WORKSHOP ON

LINEAR PROGRAMMING AND MESSAGE-PASSING APPROACHES TO HIGH-DENSITY PARITY-CHECK CODES AND HIGH-DENSITY GRAPHICAL MODELS

MARCH 1–2, 2010

TEL AVIV UNIVERSITY, ISRAEL

ORGANIZERS

Yair Be’ery Tel Aviv University, Tel Aviv, Israel
Michael Chertkov Los Alamos National Laboratory, Los Alamos, NM, USA
Stefan Ruzika University of Kaiserslautern, Kaiserslautern, Germany
Pascal O. Vontobel Hewlett-Packard Laboratories, Palo Alto, CA, USA

LOCATION

The Green House (Habayit Hayarok) at the University Club, 24 George Wise Street, Tel Aviv University. (4 km / 10–20 minutes by taxi from Alexander Suite Hotel / Melody Hotel. See also maps on Page 4.)

REGISTRATION

Registration is free but is required until February 15, 2010.
Please send an email to Ms. Irit Neulander (iritn atsymbol eng dot tau dot ac dot il), specifying first name, last name, affiliation, and days of attendance (first and/or second day).

WEB SITE

http://www.pseudocodewords.info ⇒ HDPCC Workshop 2010

MEALS

Food and drinks for lunch and coffee breaks will be provided at the workshop location.

SPONSORS

We gratefully acknowledge the financial support by the following sponsors.

Advanced Communication Center at the School of Electrical Engineering, Tel Aviv University, and the Faculty of Engineering, Tel Aviv University, Israel

Center for Mathematical and Computational Modelling, a research center at the University of Kaiserslautern, Germany
Monday, March 1, 2010

09:00–09:15  Registration

**Welcome message by the Dean, Faculty of Engineering, Tel Aviv University**

09:15–09:20  Ehud Heyman

**Introduction to the Topics of the Workshop (Chair: Yair Be’ery)**

09:20–10:20  Pascal O. Vontobel  Linear programming and message-passing approaches to parity-check codes and graphical models: successes and challenges (tutorial)

10:20–10:50  Coffee break

**Message-passing approaches to HDPC codes (Chair: Jack Wolf)**

10:50–11:15  Olgica Milenkovic  High-density error-correction via automorphism group decoding

11:15–11:40  Joakim G. Knudsen  On iterative decoding of HDPC codes using weight-bounding graph operations

11:40–12:05  Samuel Ouzan  Moderate-density parity-check codes

12:05–12:30  Michael Lentmaier  Soft-decision decoding of Reed-Solomon codes with binary and non-binary belief propagation

12:30–13:30  Lunch break  (Sandwiches, etc.)

**Linear programming approaches to HDPC codes (Chair: Stefan Ruzika)**

13:30–13:55  Akin Tanatmis  Numerical comparison of IP formulations in ML decoding and minimum distance calculation

13:55–14:20  Tadashi Wadayama  Concave penalty method for improving interior point LP decoding

14:20–14:45  Alex Yufit  Asi Lifshitz  Near-ML linear programming decoding of HDPC codes

14:45–15:15  Coffee break

**Exact and Approximate Inference (Chair: Pascal O. Vontobel)**

15:15–16:00  Michael Chertkov  On dense graphical models and their potential for coding (tutorial)

16:00–17:00  Discussion session  (Chair: Pascal O. Vontobel)

Speakers are encouraged to allocate 5 minutes of their time slot to Q&A.
### LOW- AND HIGH-DENSITY IN COMPRESSED SENSING (Chair: Olgica Milenkovic)

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<thead>
<tr>
<th>Time</th>
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<th>Topic</th>
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</thead>
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<tr>
<td>09:00–09:25</td>
<td>Ori Shental</td>
<td>Belief propagation for sparse signal reconstruction</td>
</tr>
<tr>
<td>09:25–09:50</td>
<td>Ilya Poltorak</td>
<td>Hybrid dense/sparse matrices in compressed sensing reconstruction</td>
</tr>
<tr>
<td>09:50–10:15</td>
<td>Alex Dimakis</td>
<td>A connection between compressed sensing and binary linear channel codes</td>
</tr>
</tbody>
</table>

