Energy-limited Joint Source–Channel Coding via Analog Pulse Position Modulation

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Abstract—We study the problem of transmitting a source sample with minimum distortion over an infinite-bandwidth additive white Gaussian noise channel under an energy constraint. To that end, we construct a joint source–channel coding scheme using analog pulse position modulation (PPM) and bound its quadratic distortion. We show that this scheme outperforms existing techniques since its quadratic distortion attains both the exponential and polynomial decay orders of Burnashev’s outer bound.

Index Terms—Networked Control, source coding with side information, Gaussian Channel, Communication With Feedback

I. INTRODUCTION

Recent developments in distributed sensor arrays and the internet of things raise the need for communicating a small number of measurements with small distortion over wireless media across large time spans and/or large bandwidth with limited energy.

This problem may be conveniently modeled as the classical setup of conveying \( k < \infty \) i.i.d. samples of a source over a continuous-time additive white Gaussian noise (AWGN) channel under an energy constraint per source sample.

In the limit of a large source blocklength, \( K \rightarrow \infty \), the optimal performance is known and dictated by the celebrated source–channel separation principle [1] Ch. 3.9]. For a memoryless standard Gaussian source and a quadratic distortion measure, the maximal (optimal) achievable signal-to-distortion ratio (SDR) is given by

\[
SDR = e^{2\text{ENR}},
\]

where ENR denotes the energy-to-noise ratio (ENR) of the channel. Since this source is the “least compressible” source with a given variance under a quadratic distortion measure, (1) serves as a lower bound on the achievable SDR for other sources.

While the optimal performance is known in the limit of large blocklength, it becomes much more challenging when the blocklength \( K \) is finite.

For a scalar source (\( K = 1 \)), both lower and upper bounds on the achievable SDR (equivalently, achievable distortion) have been constructed. Upper (impossibility) bounds on the SDR have been devised in [2]–[4], with Burnashev [3] providing the tightest upper bound on the SDR, for large ENRs, of

\[
SDR \leq K_1 \cdot \text{ENR}^{K_2} \cdot e^{\frac{\text{ENR}}{2}},
\]

for some positive constants \( K_1 \) and \( K_2 \) (which are not known explicitly), where ‘\( \leq \)' denotes ‘\( \leq \)' up to a multiplication by \( \{1 + o(1)\} \) where \( o(1) \rightarrow 0 \) for ENR \( \rightarrow \infty \). Separation-based schemes that attain

\[
SDR \geq K e^{\frac{\text{ENR}}{3}},
\]

have been constructed in [4]–[6]. These schemes employ scalar quantization in conjunction with orthogonal signaling—e.g., pulse position modulation (PPM)—that is known to be capacity-achieving in the power/energy limited regime [7] Ch. 8], [8] Ch. 8], [9] Ch. 2.5]. In particular, Sevínç and Tuncel [6] optimize the constant \( K \) in (3) by appealing to high-resolution quantization (a la Bennett’s asymptotic quantization [10], [11], [12] Ch. 5.6]).

However, by comparing (2) with (3), one observes a gap in the asymptotic behavior in the polynomial factor \( \text{ENR}^{K_2} \) that needs to be closed.

In this work, we construct a purely joint source–channel coding (JSCC) scheme for the setting of a scalar source and known ENR, in lieu of the available separation-based solutions: Instead of using quantization and digital PPM, we map the source sample into a shift of a rectangular pulse directly. We show that this scheme achieves Burnashev’s outer bound (2) for some positive \( K_1 \) and \( K_2 \), and is therefore strictly better than the hitherto available solutions (3) which fail to attain the additional polynomial improvement in (2).

The rest of the paper is organized as follows. We formulate the problem setup in Sec. II. We construct an analog PPM scheme and analyze its performance in Sec. III. We conclude the paper with a discussion and propose future research directions in Sec. IV.

II. PROBLEM STATEMENT

In this section, we formalize the JSCC setting that will be treated in this work.

Source. The source sample to be conveyed, \( x \in \mathbb{R} \), is distributed according to a known distribution and variance \( \sigma_x^2 \).

We will consider two specific distributions:

- Continuous uniform distribution over \([-0.5, 0.5]\).
- Standard Gaussian distribution.


Transmitter. Maps the source sample \( x \) to a continuous input waveform \( \{ s_x(t) \} \) that is subject to an energy constraint\(^1\):

\[
\frac{1}{2} \int_{-T/2}^{T/2} |s_x(t)|^2 dt \leq E, \quad \forall x \in \mathbb{R},
\]

where \( E \) denotes the transmit energy.

