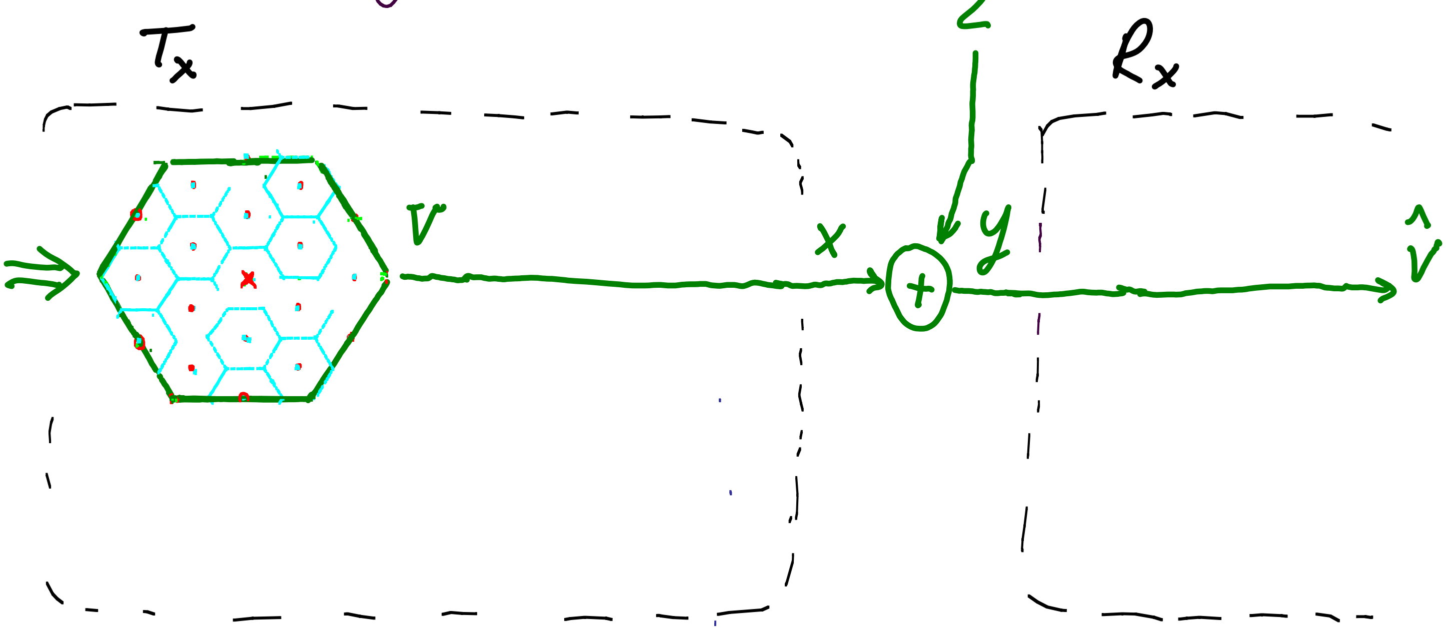


$$\mu(\Lambda, P_e) \triangleq \frac{V^{2/k}}{\sigma^2}\Big|_{@P_e}$$ = Volume-to-Noise Ratio

good lattices $\Rightarrow$ $\mu(\Lambda_k, P_e) \xrightarrow[k\to\infty]{} 2\pi e$

Lattice Dirty Paper Coding

Λ_1 = good for $\mathcal{N}(0, \sigma_z^2) \Rightarrow P_e < \varepsilon \; \forall v$



$$\text{Rate} = \frac{1}{k}\log\left(\frac{V_2}{V_1}\right)$$

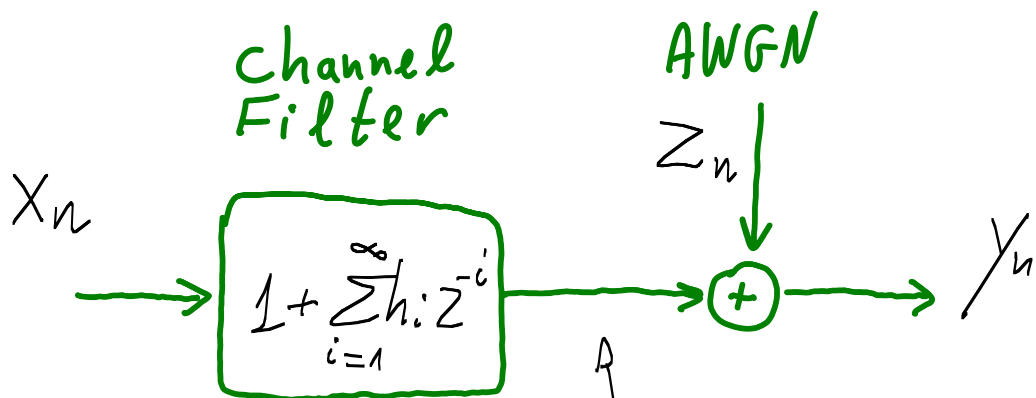
$$= \underbrace{\frac{1}{2}\log\left(\frac{P}{\sigma_z^2}\right)}_{\text{AWGN capacity @ High SNR}} - \underbrace{\frac{1}{2}\log\left(\overset{\Lambda_2}{G_k}\cdot\overset{\Lambda_1}{\mu_k}\right)}_{\text{capacity Loss}\; \xrightarrow[k\to\infty]{} 0}$$

From Dirty-Paper Coding

to

Equalization ...