10:15–10:45  **Coffee break**

### HIGH-DENSITY GRAPHICAL MODELS (Chair: Yair Weiss)

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<tr>
<th>Time</th>
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<th>Topic</th>
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<tbody>
<tr>
<td>10:45–11:10</td>
<td>Amnon Shashua</td>
<td>Convergent message passing algorithms over region graphs for approximate MAP and marginal probability estimation</td>
</tr>
<tr>
<td>11:10–11:35</td>
<td>Talya Meltzer</td>
<td>Tightening LP relaxations for MAP using message passing</td>
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<tr>
<td>11:35–12:00</td>
<td>Amir Leshem</td>
<td>Solving integer least squares problems using reconstruction from projections and redundant information</td>
</tr>
<tr>
<td>12:00–12:25</td>
<td>Jacob Goldberger</td>
<td>A MIMO detector based on Gaussian tree approximation of a fully connected graph</td>
</tr>
</tbody>
</table>

12:25–13:15  **Lunch break**  *(Sandwiches, etc.)*

13:15–14:00  **Campus tour**  *Tel Aviv University*

### LINEAR PROGRAMMING AND LDPC CODES AT LARGE (Chair: Simon Litsyn)

<table>
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<tr>
<th>Time</th>
<th>Speaker</th>
<th>Topic</th>
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<tbody>
<tr>
<td>14:00–14:25</td>
<td>David Burshtein</td>
<td>Iterative approximate linear programming decoding of LDPC codes with linear complexity</td>
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<tr>
<td>14:25–14:50</td>
<td>Nissim Halabi</td>
<td>LP decoding of regular LDPC codes in memoryless channels</td>
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<tr>
<td>14:50–15:15</td>
<td>Vitaly Skachek</td>
<td>On the pseudocodeword redundancy</td>
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<tr>
<td>15:15–15:40</td>
<td>Warren J. Gross</td>
<td>Stochastic decoding of LDPC Codes over GF(q)</td>
</tr>
</tbody>
</table>

15:40–16:00  **Coffee break**

16:00–17:00  **Discussion session**  *(Chair: Michael Chertkov)*

*End of workshop*  *Have a safe trip home!*

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Speakers are encouraged to allocate 5 minutes of their time slot to Q&A.
Workshop Location

The Green House (Habayit Hayarok) at the University Club
24 George Wise Street
Tel Aviv University

Building 39 on the above Tel Aviv University campus map
WORKSHOP ON
Linear Programming and Message-Passing Approaches to High-Density Parity-Check Codes and High-Density Graphical Models
March 1–2, 2010
Tel Aviv University, Israel

Abstracts
(in alphabetic order by speaker surname)

Sponsors
We gratefully acknowledge the financial support by the following sponsors.

Advanced Communication Center at the School of Electrical Engineering, Tel Aviv University, and the Faculty of Engineering, Tel Aviv University, Israel

Center for Mathematical and Computational Modelling, a research center at the University of Kaiserslautern, Germany
The problem of low complexity linear programming (LP) decoding of low-density parity-check (LDPC) codes is considered. An iterative algorithm, similar to min-sum and belief propagation, for efficient approximate solution of this problem was proposed by Vontobel and Koetter. In this paper the convergence rate and computational complexity of this algorithm are studied using a scheduling scheme that we propose. In particular we are interested in obtaining a feasible vector in the LP decoding problem that is close to optimal in the following sense. The distance, normalized by the block length, between the minimum and the objective function value of this approximate solution can be made arbitrarily small. It is shown that such a feasible vector can be obtained with a computational complexity which scales linearly with the block length. Combined with previous results that have shown that the LP decoder can correct some fixed fraction of errors we conclude that this error correction can be achieved with linear computational complexity. This is achieved by first applying the iterative LP decoder that decodes the correct transmitted codeword up to an arbitrarily small fraction of erroneous bits, and then correcting the remaining errors using some standard method. These conclusions are also extended to generalized LDPC codes.

In this review talk I will discuss recent and not very recent results concerning exact and approximate inference over sparse and not very sparse (e.g. planar and surface) graphical models and speculate about possible applications of these results in various information theoretic contexts (mainly decoding).

This is a tale of two linear programming decoders, namely channel coding linear programming decoding and compressed sensing linear programming decoding. So far, they have evolved quite independently. Our aim is to show that there is a tight connection between the compressed sensing measurement matrices (which are real valued) and the parity check matrices of binary linear codes (over finite fields). Our connection is a formal proof that good channel code matrices always correspond to good measurement matrices for compressed sensing, for various notions of good performance. This connection allows one to translate performance guarantees from one setup to the other.
**Authors**  Jacob Goldberger (Bar-Ilan University, Israel)
                Amir Leshem (Bar-Ilan University, Israel)

**Title**  A MIMO Detector based on Gaussian Tree Approximation of a Fully Connected Graph

**Abstract**  In the talk we describe a new algorithm for the linear least squares problem where the unknown variables are constrained to be in a finite set. The factor graph that corresponds to this problem is a complete graph. Hence, applying the BP algorithm yields very poor results. Our method is based on an optimal tree approximation of the Gaussian density of the unconstrained linear system. It is shown that even though the approximation is not directly applied to the exact discrete distribution, applying the BP algorithm to the modified loop-free factor graph outperforms current methods.

