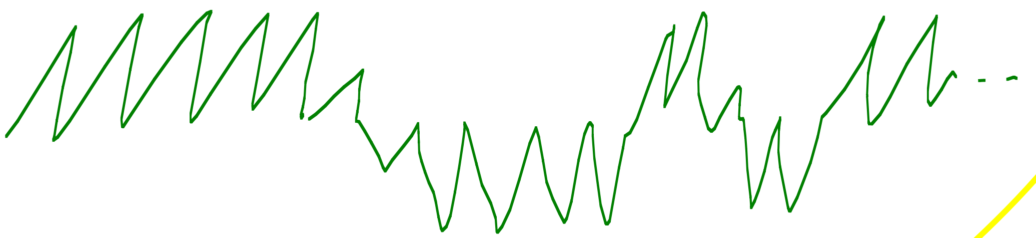
Lattice Modulation



Rami Zamir @ ETH

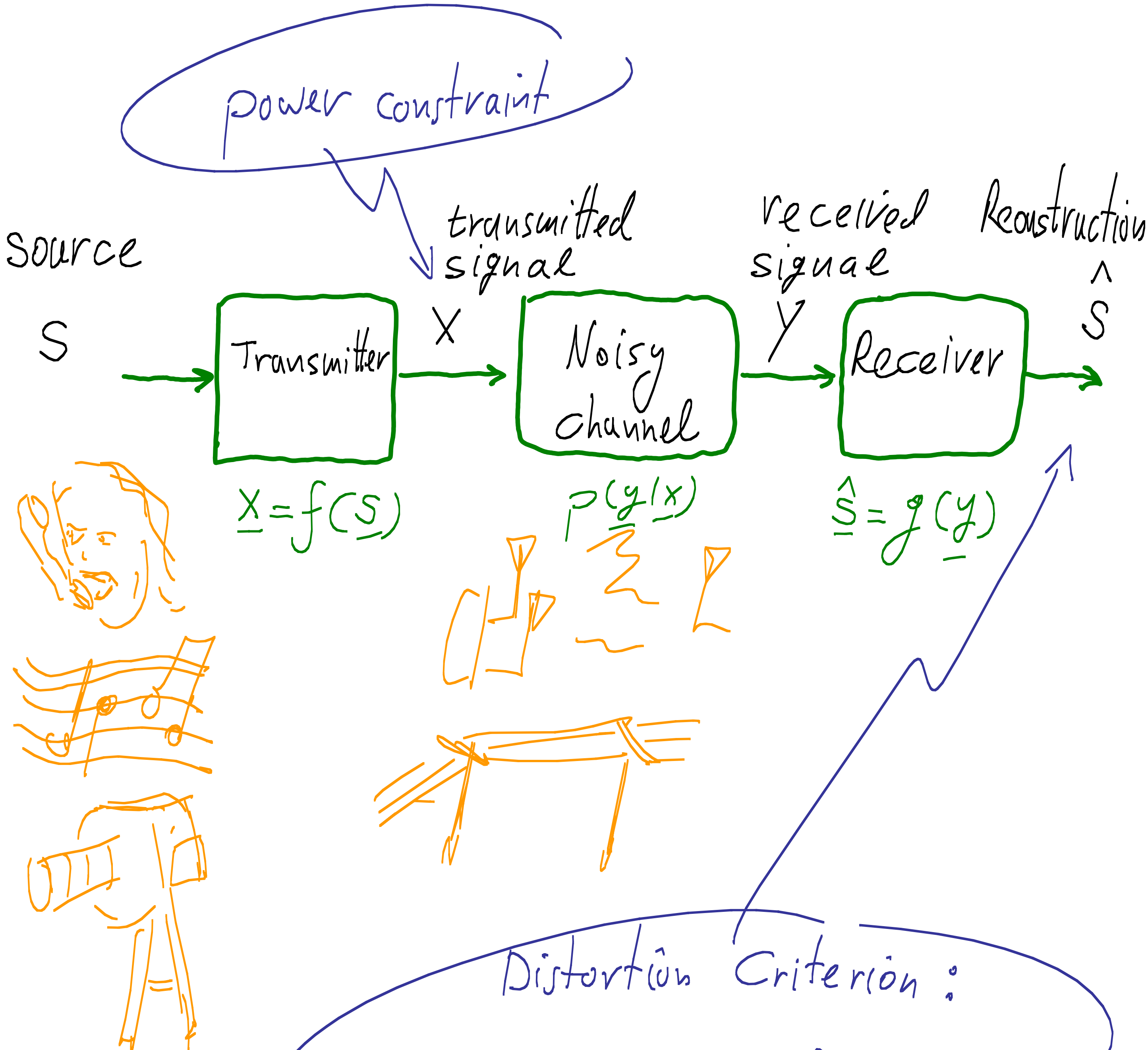
I

Shannon's Source/Channel Separation

&

Analog Modulation

Point-to-Point Communication Set-up



Distortion Criterion:

$$D = E(\hat{S} - S)^2$$

Optimum Performance Theoretically Attainable

$$D^{OPTA} = \inf \text{ Distortion}$$

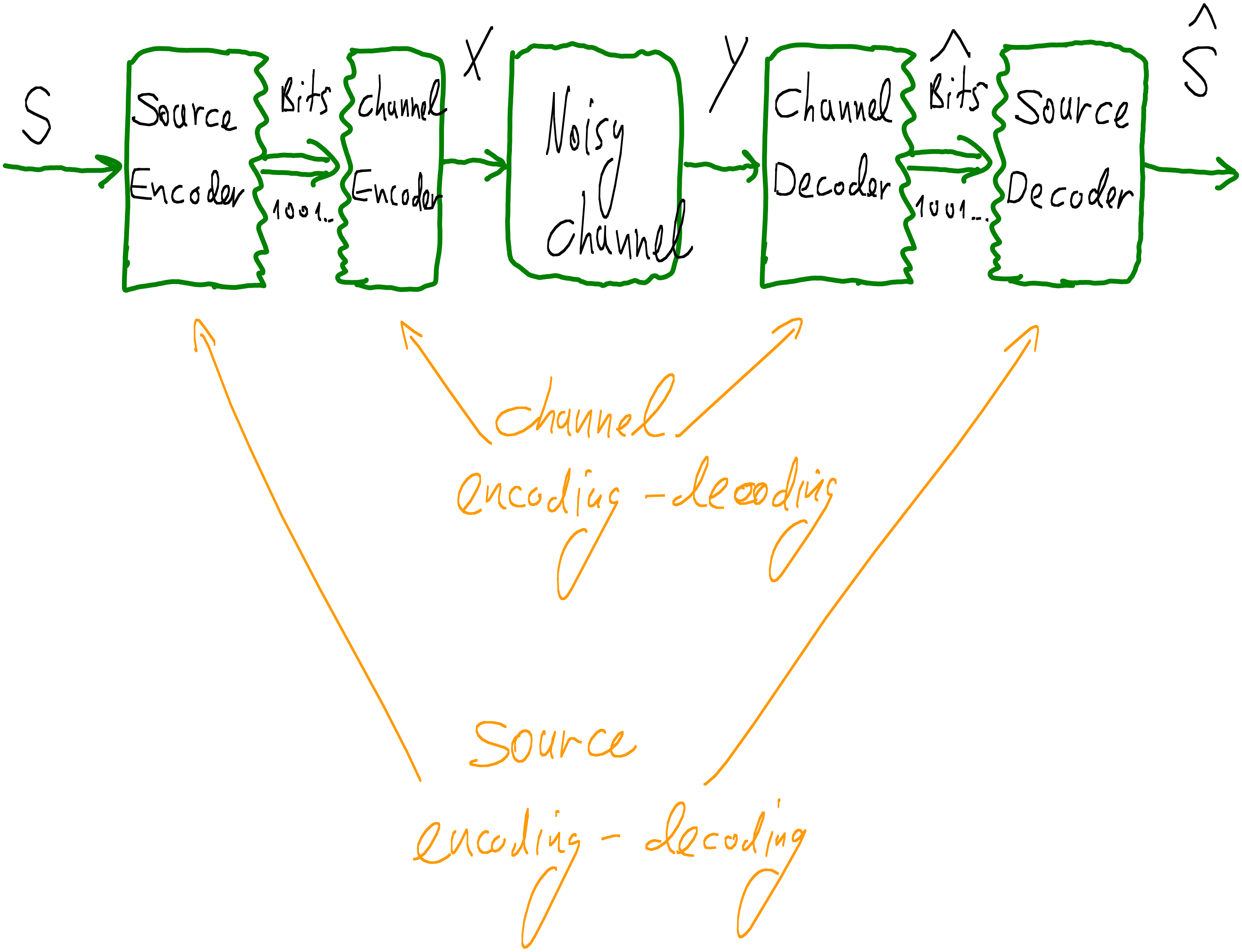
over all ----

- encoding-decoding mappings
- of any block length
- under the power constraint

unlimited system complexity

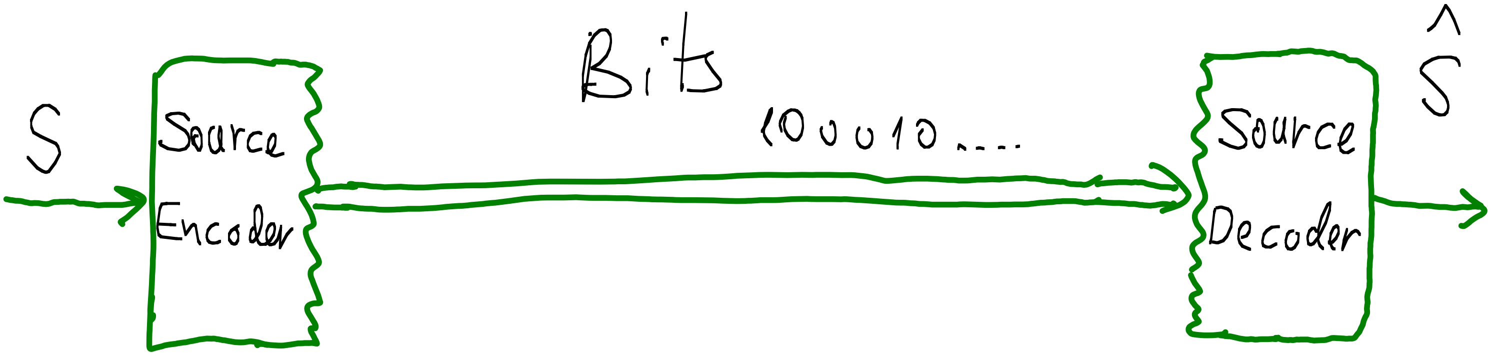
Separation Principle:

Digital Communication achieves OPTA !