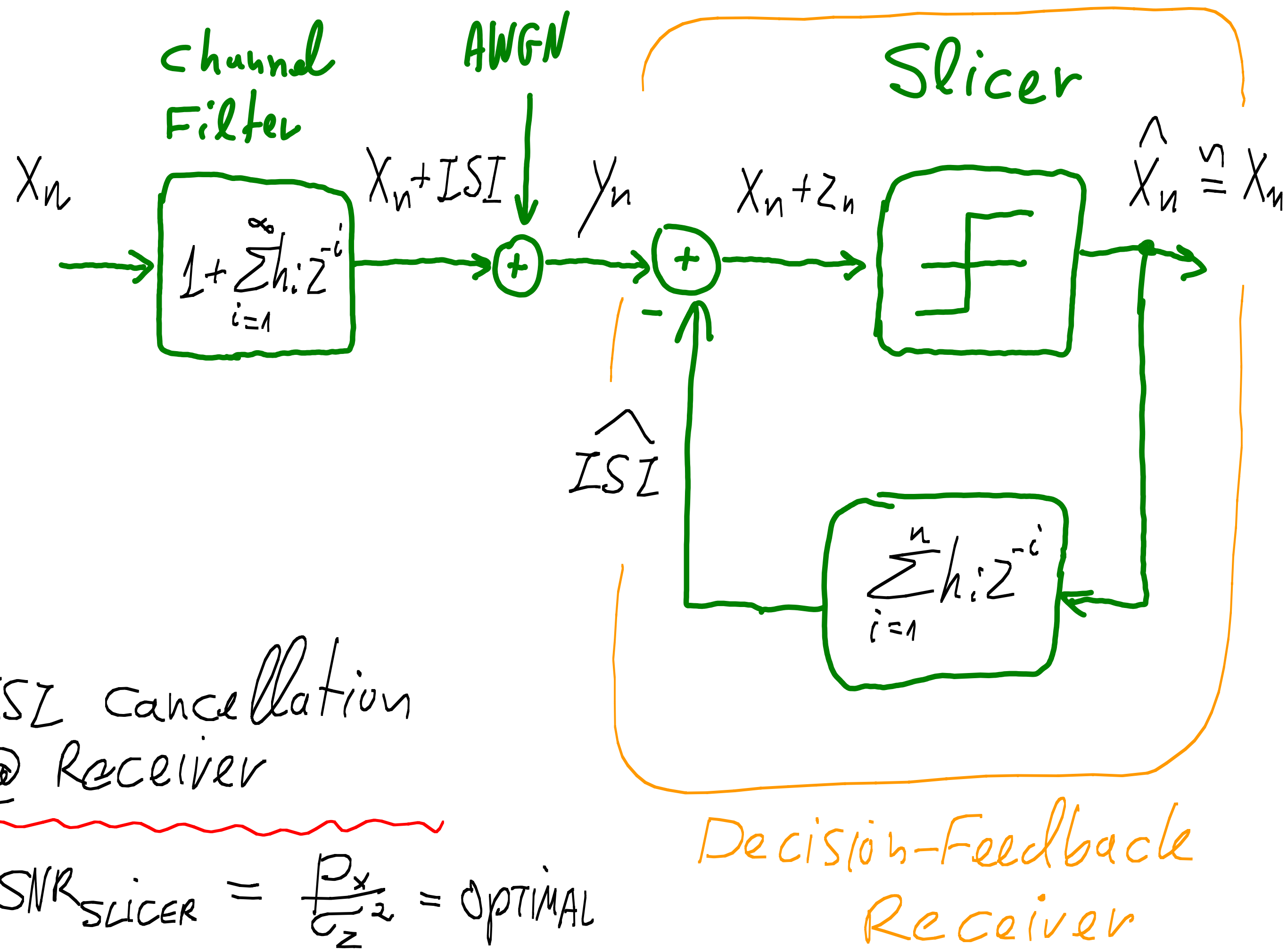
Inter-Symbol Interference Channel
&
Decision-Feedback Equalization