**Author**  Warren J. Gross (McGill University, Canada)

**Title**  Stochastic Decoding of LDPC Codes over $\text{GF}(q)$

**Abstract**  Nonbinary LDPC codes have been shown to outperform currently used codes for magnetic recording and several other channels. Currently proposed nonbinary decoder architectures have very high complexity for high-throughput implementations and sacrifice error-correction performance to maintain realizable complexity. In this paper we present an alternative decoding algorithm based on stochastic computation that has a very simple implementation and minimal performance loss when compared to the sum-product algorithm. We demonstrate the performance of the algorithm when applied to a $\text{GF}(16)$ code and provide details of the hardware resources required for an implementation.

**Authors**  Nissim Halabi (Tel Aviv University, Israel)
                Guy Even (Tel Aviv University, Israel)

**Title**  LP Decoding of Regular LDPC Codes in Memoryless Channels

**Abstract**  We study error bounds for linear programming decoding of regular LDPC codes with logarithmic girth. For memoryless binary-input AWGN channels, we prove thresholds for exponentially small decoding errors. Specifically, for $(3,6)$-regular LDPC codes, we prove thresholds of $\sigma = 0.735$ provided that the girth is greater than 88 and $\sigma = 0.605$ for girth at least 4. For memoryless binary-input output-symmetric channels, we prove exponentially small error bounds.

Our proof is an extension of a recent paper of Arora, Daskalakis, and Steurer [STOC 2009] who presented a novel probabilistic analysis of LP decoding over binary symmetric channels. Their analysis works with the primal LP representation and has an explicit connection to message passing algorithms. We extend this analysis to any memoryless binary-input output-symmetric (MBIOS) channel.

Our results improve thresholds presented by Koetter and Vontobel [ITAW 2006]. They proved a threshold of $\sigma = 0.5574$ for LP decoding of $(3,6)$-regular codes working with the dual LP for girth at least 4. In that analysis the threshold does not increase with the girth.
In this work, we extend our previous work on iterative soft-input soft-output (SISO) decoding of high density parity check (HDPC) codes. Edge-local complementation (ELC) is a local graph operation which can be used to give structural redundancy during decoding with the sum-product algorithm (SPA). We describe the specific local subgraphs required for ELC to not increase the weight of the Tanner graph beyond a specified upper bound. We call this controlled operation weight-bounding ELC (WB-ELC). A generalized iterative SISO HDPC decoder is described, which can be configured to employ different operations; most importantly, ‘flooding’ SPA, and iterative ‘permutation decoding.’ The latter is a state-of-the-art decoding algorithm for HDPC codes, using permutations from the automorphism group of the code. We observe performance improvements when the SISO HDPC decoder is configured to use WB-ELC, as compared to using ELC, or even permutation decoding. We also propose a method for reducing the weight of a systematic parity check matrix for the code. Using simple heuristics, we are able to find reduced weight matrices for various codes, in some cases reaching the minimum weight given by the dimensions and minimum distance of the code.

Applying the standard belief propagation (BP) decoding algorithm to Reed-Solomon (RS) codes of short or medium block lengths leads to poor error-correcting performance. The high-density parity-check (HDPC) matrices result in Tanner graphs with many short cycles that prevent the convergence of the iterative decoder to the correct codeword. In this work we compare several modifications of BP decoding that operate on a set of different parity-check matrices in order to improve performance. Focusing on performance-cost trade-offs, we investigate quantization effects and explore complexity reducing modifications for efficient implementations in software or hardware.

While the adaptive BP (ABP) algorithm updates the parity-check matrices sequentially during the decoding iterations, the multiple bases BP (MBBP) algorithm operates on several matrices in parallel. Optionally, the hard decision output of the iterative decoder can be used as input of a second decoding stage based on, e.g., the Berlekamp-Massey algorithm. We also consider non-binary BP decoding of RS codes and apply the ABP and MBBP principle to higher order Galois fields (GF-ABP and GF-MBBP). Our results show that for large block lengths the binary decoders tend to outperform the non-binary ones. For small block lengths, on the other hand, the GF-MBBP can be an attractive solution for future many-core systems due to its inherent parallel nature.
**Title**

_Solving Integer Least Squares Problems using Reconstruction from Projections and Redundant Information_

**Abstract**

In this talk we will describe two techniques for solving integer least squares problems using redundant information. The first technique is based on Bayesian reconstruction from projections, the second is by using a pseudo prior to assist a very loopy belief propagation converging.