Separation Principle:

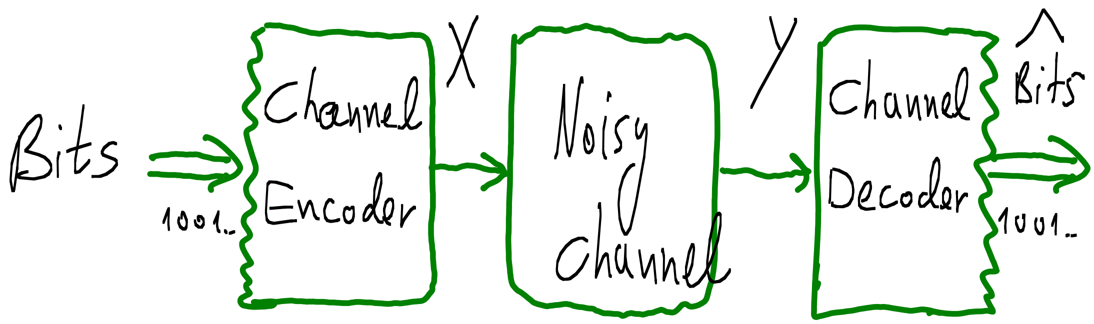
Source Compression



$$\text{Minimum Bit Rate} = \begin{cases} \text{Entropy}, & \text{@ lossless case} \\ \text{Rate-Distortion function}, & \text{@ lossy case} \end{cases}$$

function of source statistics
& distortion constraint

Separation Principle:
Channel Coding



$$\text{Maximum Bit Rate} = \text{Capacity}(\underline{p})$$

function of channel statistics
& input constraint

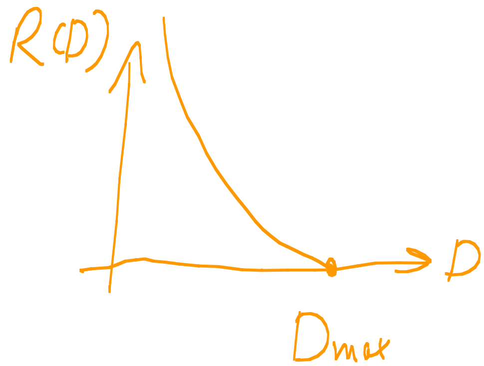
Joint Source/Channel Coding Theorem

In general

$$R(D) < R < C$$

rate-distortion function — Bit Rate — Capacity

But

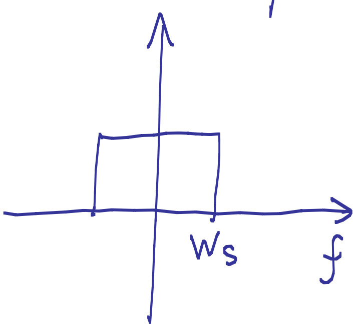


$\Rightarrow$ @ Optimum

$$\boxed{R(D^{OPTA}) = C}$$

Example: Quadratic - Gaussian

Source spectrum

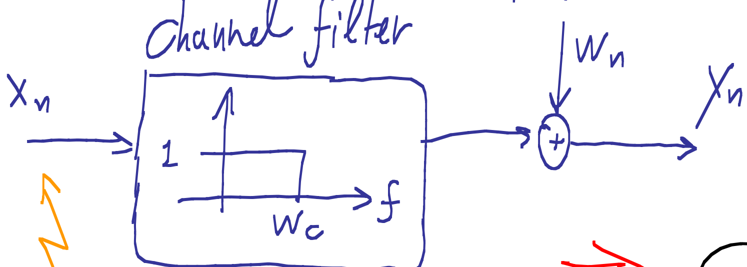


distortion measure = MSE

SNR_s

$$\Rightarrow \quad R(D) = W_s \cdot \log\left(\frac{\sigma_s^2}{D}\right)$$

bit per sample



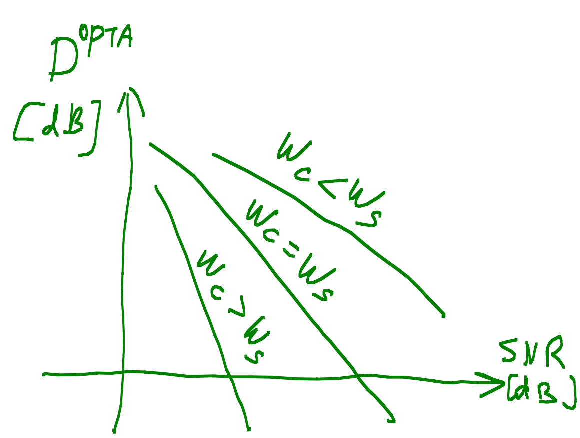
P = power constraint

$$SNR_c \triangleq \frac{P}{\sigma_w^2}$$

$$\Rightarrow \quad C = W_c \cdot \log(1 + SNR_c)$$

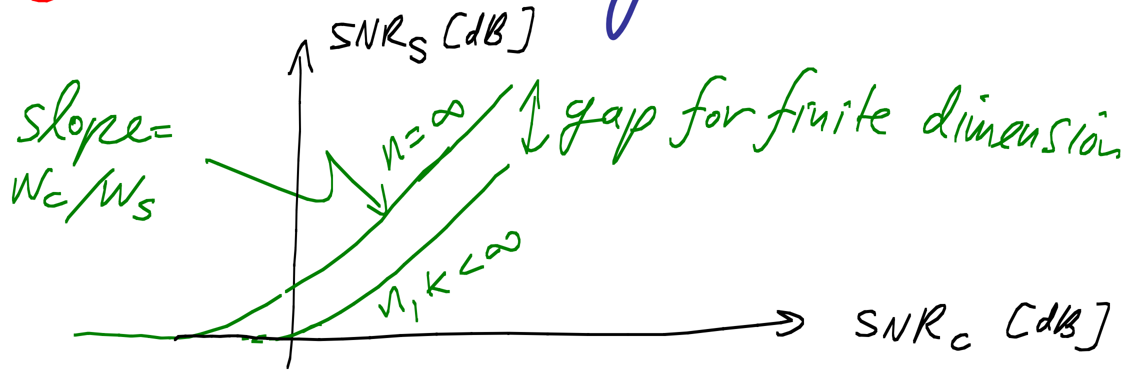
bit per channel use

$$\frac{\sigma_s^2}{D^{OPTA}} \triangleq SNR_s^{OPTA} = (1 + SNR_c)^{W_c/W_s}$$



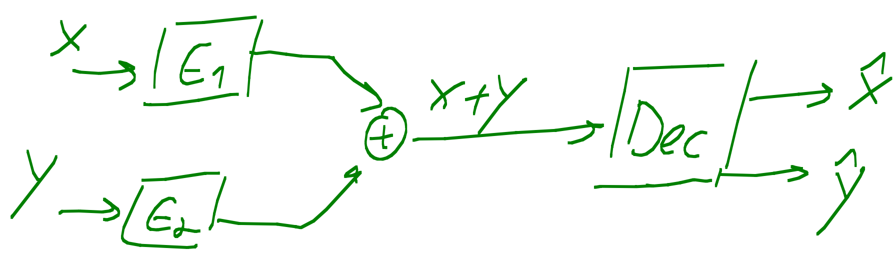
When is Digital (= Separate S/C Coding) Not Optimal?

1. **Finite block length**: see Ziv-Zakai bounds



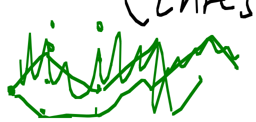
2. **Network problems**: e.g. joint Slepian-Wolf-MAC