$$X_n + \underbrace{\sum_{i=1}^{\infty} h_i X_{n-i}}_{ISI}$$

$$\Rightarrow \quad Y_n = X_n + ISI + Z_n$$

Inter-Symbol Interference Channel & Decision-Feedback Equalization



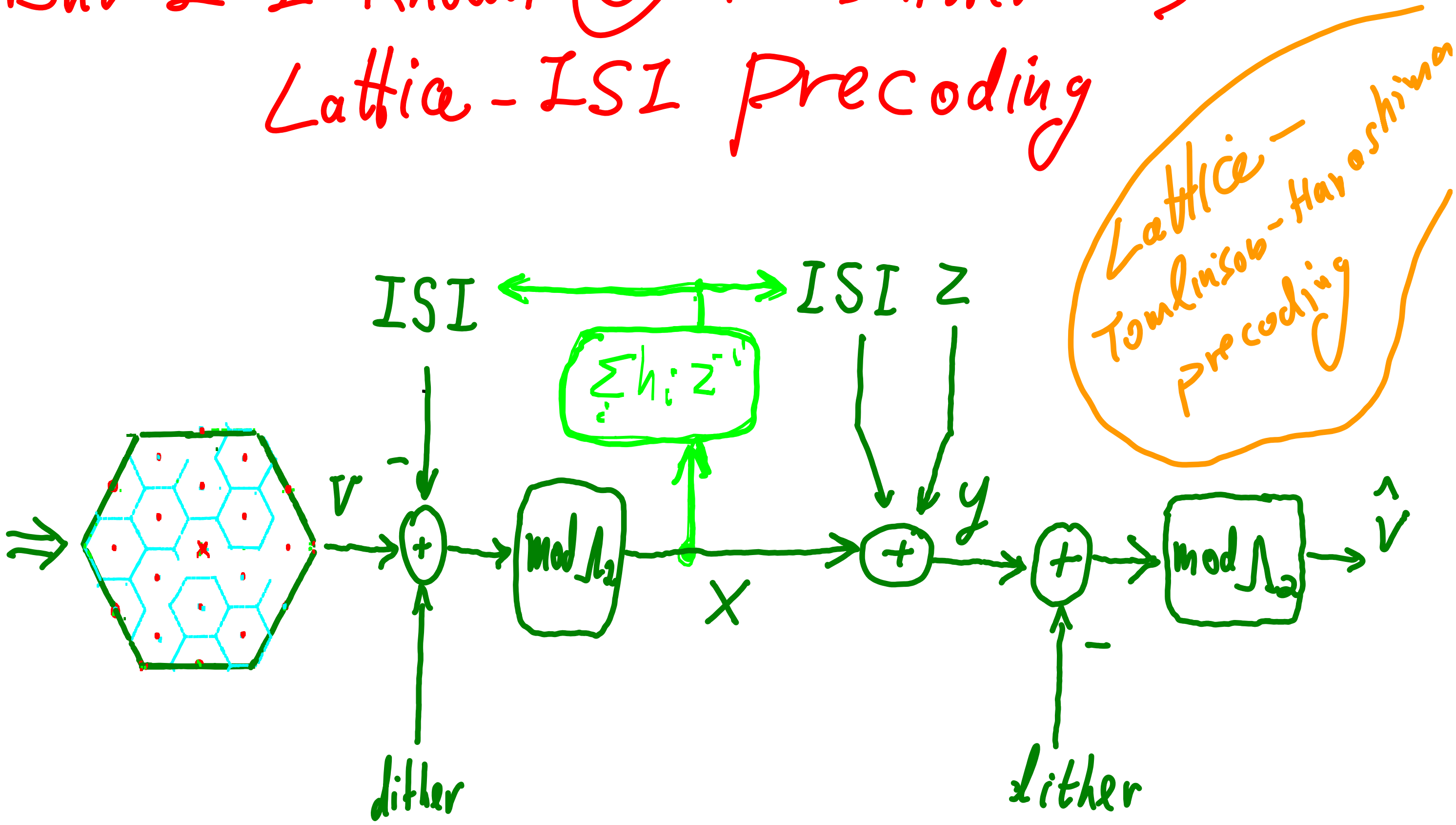
ISI cancellation @ Receiver

⇒ $SNR_{SLICER} = \frac{P_x}{\sigma_z^2}$ = OPTIMAL

⇒ Error Propagation

⇒ Zero-delay decoding

But ISI Known @ Transmitter ⇒ Lattice-ISI Precoding



- ISI Cancellation @ Transmitter:
  ⇒ no error propagation
- mod-Λ subtraction:
  ⇒ no power amplification
- Nested Lattices $\Lambda_1 \supset \Lambda_2$:
  ⇒ $\text{Rate} = \frac{1}{2}\log\left(\frac{P_x}{\sigma_z^2}\right) - \frac{1}{2}\log\left(G(\Lambda_2)\cdot\mu(\Lambda_1)\right)$

From the

Dirty Paper Channel

to the plain old

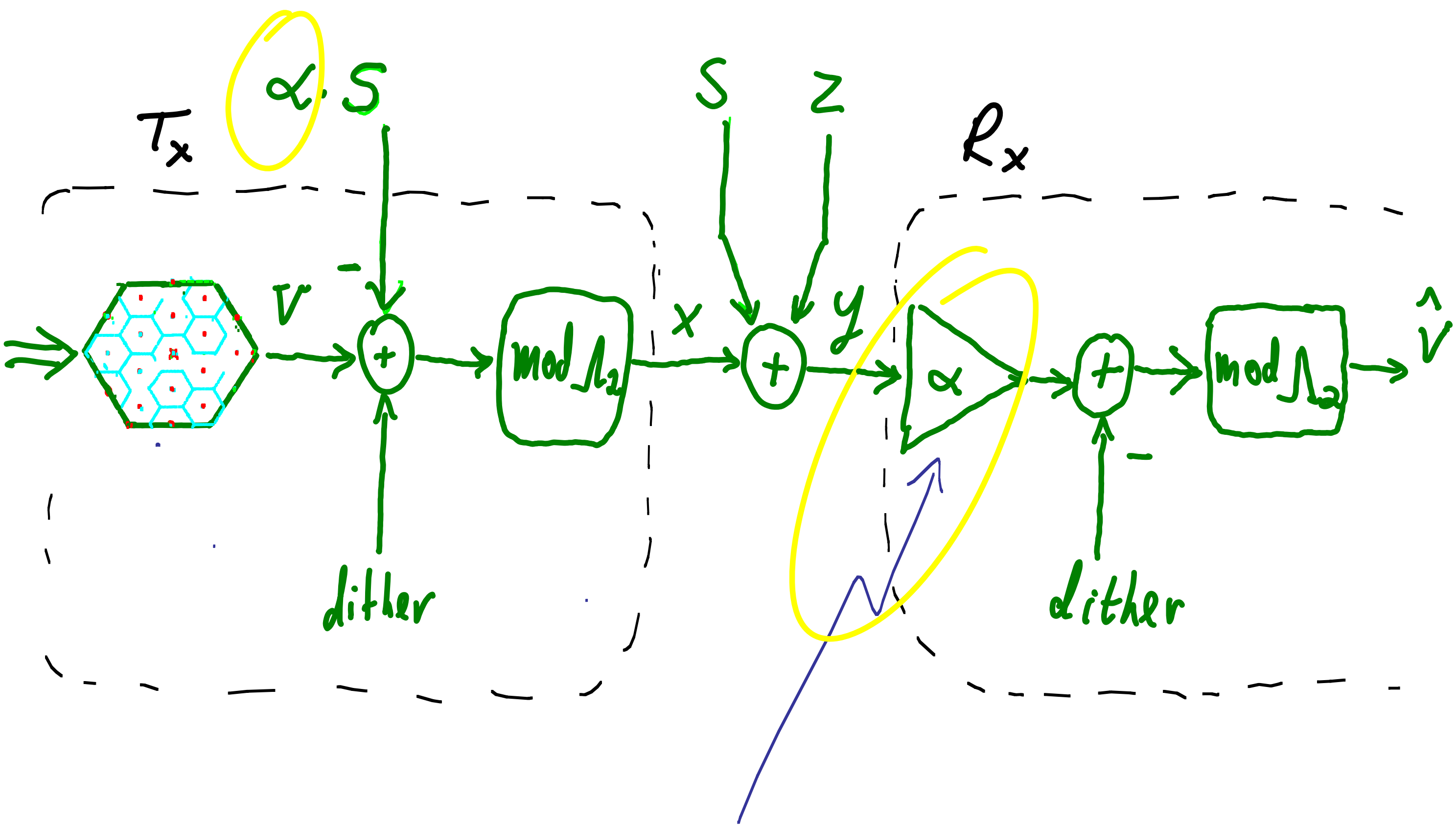
AWGN Channel ...