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**Authors**

Amir Leshem (Bar-Ilan University, Israel)
Jacob Goldberger (Bar-Ilan University, Israel)

Talya Meltzer (Hebrew University of Jerusalem, Israel)
David Sontag (Massachusetts Institute of Technology, USA)
Amir Globerson (Hebrew University of Jerusalem, Israel)
Tommi Jaakkola (Massachusetts Institute of Technology, USA)
Yair Weiss (Hebrew University of Jerusalem, Israel)

**Title**

_Tightening LP Relaxations for MAP using Message Passing_

**Abstract**

Linear Programming (LP) relaxations have become powerful tools for finding the most probable (MAP) configuration in graphical models. These relaxations can be solved efficiently using message-passing algorithms such as belief propagation and, when the relaxation is tight, provably find the MAP configuration. The standard LP relaxation is not tight enough in many real-world problems, however, and this has lead to the use of higher order cluster-based LP relaxations. The computational cost increases exponentially with the size of the clusters and limits the number and type of clusters we can use. We propose to solve the cluster selection problem monotonically in the dual LP, iteratively selecting clusters with guaranteed improvement, and quickly re-solving with the added clusters by reusing the existing solution. Our dual message-passing algorithm finds the MAP configuration in protein side-chain placement, protein design, and stereo problems, in cases where the standard LP relaxation fails.

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**Authors**

Thorsten Hehn (University of Erlangen-Nuremberg, Germany)
Stefan Laendner (University of Erlangen-Nuremberg, Germany)
Olgica Milenkovic (University of Illinois, Urbana-Champaign, USA)
Johannes Huber (University of Erlangen-Nuremberg, Germany)

**Title**

_HIGH-DENSITY ERROR-CORRECTION VIA AUTOMORPHISM GROUP DECODING_
Authors: Samuel Ouzan (Tel Aviv University, Israel)
Yair Be’ery (Tel Aviv University, Israel)

Title: Moderate-Density Parity-Check Codes

Abstract: We propose a new type of short to moderate block-length, linear error-correcting codes, called moderate-density parity-check (MDPC) codes. The number of ones of the parity-check matrix of the codes presented is typically higher than the number of ones of the parity-check matrix of low-density parity-check (LDPC) codes. But, still lower than those of the parity-check matrix of classical block codes. The proposed MDPC codes are cyclic and are designed by constructing idempotents using cyclotomic cosets. The construction is simple and allows finding short block-length, high-rate codes with good minimum distance. Inspired by some recent iterative soft-input soft-output (SISO) decoders used in the context of classical block codes, we propose a low complexity, efficient, iterative decoder called Auto-Diversity (AD) decoder. The AD decoder is based on the belief propagation (BP) decoder and takes advantage of fundamental properties of the automorphism group of the constructed cyclic code.

Authors: Ilya Poltorak (Technion – Israel Institute of Technology, Israel)
Dror Baron (Technion – Israel Institute of Technology, Israel)
Deanna Needell (Stanford University, USA)

Title: Hybrid Dense/Sparse Matrices in Compressed Sensing Reconstruction

Abstract: Compressed sensing is an emerging area that reconstructs sparse input vectors from a small number of linear measurements. We leverage earlier results on fast reconstruction using a sparse measurement matrix where measurements are noiseless. Our first contribution is to propose a “two part” reconstruction approach that combines different algorithms and/or matrix structures. Next, we fuse two algorithms with complementing advantages to reconstruct well from slightly noisy measurements; our measurement matrix has a corresponding hybrid dense/sparse form.