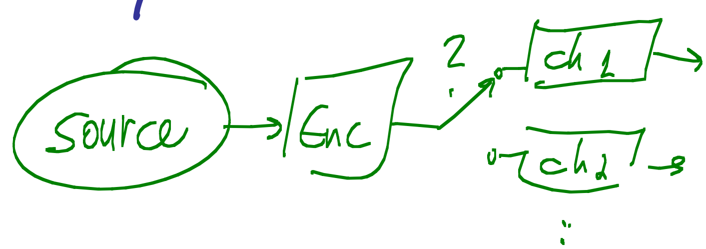
$y \backslash x$	0	1
0	1/3	1/3
1	1/3	0



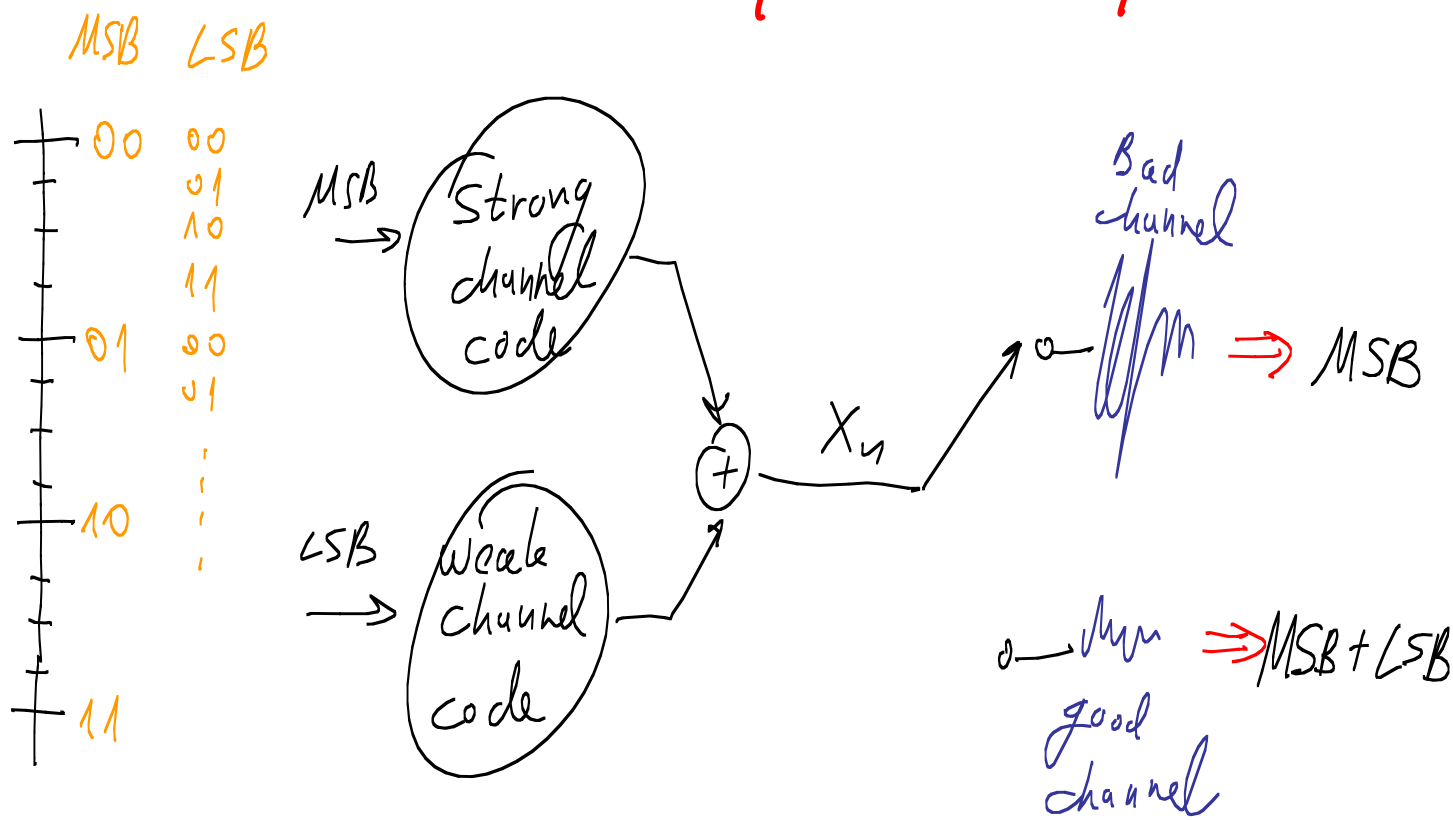
3. **Channels with unknown parameter**

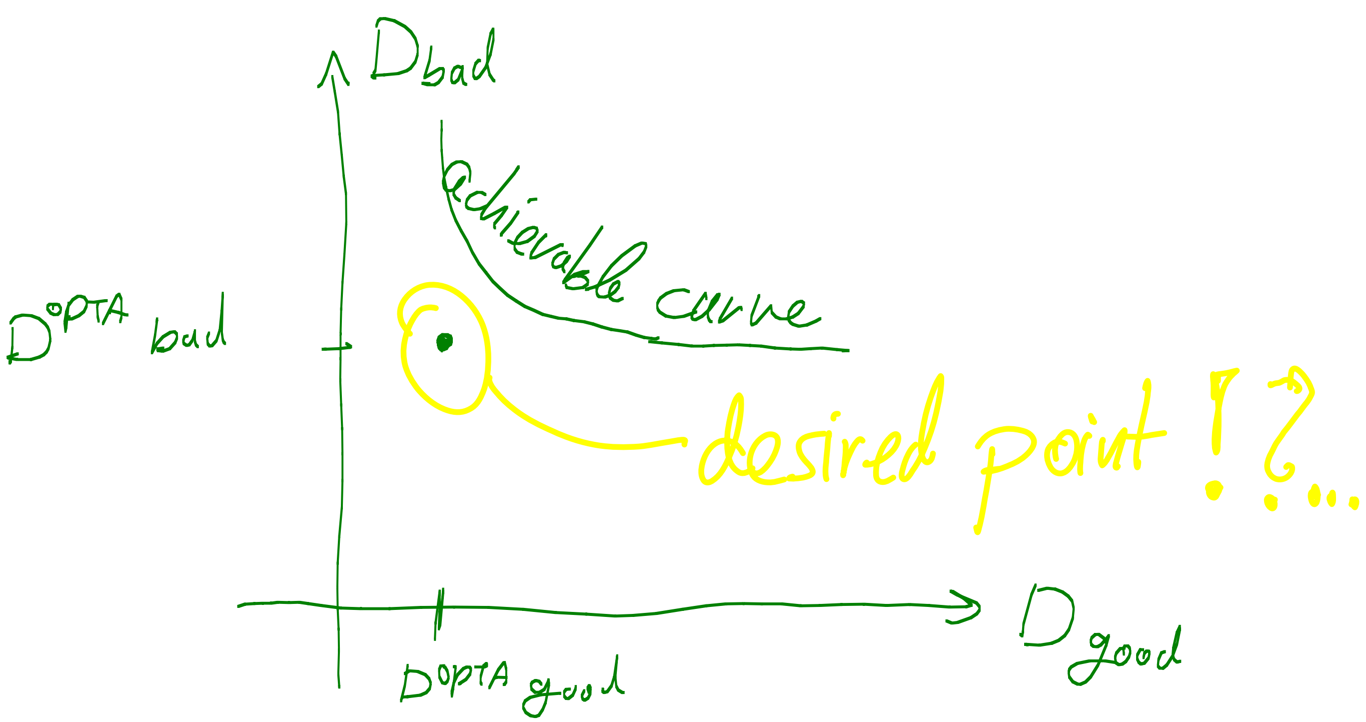
e.g. un-known SNR
(threshold effect)





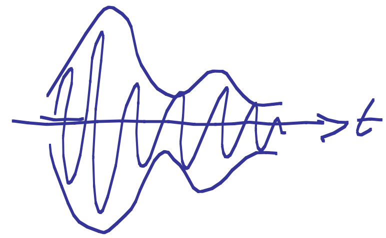
Digital solution for Unknown SNR:
"Un-equal Error Protection"



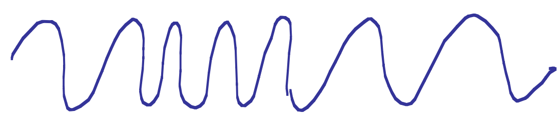


Analog Communication

Amplitude Modulation

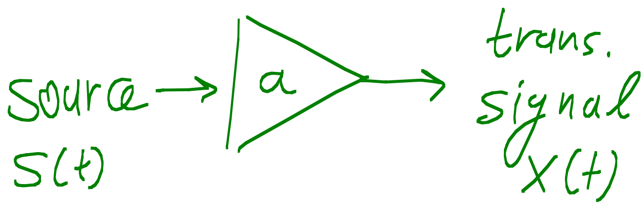


Frequency Modulation



Simplified model $\quad X(t) = a \cdot S(t)$

or



$$X(t) = a(t) * S(t)$$

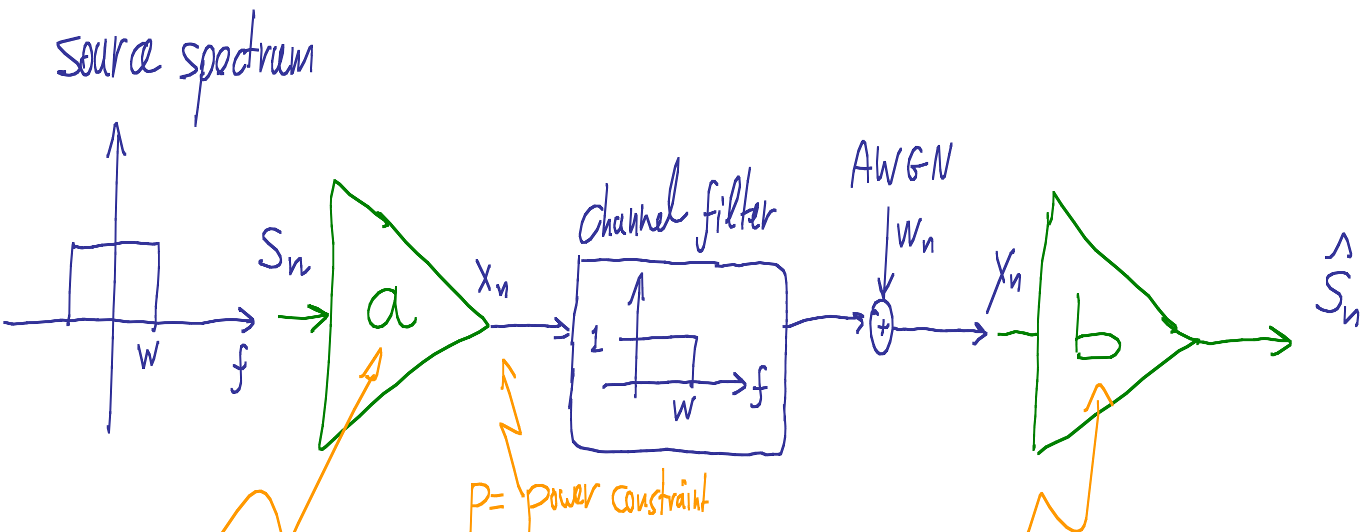
$\Rightarrow$ Continuous (linear) mapping from source to transmission.

* Observation: Distortion generated by nature

Ideal Case: Perfect Source-Channel Matching

White Gaussian source & channel,

$W_S = W_S = W \Rightarrow$ one source sample per channel use



Source power matching

$a = \sqrt{P/\sigma_S^2}$

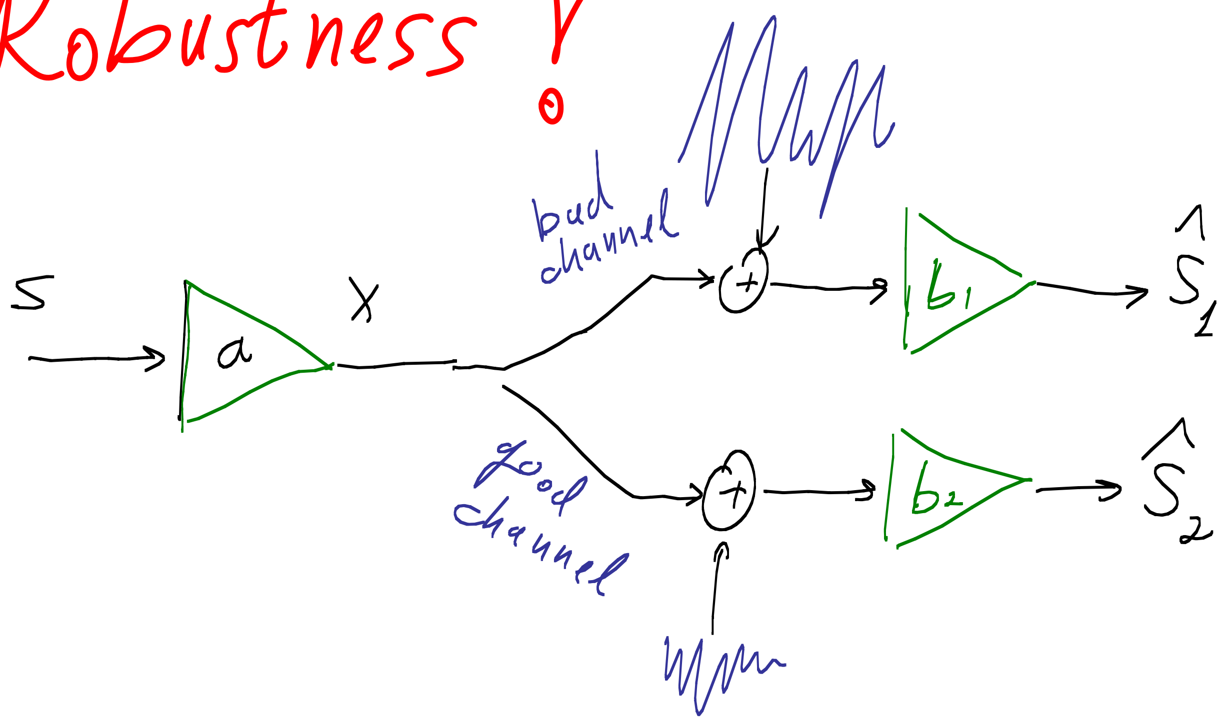
Wiener estimation (MMSE)

$b = \frac{SNR_c}{1+SNR_c} \cdot \frac{1}{a}$

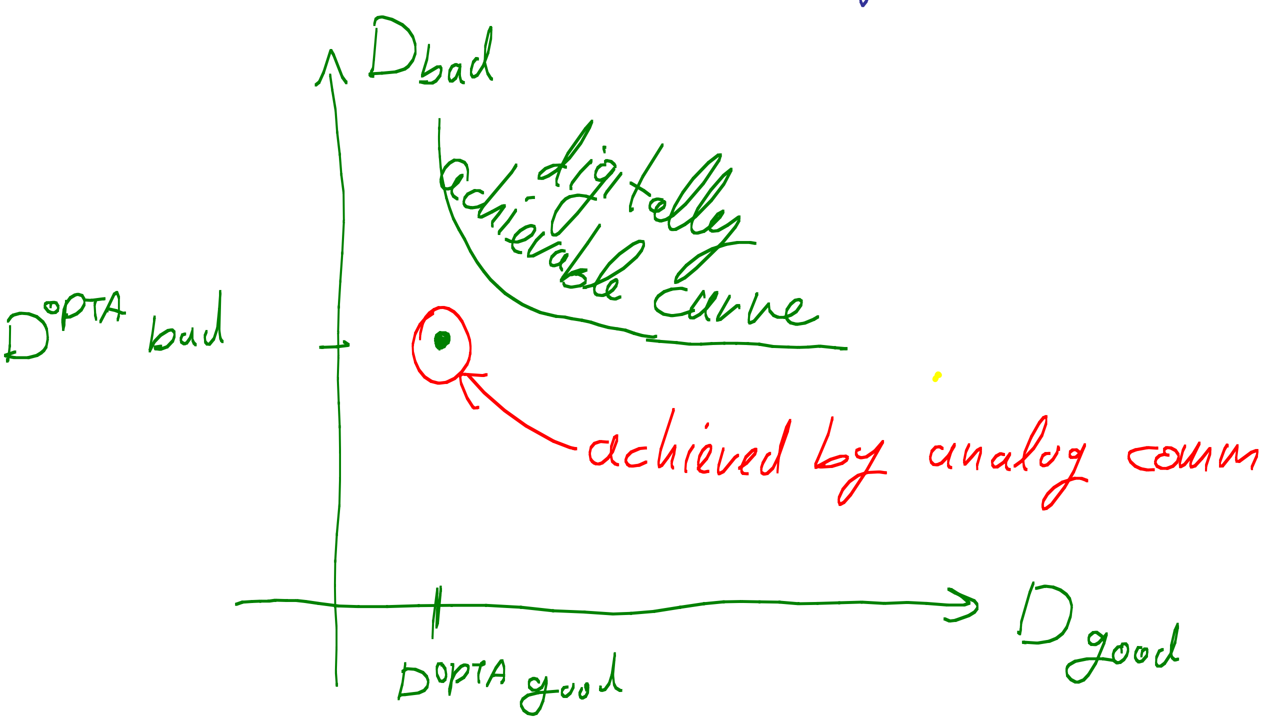
$\Rightarrow \quad SNR_S = 1 + SNR_c \Rightarrow$ OPTA!