+

Extension to general SNR

Achieving $\frac{1}{2}\log(1+SNR)$ @ general SNR

$(SNR = P/\sigma_z^2)$

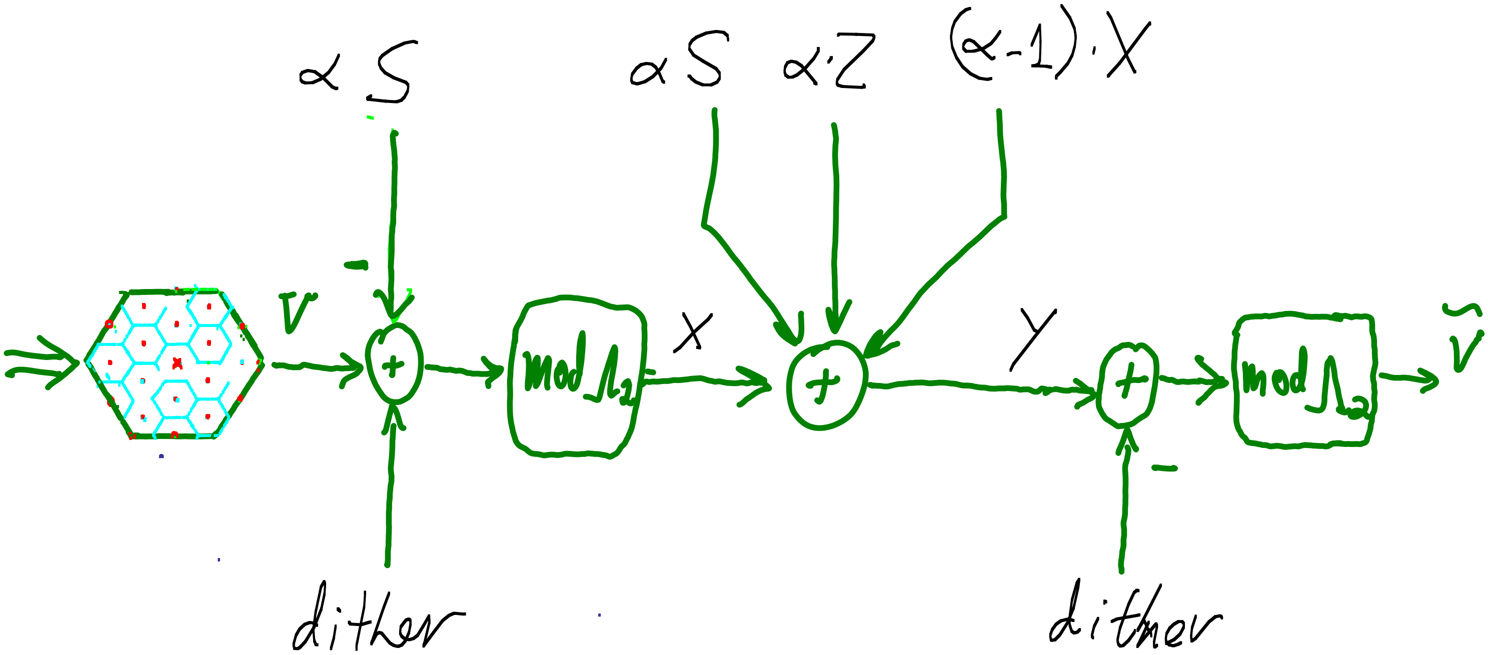


We shall see later that a good choice for α is:

$$\alpha = \text{MMSE (Wiener) Coefficient}$$

$$= \frac{P}{P+\sigma_z^2} \simeq 1 \text{ @ HSNR}$$

Achieving $\frac{1}{2}\log(1+SNR)$ @ general SNR



$$Y = X + S + Z \quad \& \quad X = [V - \alpha S + \text{dither}] \bmod \Lambda_2$$

$$\Rightarrow \alpha Y = X + \alpha S + \alpha Z + (\alpha-1)\cdot X$$

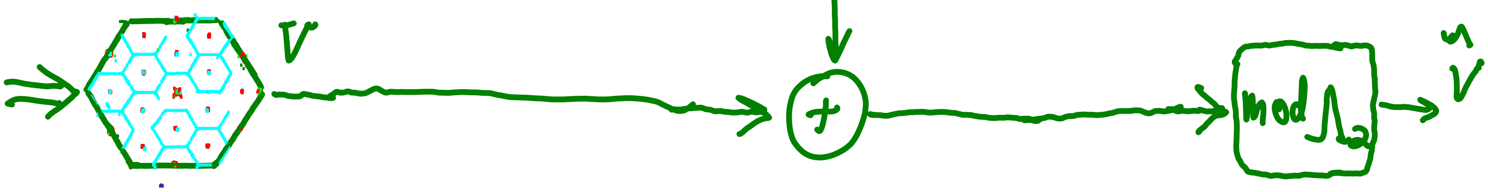
$$\tilde{V} = [\alpha Y - \text{dither}] \bmod \Lambda_2$$

$$= [V + \underbrace{\alpha Z + (\alpha - 1) X}_{\triangleq Z_{eq}}] \bmod \Lambda_2$$

iterated Modulo property

Achieving $\frac{1}{2}\log(1+SNR)$ @ general SNR

$$Z_{eq} = \alpha\cdot Z + (\alpha-1)\cdot X$$

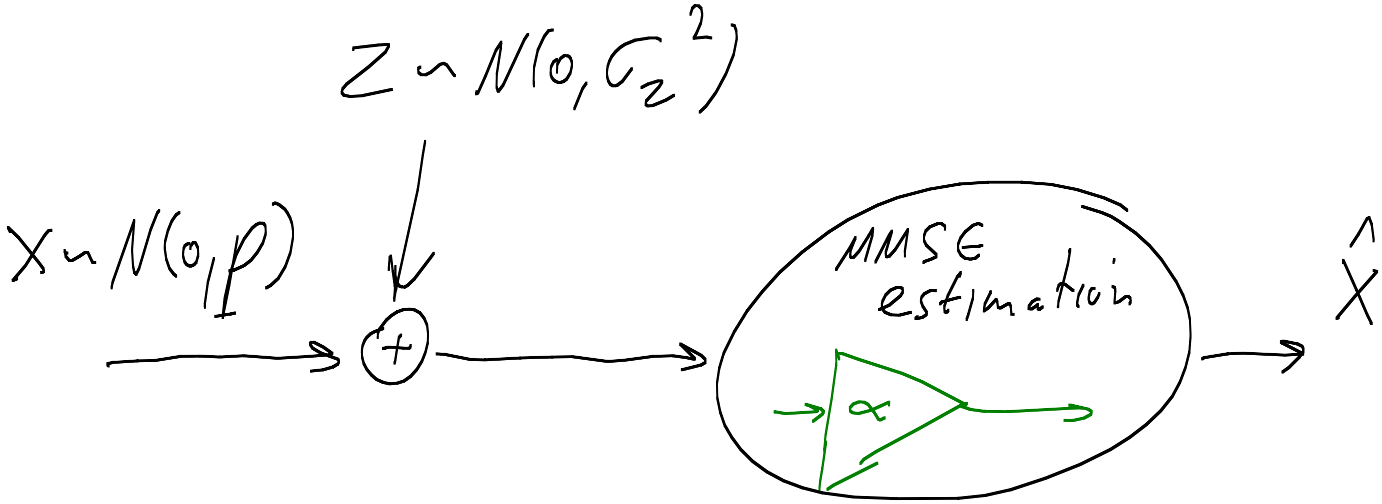


dithered quantization $\Rightarrow$ $V \perp\!\!\!\perp X$

$\Rightarrow$ $V \perp\!\!\!\perp Z_{eq}$

Equivalent Modulo-Λ Additive Noise Channel

How to Choose α ?
Wiener Estimation



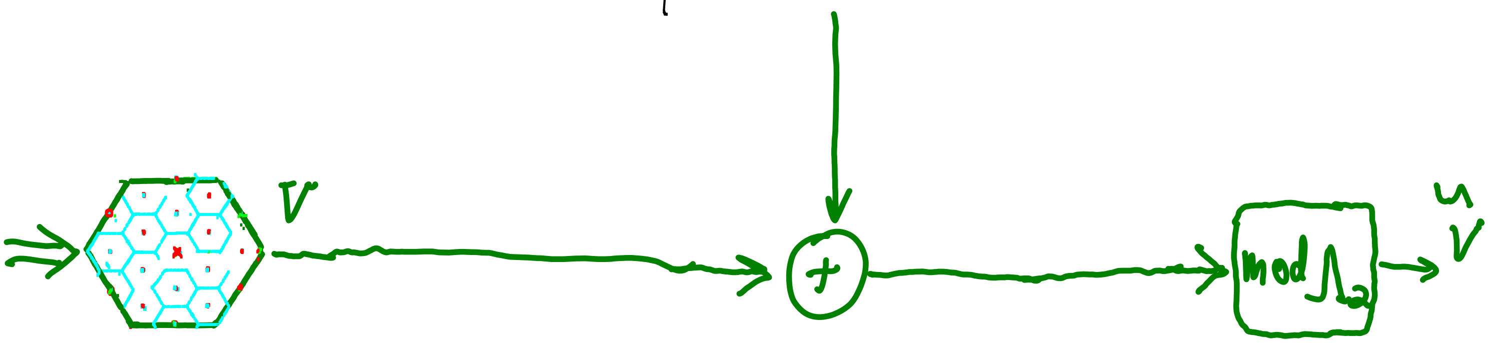
$$\min E(\hat{X} - X)^2 \Rightarrow \alpha^{opt} = \frac{P}{P + \sigma_z^2}$$