Author: Amnon Shashua (Hebrew University of Jerusalem, Israel)

Title: Convergent Message Passing Algorithms over Region Graphs for Approximate MAP and Marginal Probability Estimation

Abstract: Approximate inference over probability distributions represented by an MRF can be mapped onto a variational framework of an approximate free energy with marginal consistency constraints. In this work we present the free energy approximation within the Fenchel Duality framework and derive both convex and non-convex approximations of the free energy with convergent guaranteed message-passing algorithms. The family of algorithms, called Region-Norm-Product, includes in particular existing schemes such as Generalized-Belief-Propagation.
**Author** Ori Shental (Qualcomm, San Diego, USA)

**Title** Belief Propagation for Sparse Signal Reconstruction

**Abstract** A standard approach in signal processing for sparse signal reconstruction is solving a least-squares (squared $l_2$ norm) problem regularized by a sparseness-inducing $l_1$ norm. This canonical framework is closely related to several popular optimization problems emerging ubiquitously in signal processing, and particularly in compressed sensing. In this contribution, we propose a message-passing algorithm for solving this important class of convex optimization problems, based on a modification of the Gaussian Belief Propagation algorithm. While having a comparable computational complexity to previous reconstruction algorithms, e.g., Gradient Projection for Sparse Reconstruction (GPSR), the message-passing nature of our algorithm allows for a distributed implementation in a multiprocessor environment with relative ease.

**Authors** Jens Zumbrägel (University College Dublin, Ireland)
Mark F. Flanagan (University College Dublin, Ireland)
**Vitaly Skachek** (Nanyang Technological University, Singapore)

**Title** On the Pseudocodeword Redundancy

**Abstract** We define the BEC, AWGN, BSC, and max-fractional pseudocodeword redundancy $\rho(C)$ of a code $C$ as the smallest number of rows in a parity-check matrix such that the corresponding minimum pseudoweight is equal to the minimum Hamming distance of $C$. It was shown by Schwartz and Vardy that for any binary linear code, the BEC pseudocodeword redundancy is finite. In this work we show that most binary linear codes do not have a finite AWGN, BSC, and max-fractional pseudocodeword redundancy. We also provide bounds on the pseudocodeword redundancy for some families of codes, including codes based on designs. We also obtain the exact values of AWGN and BSC pseudocodeword redundancies for some short codes.

**Authors** Stefan Ruziška (University of Kaiserslautern, Germany)
**Akin Tanatmis** (University of Kaiserslautern, Germany)
Horst W. Hamacher (University of Kaiserslautern, Germany)
Norbert Wehn (University of Kaiserslautern, Germany)
Frank Kienle (University of Kaiserslautern, Germany)
Mayur Punekar (University of Kaiserslautern, Germany)

**Title** Numerical Comparison of IP Formulations in ML Decoding and Minimum Distance Calculation

**Abstract** For binary linear codes with short and medium block length, Maximum Likelihood (ML) decoding can be achieved by solving the associated integer programming (IP) problem with a general purpose solver. We analyze several IP formulations and computationally compare them on various LDPC, Turbo, BCH codes. Most of these formulations are obtained by forcing integrality on linear programming (LP) decoding formulations proposed in the literature. IP can also be used for computing the Minimum Distance (MD) of a code. We present results on MD of WiMax, WiFi & WiMedia LDPC codes, 3GPP LTE TC & WiMax dbTC used in current communication standards.
Linear Programming and Message-Passing Approaches to Parity-Check Codes and Graphical Models: Successes and Challenges (Tutorial)

Abstract

Linear programming and message-passing based approaches have been very successfully applied to sparse graphical models, in particular to graphical models representing low-density parity-check matrix codes. However, these approaches have also been applied successfully to some dense graphical models.

In this tutorial talk we will review the basics of factor graphs, message-passing iterative algorithms (belief propagation), and linear programming relaxations and explain why the latter two are tightly connected. Afterwards, we discuss dense graphical models where linear programming and message-passing based approaches work well and give reasons for this. We conclude by posing some open questions with respect to dense graphical models.

Concave Penalty Method for Improving Interior Point LP Decoding

Abstract

In this talk, a novel method to improve decoding performance of an interior point LP decoding algorithm for LDPC codes will be presented. The main idea of the proposed algorithm is to include a concave penalty term into the merit function which is used in an interior point method. The concave penalty term gives the smallest value if the search point stays at an integral point; i.e., the penalty term prefers an integral vertex of the fundamental polytope. From some computer experiments, it has been observed that the new algorithm provides better BER/FER performance than that of the conventional sum-product algorithm in some cases.

Near-ML Linear Programming Decoding of HDPC codes

Abstract

We propose several improvements for Linear Programming (LP) decoding algorithms for High Density Parity Check (HDPC) codes. First, we use the automorphism groups of a code to create parity check matrix diversity and to generate valid cuts from redundant parity checks. Second, we propose an efficient mixed integer decoder utilizing the branch and bound method. We further enhance the proposed decoders by removing inactive constraints and by adapting the parity check matrix prior to decoding according to the channel observations. Based on simulation results the proposed decoders achieve near-ML performance with reasonable complexity.