* Observation: Transmitter independent of channel

Robustness !



s
a
x
bad channel
good channel
b_1
b_2
$\hat{s}_1$
$\hat{s}_2$



D_{bad}
digitally achievable curve
D^{OPTA}_{bad}
achieved by analog comm
D_{good}
D^{OPTA}_{good}

When is analog Comm not optimal?

Source-Channel Mismatch

1. Cost-function mismatch

   e.g., Source = Gaussian d = MSE

   channel = Gaussian input cost = pick limited

2. Bandwidth mismatch

   $W_c \neq W_s$

3. Color mismatch

   source spectrum $\neq$ optimum input for C

   or

   channel spectrum $\neq$ optimium test channel for $R(D)$

4. distribution mismatch, etc. ...

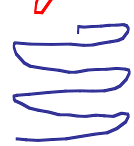
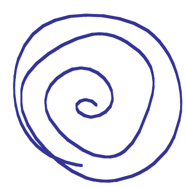
Bandwidth $\longleftrightarrow$ SNR Conversion

preserve mutual information rate

$$I(\text{source}\,;\,\text{reconstruction}) = I(\text{channel})$$

$$W_s \log(SNR_s) = W_c \log(1 + SNR_c)$$

$$SNR_s^{\,W_s} \simeq SNR_c^{\,W_c}$$

* one of Shannon's early motivations...
* Space-filling-curves , , FM

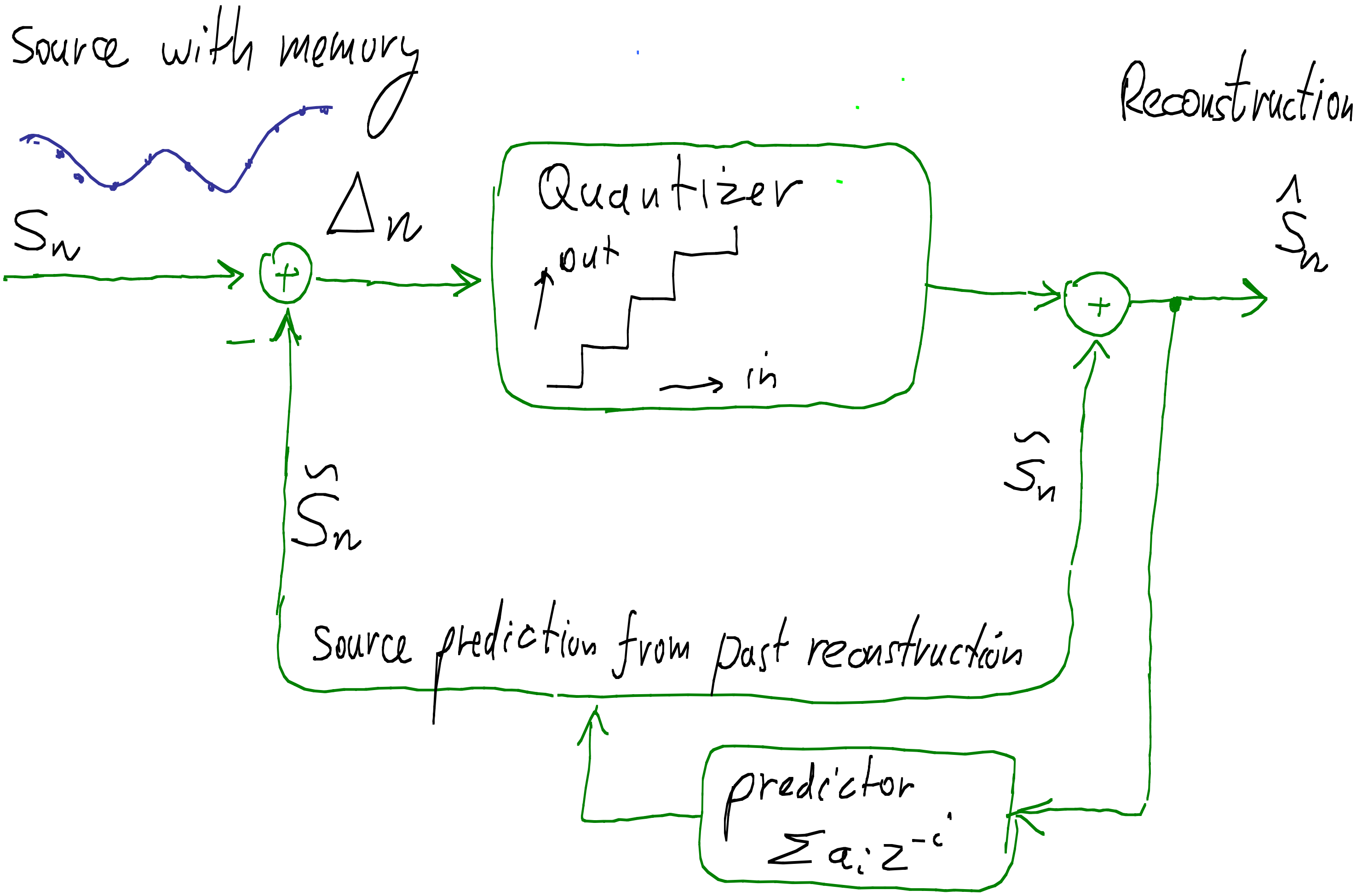
II

Analog Matching

Yuval Kochman, PhD @ TAU

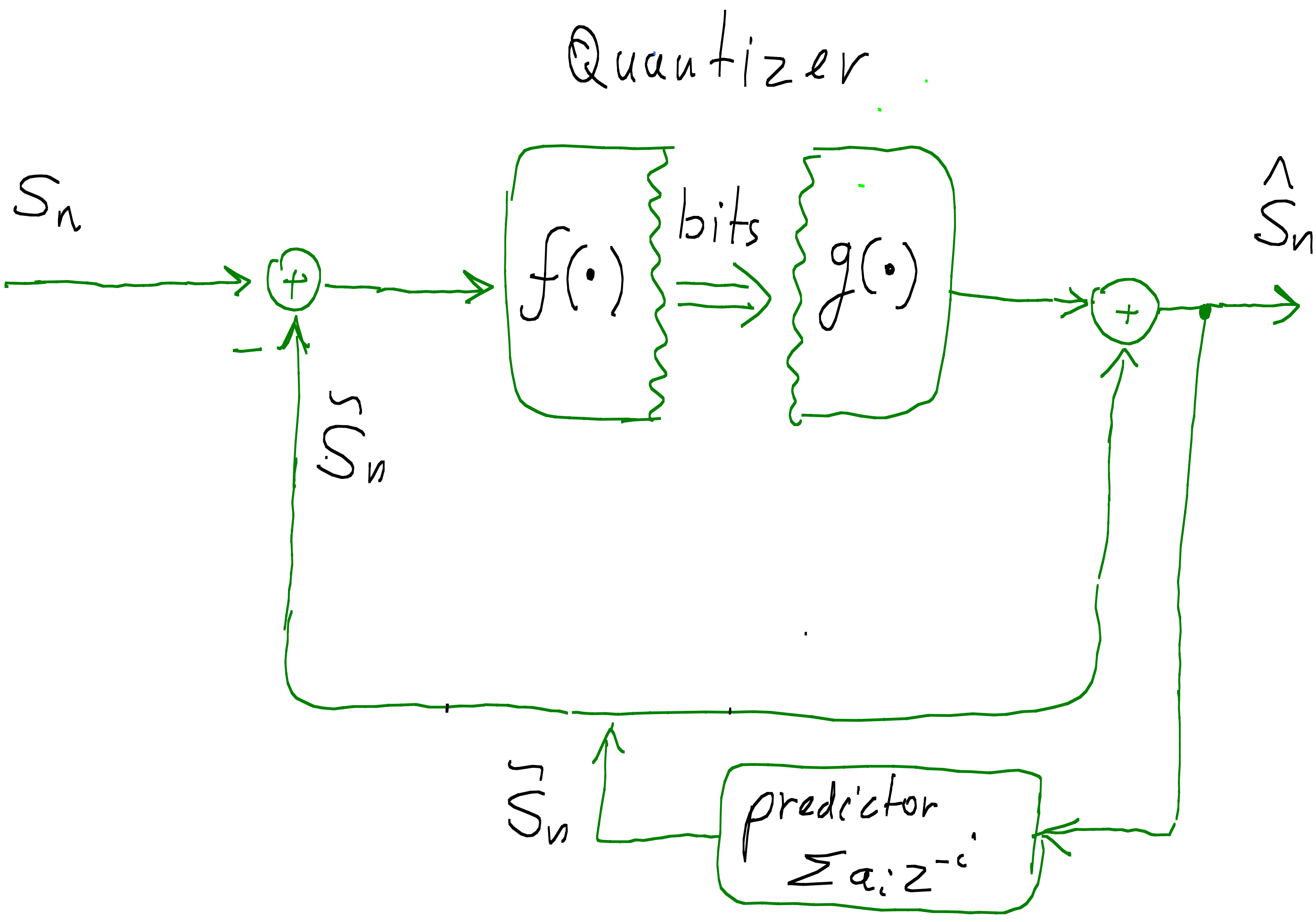
Wyner - Ziv - D.P.C.M.