$$\Rightarrow MSE = \frac{P \cdot \sigma_z^2}{P + \sigma_z^2}$$

$$\text{error} = \underbrace{(\alpha - 1) \cdot X}_{\text{"self noise"}} + \underbrace{\alpha \cdot Z}_{\text{residual "natural" noise}}$$

Achieving $\frac{1}{2}\log(1+SNR)$ @ general SNR

$$Z_{eq} = \alpha \cdot Z + (\alpha - 1)\cdot X$$



Equivalent Modulo-Λ Additive Noise Channel

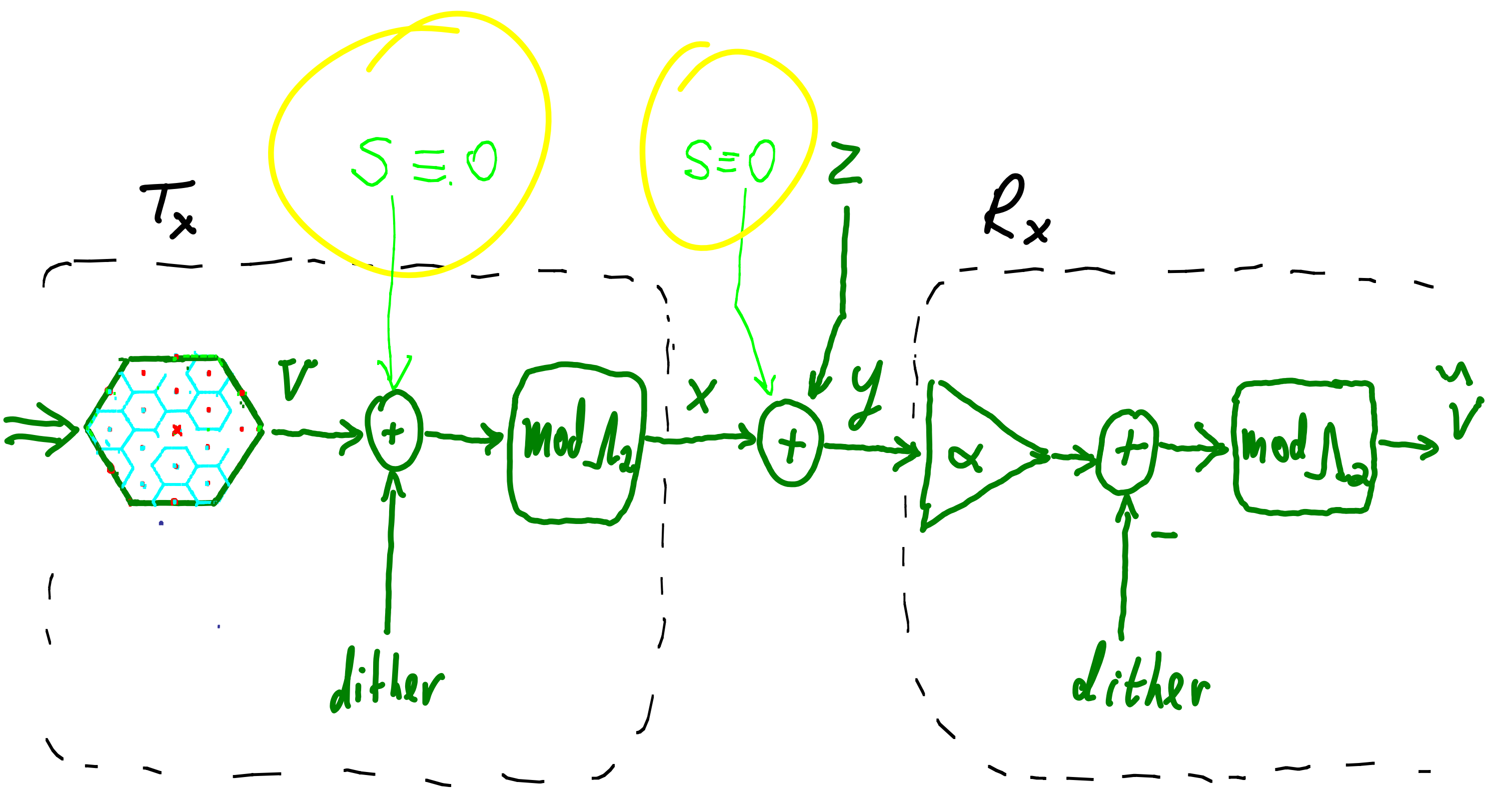
$\alpha = \alpha_{wiener}$

$$\Rightarrow Var\{Z_{eq}\} = \frac{P\sigma_z}{P+\sigma_z^2} \overset{?}{\sim} N\left(0, \frac{P\sigma_z^2}{P+\sigma_z^2}\right)$$