predective-coding for saving bits on source memory



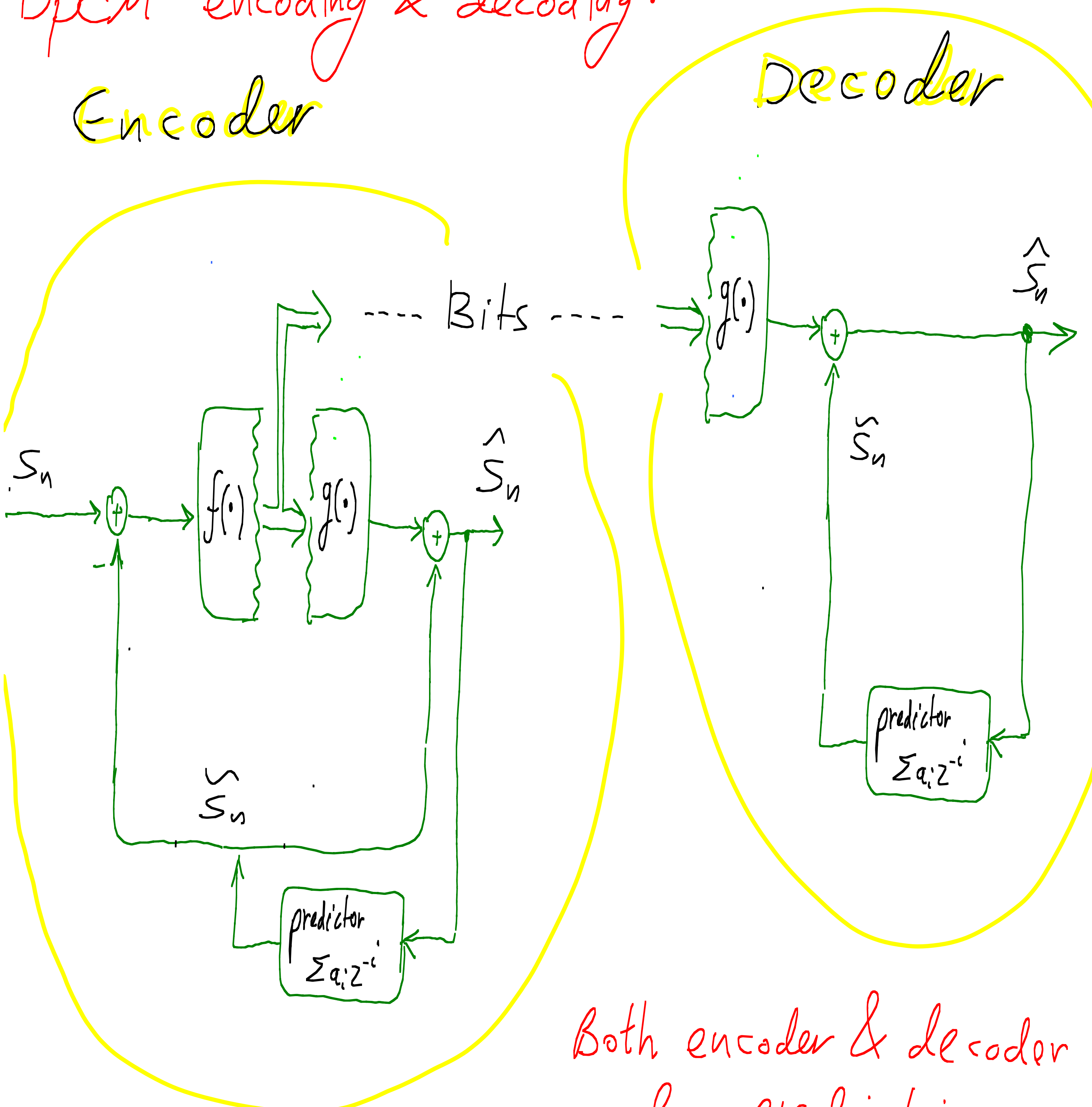
Wyner - Ziv - D.P.C.M.

DPCM encoding & decoding



Wyner-Ziv - D.P.C.M.

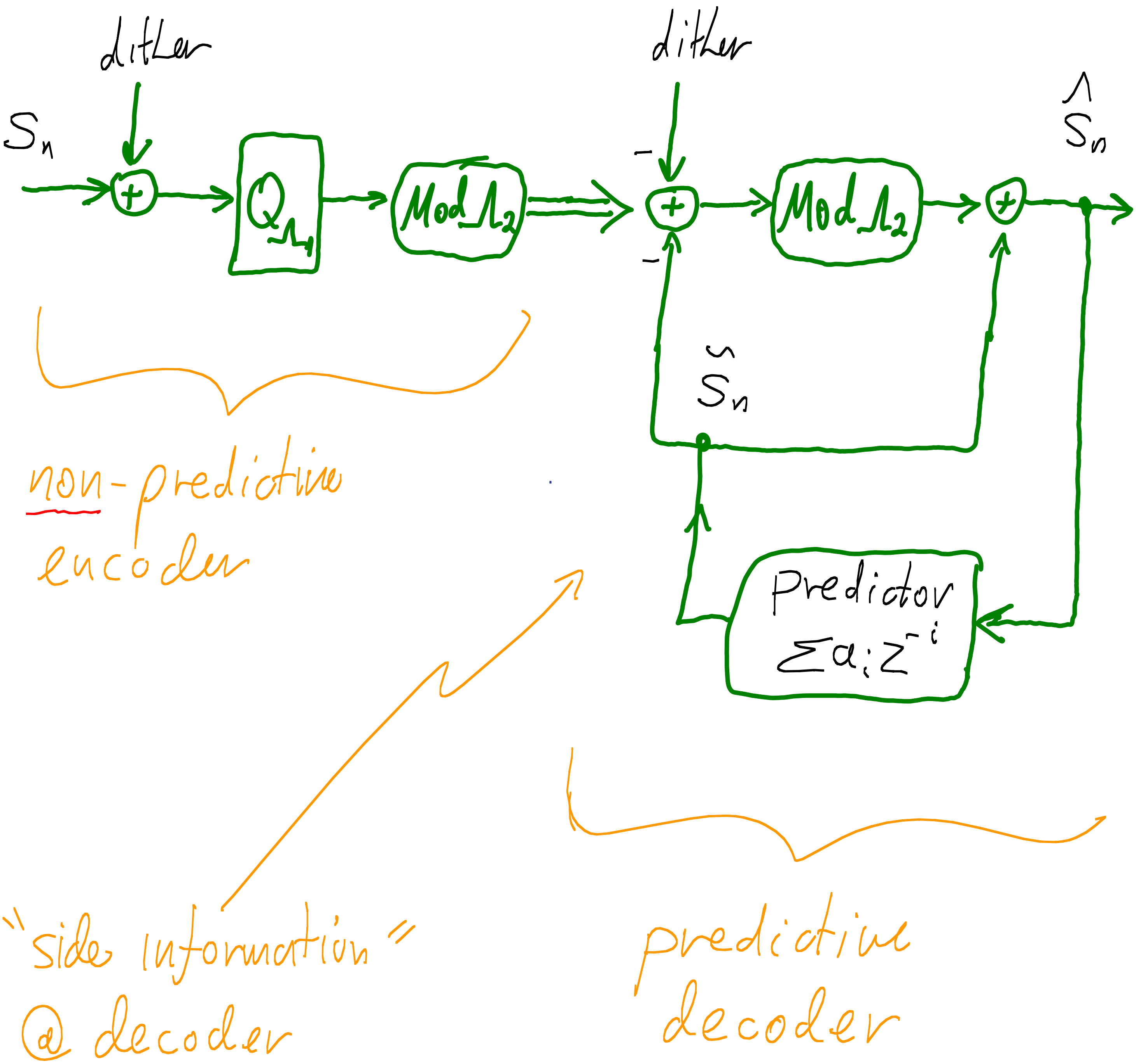
DPCM encoding & decoding



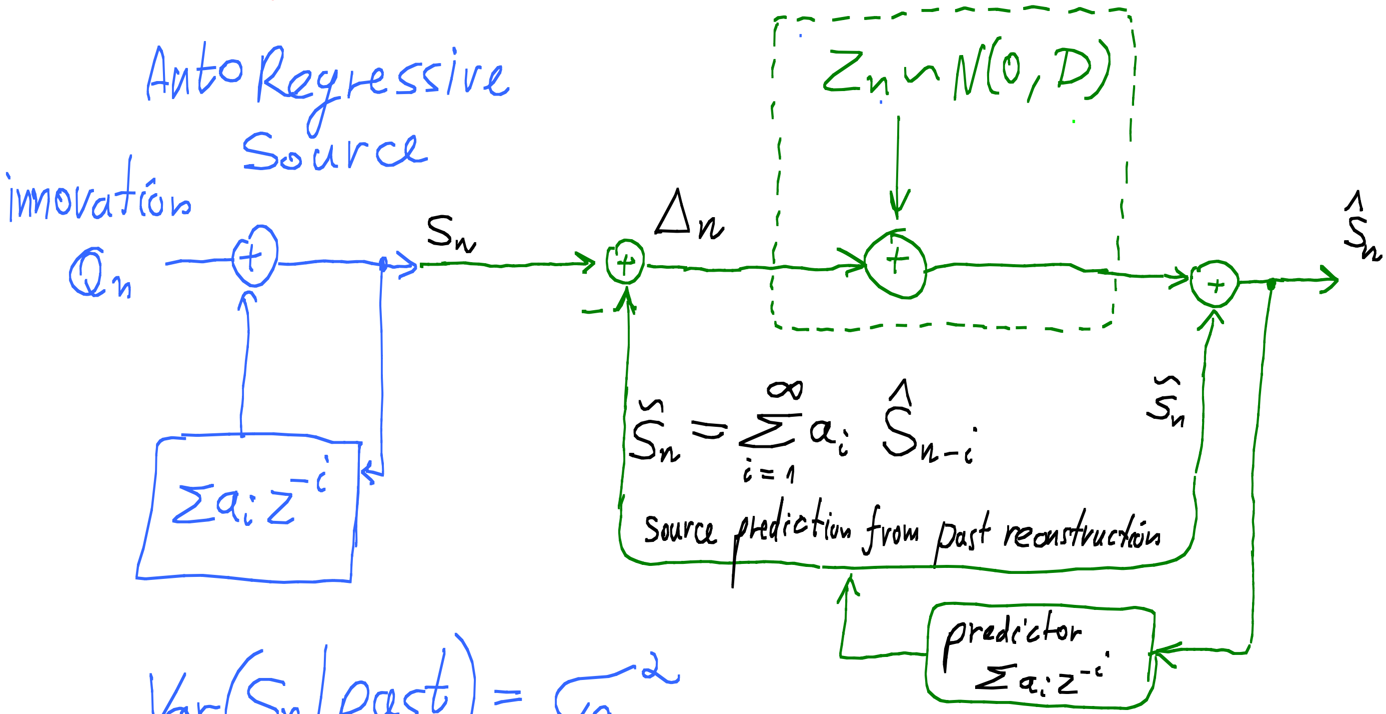
Both encoder & decoder apply prediction

Wyner-Ziv-D.P.C.M.

Decoder-only prediction



Analysis for DPCM



$$Var(S_n | past) = \sigma_Q^2$$

* DPCM error identity $\Rightarrow$ $\hat{S}_n - S_n = Q(\Delta_n) - \Delta_n = Z_n$

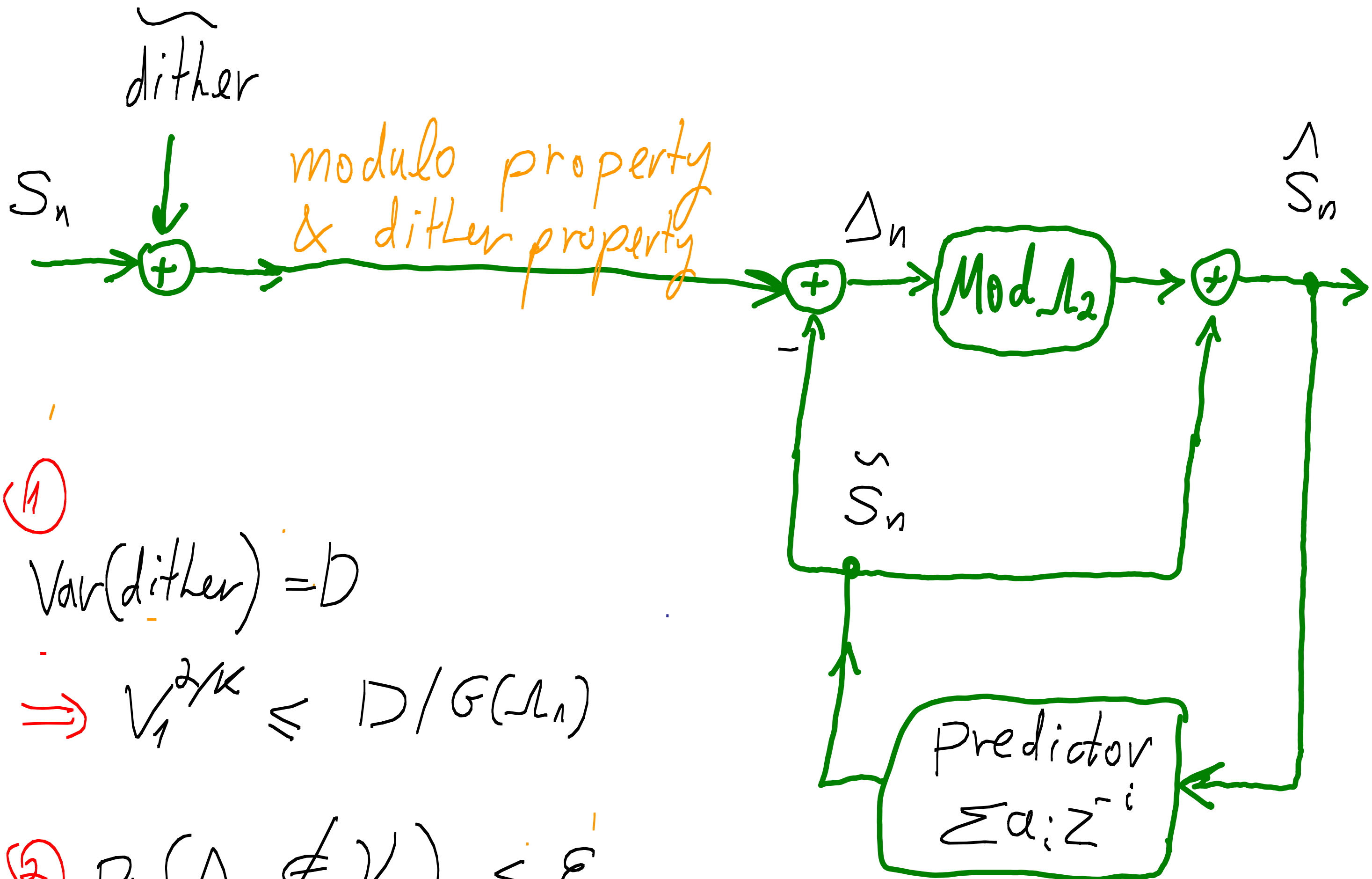
* Information preservation identity @ optimal prediction $\Rightarrow$ $\underbrace{\bar{I}(S_n; \hat{S}_n)}_{\text{information rate}} = \underbrace{I(\Delta_n; \Delta_n + Z_n)}_{\text{scalar information}}$

$\Downarrow$

HRQ $\Rightarrow$ $\Delta_n \simeq Q_n$ $\Rightarrow$ $Var(\Delta_n) \cong Var(S_n / past)$

$$\Rightarrow \boxed{\bar{I}(S_n; \hat{S}_n) \cong \frac{1}{2} \log\left(\frac{Var(S_n | past)}{D}\right) = R(D)}$$

Analysis for Wyner - Ziv - D.P.C.M.



(1)

$\mathrm{Var}(\text{dither}) = D$

$\Rightarrow V_1^{2/K} \le D / G(\Lambda_1)$

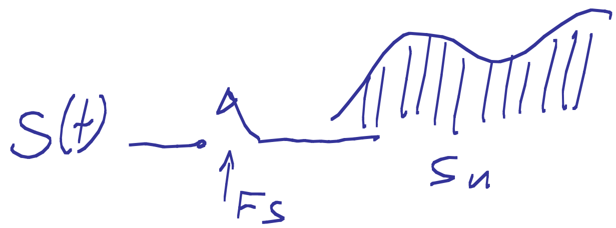
(2) $\Pr(\Delta_n \notin V_2) < \varepsilon$

$\Rightarrow V_2^{2/K} \ge \mu(\Lambda_2, \varepsilon) \cdot \mathrm{Var}(S_n|\text{past})$

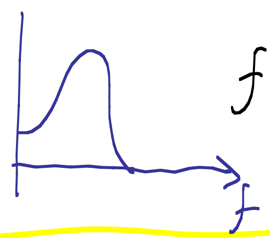
$$(1)+(2) \Rightarrow \text{Rate} \ge \overbrace{\frac{1}{2}\log\left(\frac{\mathrm{Var}(S_n|\text{past})}{D}\right)}^{R(D)\ @\ \text{HRQ}} + \overbrace{\frac{1}{2}\log\left(G(\Lambda_1)\cdot\mu(\Lambda_2,\varepsilon)\right)}^{\text{good lattices} \to 0}$$

Remarks

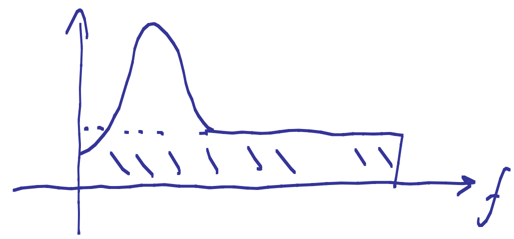
1. Oversampling



predict



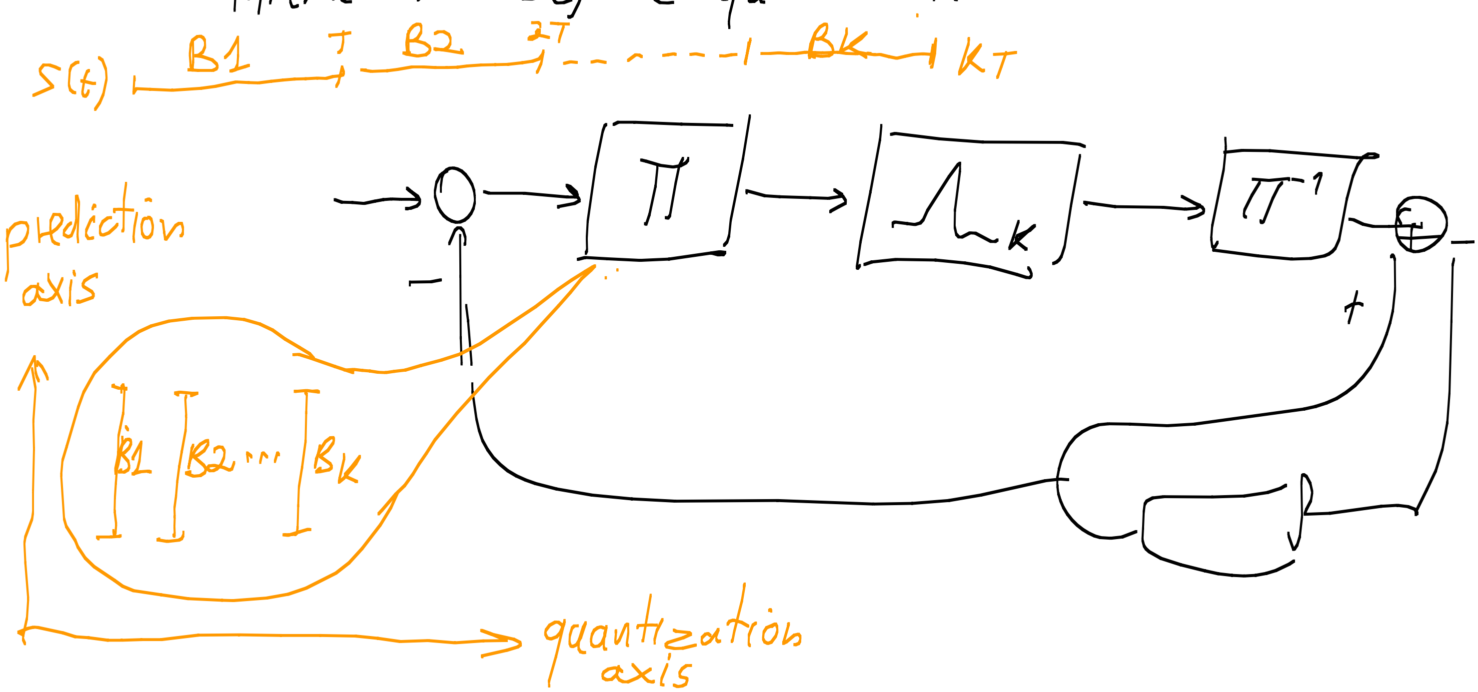
from



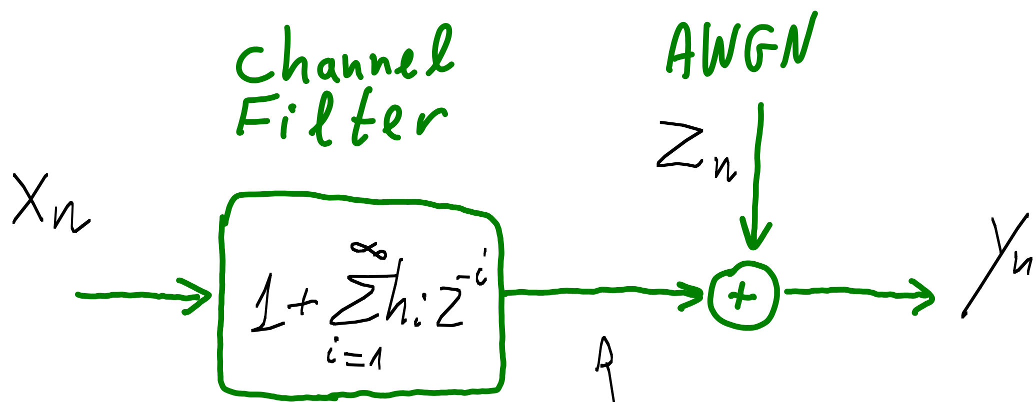
keep rate-per-samle $*$ $F_S \equiv$ constant