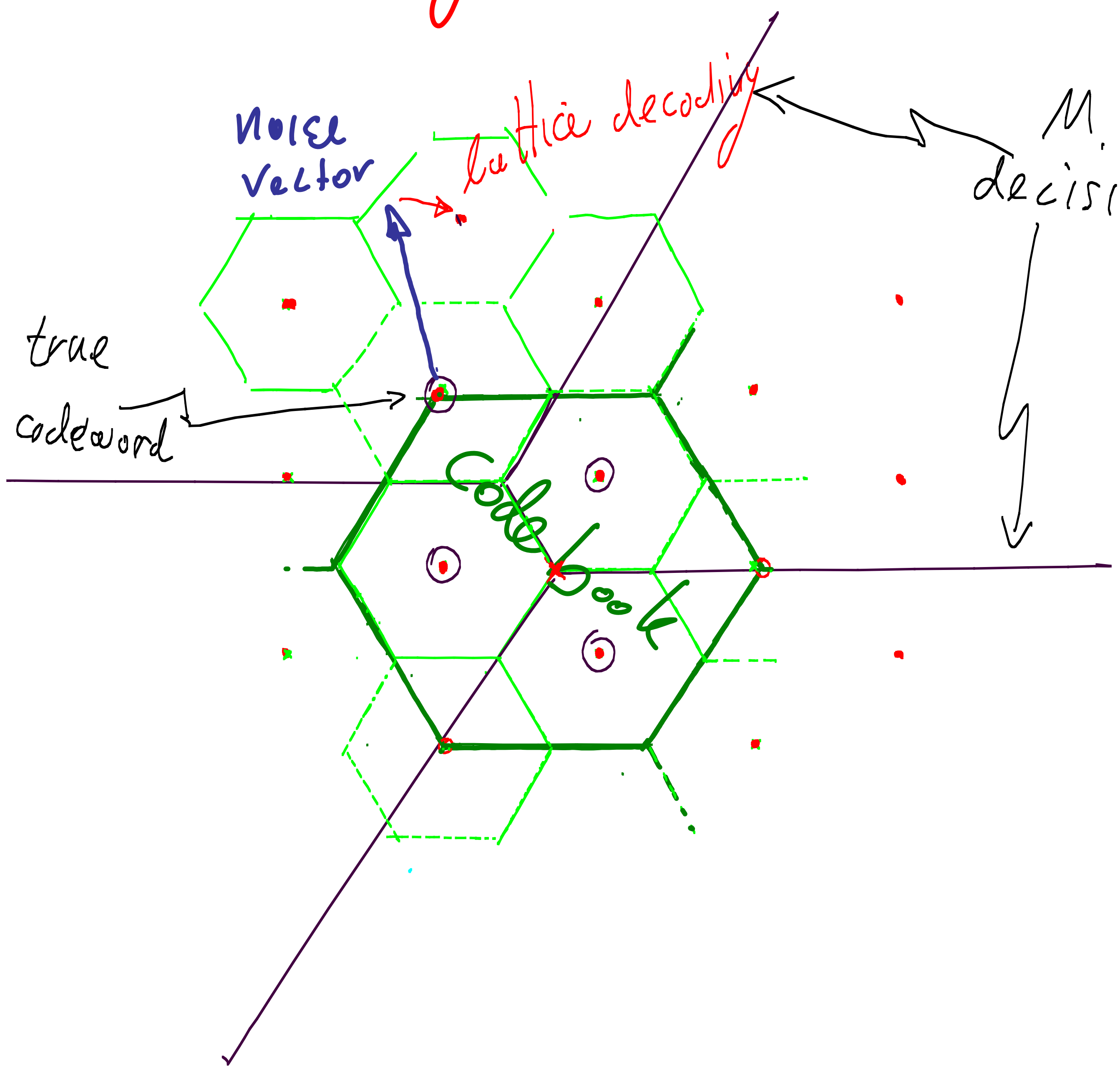
$$\Rightarrow Rate = \frac{1}{k}\log\left(\frac{V_2}{V_1}\right) \overset{?}{\simeq} \frac{1}{2}\log\left(1 + \frac{P}{\sigma_z^2}\right)$$

Achieving $\frac{1}{2}\log(1+SNR)$ over the AWGNC with Lattice Encoding & Decoding