2. Sequential Lattice quantization

interleave before quantization



Inter-Symbol Interference Channel
&
Decision-Feedback Equalization

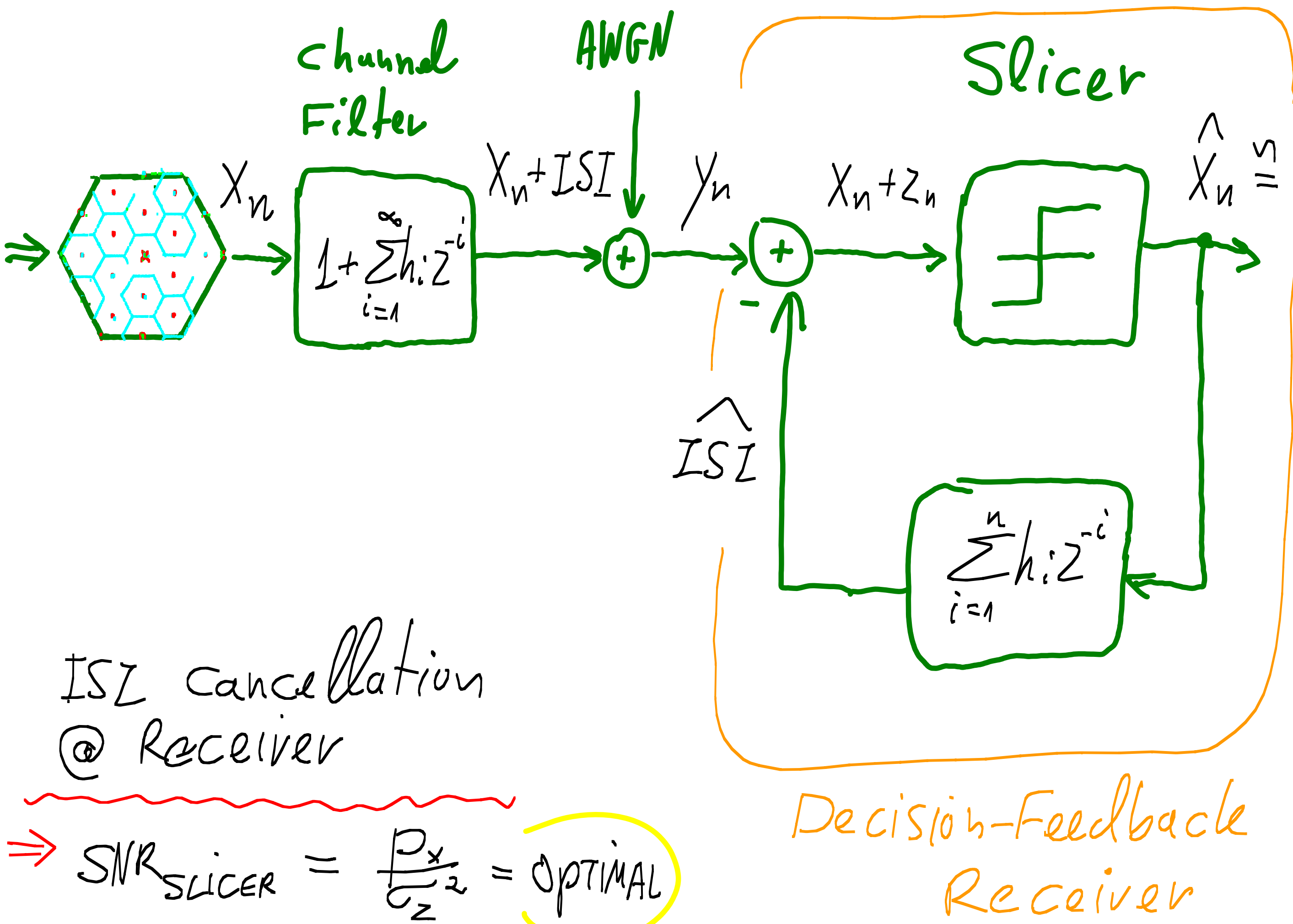


monic minimum-phase filter (w.l.o.g)

$$x_n + \underbrace{\sum_{i=1}^{\infty} h_i x_{n-i}}_{ISI}$$

$\Rightarrow$ $y_n = x_n + ISI + z_n$

Inter-Symbol Interference Channel & Decision-Feedback Equalization



ISI cancellation
@ Receiver

⇒ $SNR_{SLICER} = \frac{P_x}{\sigma_z^2}$ = OPTIMAL

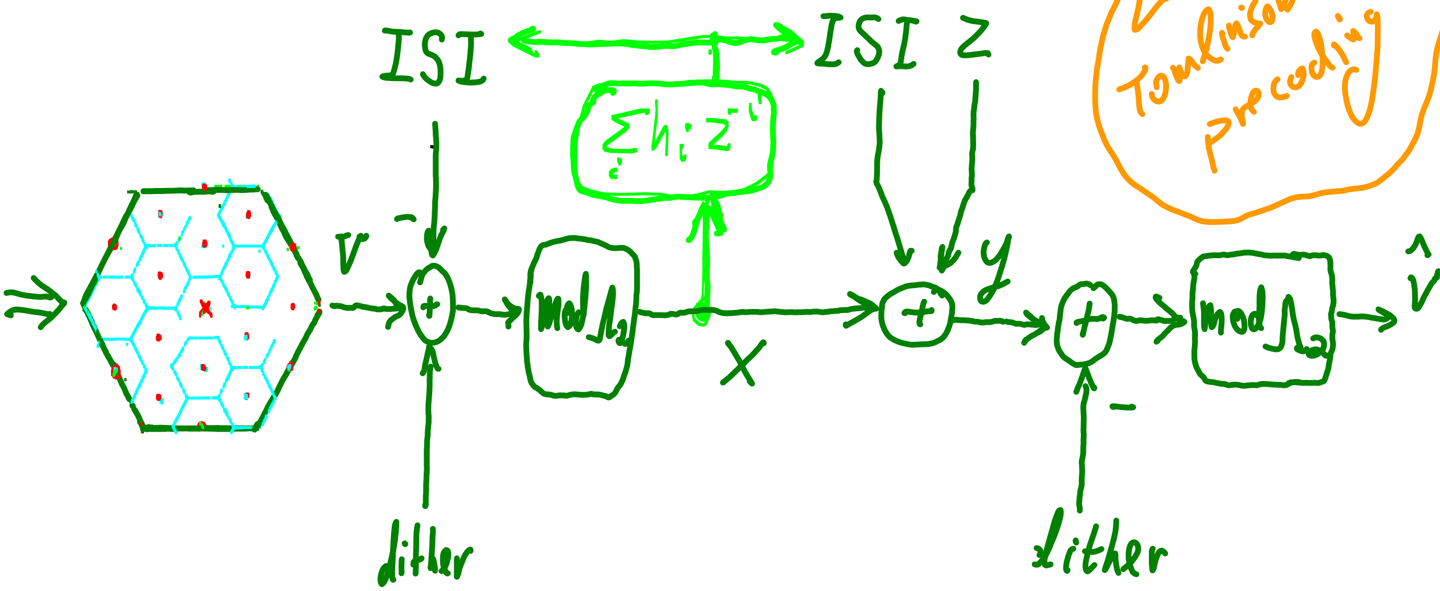
⇒ Error Propagation

⇒ Zero-delay decoding

$\left(C \simeq \frac{1}{2} \log (P/\sigma_z^2) \right.$
@ HSNR, MinPhase $\left. \right)$

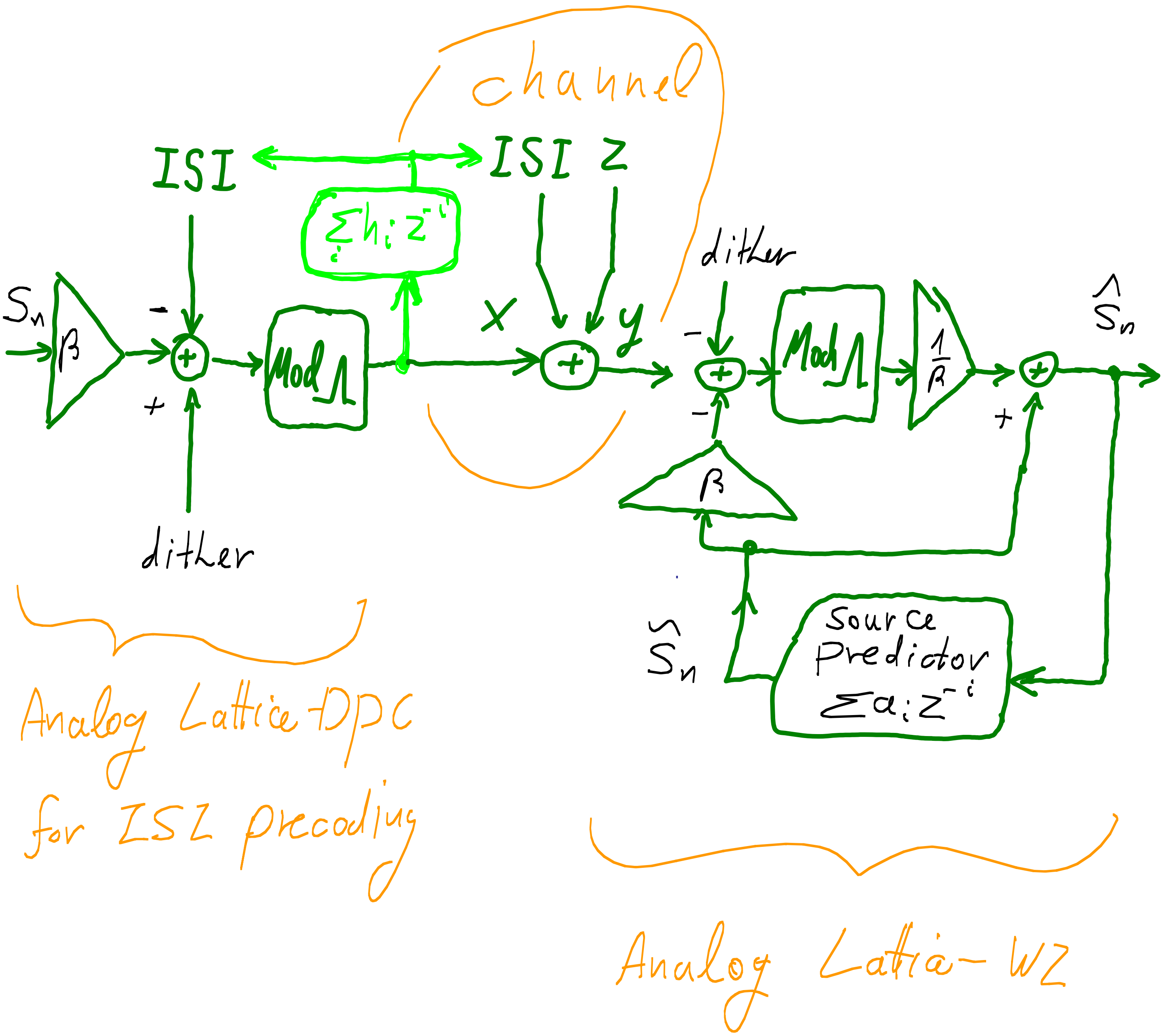
But ISI Known @ Transmitter ⇒ Lattice-ISI Precoding

Lattice-Tomlinson-Harashima precoding



- ISI Cancellation @ Transmitter:
  ⇒ no error propagation
- mod-Λ subtraction:
  ⇒ no power amplification
- Nested Lattices $\Lambda_1 \supset \Lambda_2$:
  ⇒ $$\text{Rate} = \underbrace{\frac{1}{2}\log\left(\frac{P_x}{\sigma_Z^2}\right)}_{C \text{ @ HSNR, Min Phase}} - \underbrace{\frac{1}{2}\log\left(G(\Lambda_2)\cdot\mu(\Lambda_1)\right)}_{\to 0 \text{ for good lattices}}$$

"Analog Matching" System:
Merge WZ-DPCM & DPC



$$\Rightarrow \frac{D}{D^{OPTA}} = G(\Lambda)\cdot \mu(\Lambda, P_e) \quad @ \text{ HSNR},\ W_c = W_s,\ \text{no error}$$

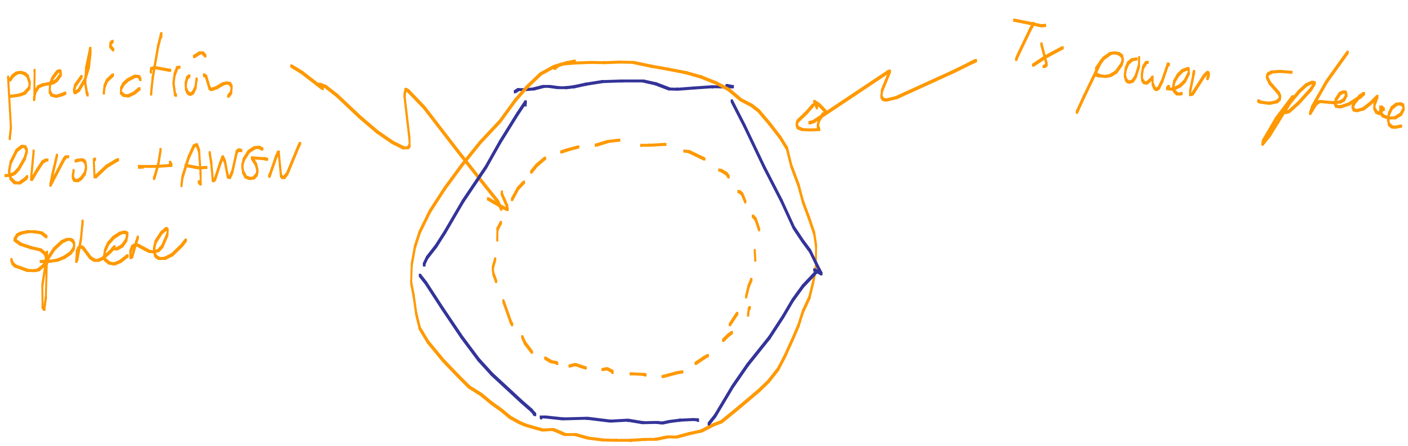
$\to 1$ for good lattices

Remarks:

1. β^2 = source-channel power-matching & margin

$$\beta^2 < \frac{P}{Var(S_n|past)} \cdot \frac{1}{G(\Lambda)\cdot \mu(\Lambda, p_e)}$$

2. margin (Loss in distortion)



3. Extension to general SNR & non-Minimum-phase channel Filter

use source & channel "water-pouring" & Wiener filters

4. Extension to mismatch bandwidth

Oversample to $F_s = \max\{W_s, W_c\}$
use w.p. & Wiener filters @ F_s

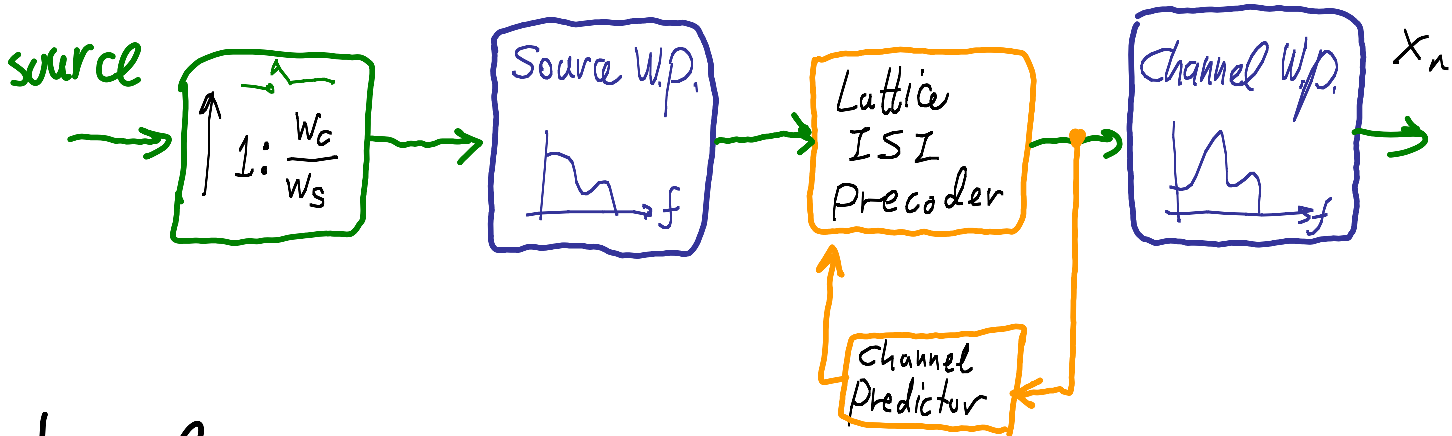
5. Robustness:

If $W_c = S_W$, $P \gg \sigma_z^2$
$\Rightarrow$ Tx is independent of noise power

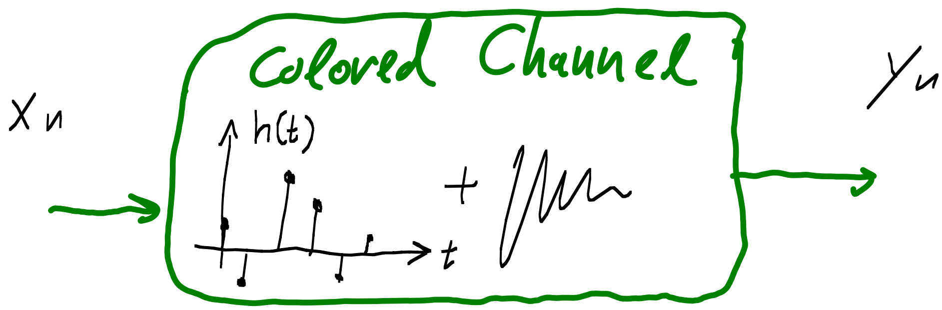
6. Scalar case

Analog Matching: General Case
(SNR, Channel Filter, W_c, W_s)

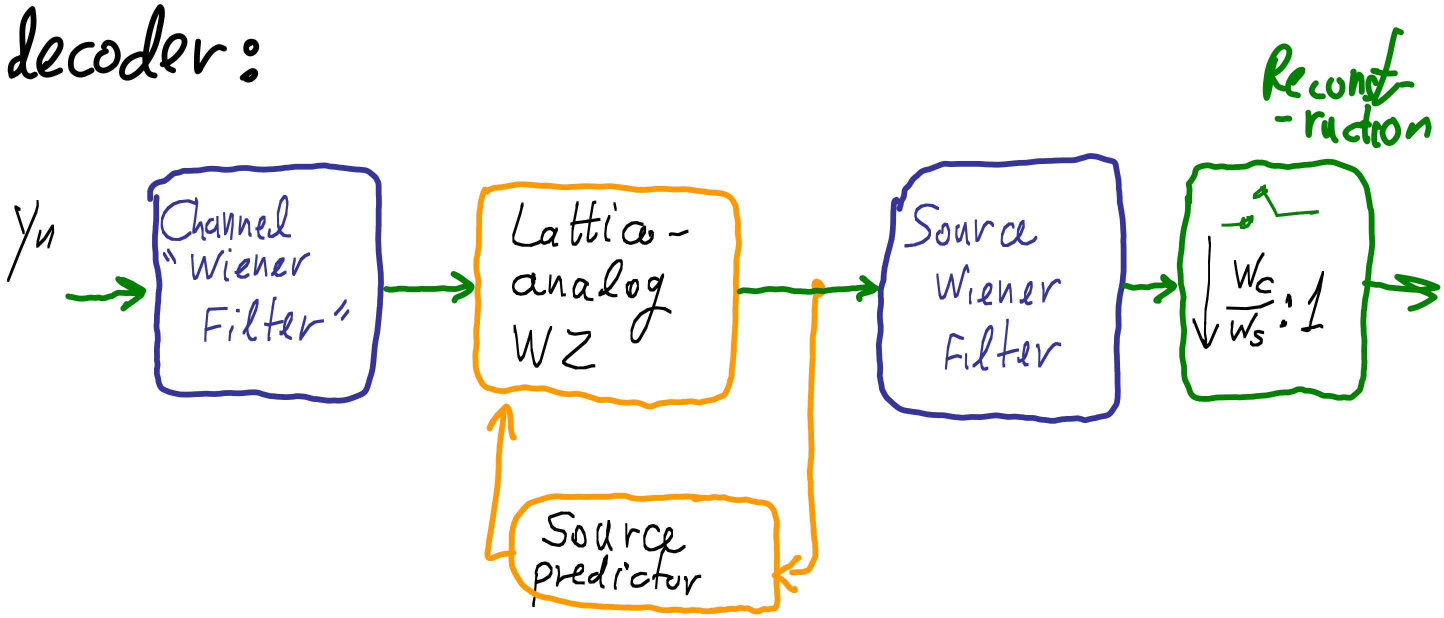
encoder:



channel:

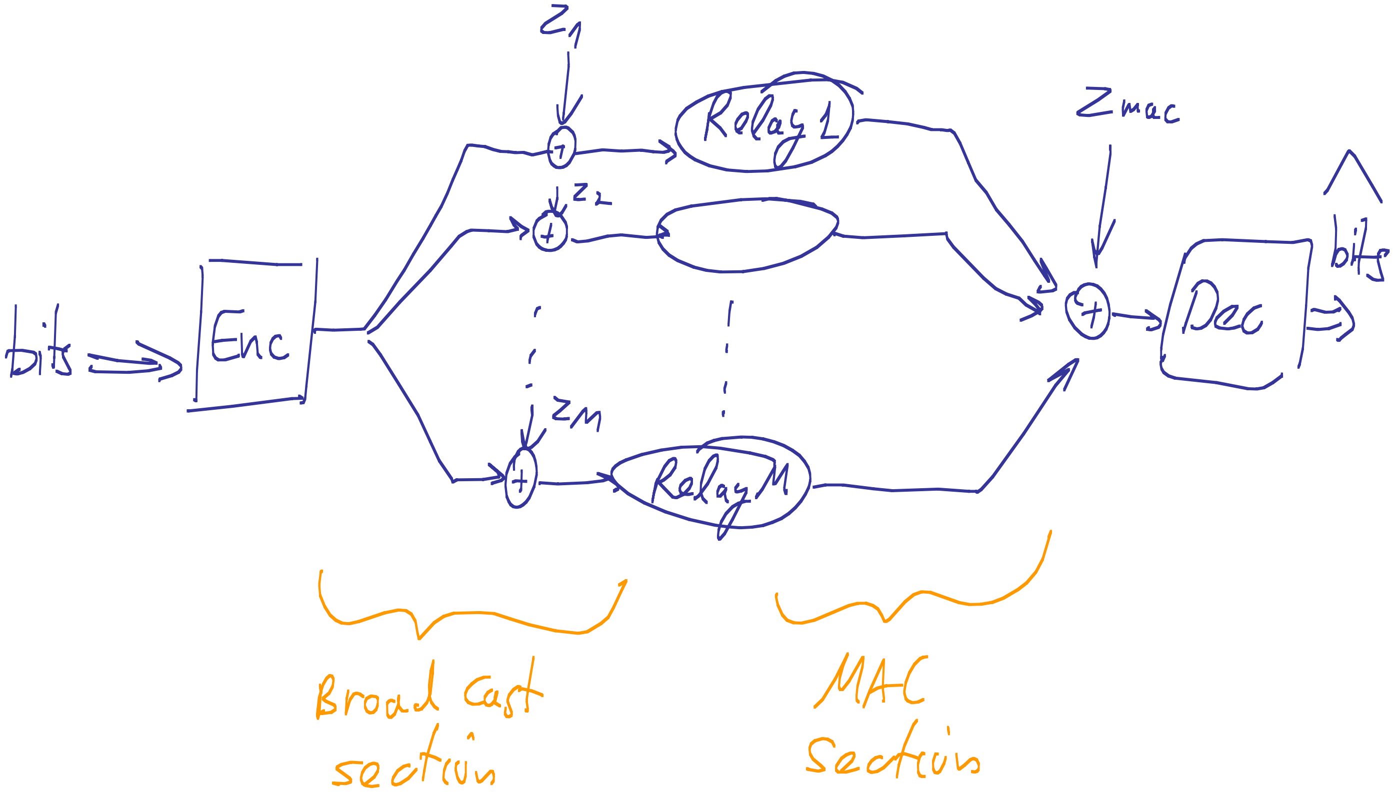


decoder:



7. Application:

Rematch & Forward for colored parallel Relays



$W_{MAC} \neq W_{BC}$

Analog-matching replaces Amplify & Forward for mis-match BW.

The End

Thanks.

Rami Zamir
Tel Aviv University