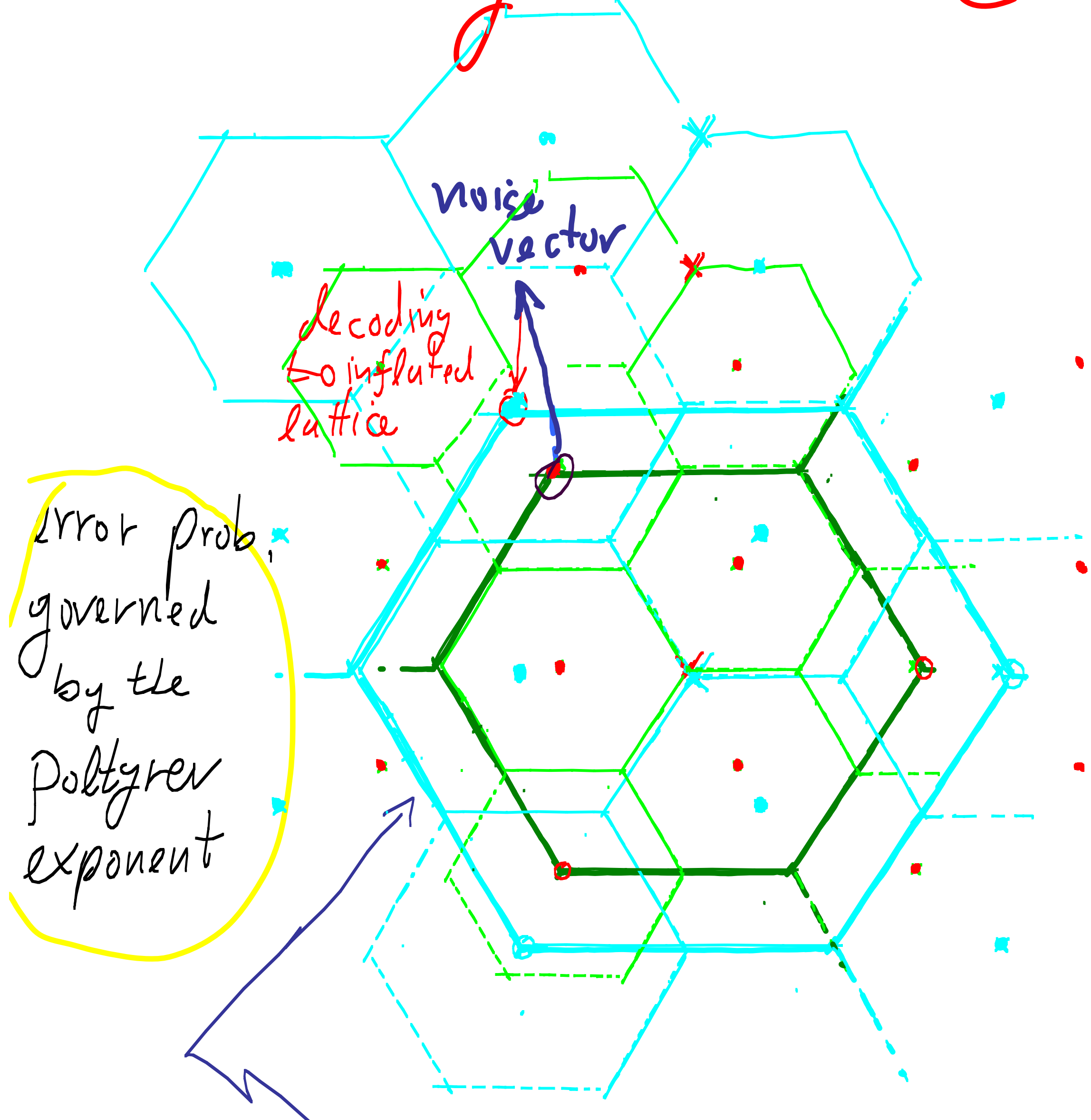
[Erez & Z]



Achieving $\frac{1}{2}\log(1+SNR)$ over the AWGNC with Lattice Encoding & Decoding: Why Not Obvios @ Low SNR...



Achieving $\frac{1}{2}\log(1+SNR)$ over the AWGNC with Lattice Encoding & Decoding: Why Not Obvious @ Low SNR...

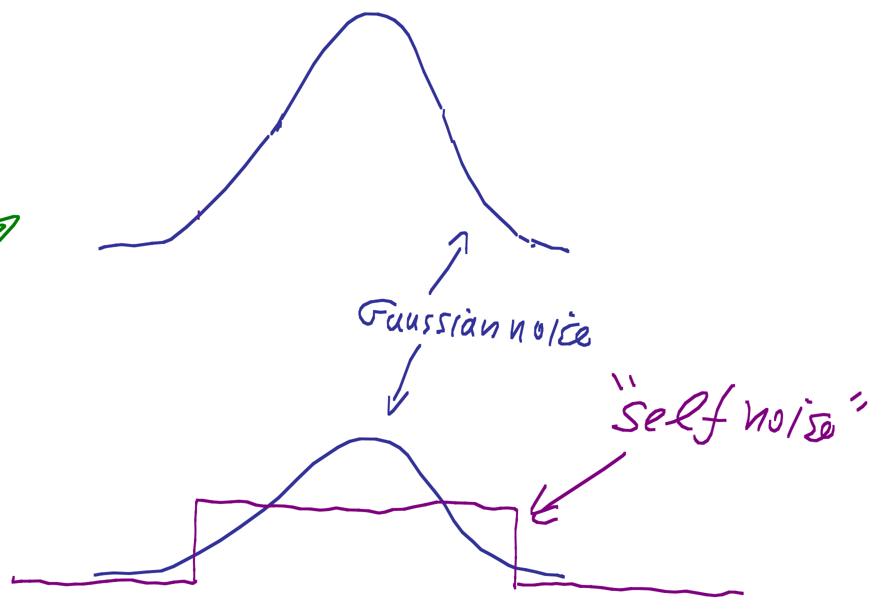


Lattice @ Decoder Inflated by $\frac{1}{\alpha}$

But Noise is Not Quite Gaussian ...

$\alpha = 1$

pure Gaussian

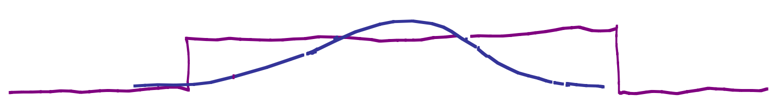


$\alpha = \alpha_{wiener}$

minimum-energy mixture

$\alpha < \alpha_{wiener}$

reduced-tail mixture



$\Rightarrow$ inflation-coefficient α affects equivalent noise distribution

Effect of α on Lattice-Decoding