Channel. \( s_x \) is transmitted over a continuous-time AWGN channel:

\[
r(t) = s_x(t) + n(t), \quad t \in \left[ -\frac{T}{2}, \frac{T}{2} \right],
\]

where \( n \) is a continuous-time AWGN with two-sided spectral density \( N_0/2 \), and \( r \) is the channel output signal.

Receiver. Receives the channel output signal \( r \), and constructs an estimate \( \hat{x} \) of \( x \).

Distortion. The average quadratic distortion between \( x \) and \( \hat{x} \) is defined as:

\[
D \triangleq \mathbb{E}[(x - \hat{x})^2],
\]

and the corresponding signal-to-distortion ratio (SDR)—by

\[
\text{SDR} \triangleq \frac{\mathbb{E}[x^2]}{D}.
\]

Regime. We concentrate on the energy-limited regime, viz. the channel input is not subject to a power or a bandwidth constraint, but rather to an energy constraint.

As specified in (4), the channel input is subject to an energy constraint \( E \), the capacity of which is equal to \([13\ Ch.\ 9.3]\)

\[
C = \text{ENR},
\]

where \( \text{ENR} \triangleq E/N_0 \) is the energy-to-noise ratio (ENR), and the capacity is measured in nats. Note that the available bandwidth is unconstrained (i.e., infinite).

III. MAIN RESULT

In this section we present a scheme that employ analog PPM and derive upper bounds on its distortion for uniform and Gaussian sources in Secs. III-A and III-B respectively.

The performance of analog PPM—the main tool that we employ in this work—hinges on the ability of the receiver to estimate the position of a transmitted pulse with known shape corrupted by AWGN: Consider the problem statement of Sec. II with the following specific modulation:

\[
s_x(t) = \sqrt{E} \phi(t - x\Delta)
\]

where \( \phi(t) \) is a predefined pulse with unit energy and \( \Delta \) is a scaling parameter.

This fundamental problem received much attention over the years because of its importance in classical applications such as radar and in emerging applications such as the internet of things. Nonetheless, closed-form expressions for the optimal receiver and its distortion \((5)\) remain an open problem, in general. Consequently, various bounds and approximations have been derived over the years; see e.g., [14]–[18], [7, Ch. 8] (and the reference therein). Interestingly, the available results suggest that the shape of the pulse \( \phi \) may have a big effect on the achievable performance as well as the distribution of \( x \); specifically, a rectangular pulse is known to achieve good performance as we detail next.

Thus, we concentrate on the case of a rectangular pulse:

\[
\phi(t) = \begin{cases} \sqrt{\frac{E}{\Delta}}, & |t| \leq \frac{\Delta}{2 \\ 0, & \text{otherwise} \end{cases}
\]

for some parameter \( \beta > 1 \) which is sometimes referred to as effective dimensionality. Clearly, \( T = \Delta + \Delta/\beta \).

The optimal receiver is the MMSE estimator \( \hat{x} \) of \( x \) given the entire output signal:

\[
\hat{x}_{\text{MMSE}} = \mathbb{E}[x|r].
\]

A. Upper Bound on the Distortion for a Uniform Source

We construct here an upper bound on the achievable distortion of the proposed analog PPM scheme for a uniform source. For the sake of analysis, we examine the performance of the (suboptimal) maximum a posteriori (MAP) estimator instead of the (optimal) MMSE estimator \([7]\), which, for the case of a uniform source, reduces to the maximum-likelihood (ML) estimator:

\[
\hat{x}_{\text{MAP}} = \arg\max_{\hat{x}} \int_{|\hat{x}| \leq 0.5} \mathbb{E}\left[(r(t) - s_t(t))^2 \right] dt
\]

\[
\hat{x}_{\text{MAP}} = \arg\min_{\hat{x}} \int_{|\hat{x}| \leq 0.5} \mathbb{E}\left[(r(t) - s_t(t))^2 \right] dt.
\]

Since the waveform \( s_x \) of (6) has energy \( E \) for all \( x \), the ML estimator reduces further to maximum correlation:

\[
\hat{x}_{\text{MAP}} = \arg\max_{\hat{x}} \int_{|\hat{x}| \leq 0.5} r(t) s_{\hat{x}}(t) dt
\]

\[
\hat{x}_{\text{MAP}} = \arg\max_{\hat{x}} \int_{|\hat{x}| \leq 0.5} r(t) \phi(t - \hat{x}\Delta) dt
\]

\[
\hat{x}_{\text{MAP}} = \arg\max_{\hat{x}} \int_{|\hat{x}| \leq 0.5} R_{r,\phi}(\hat{x}\Delta),
\]

where

\[
R_{r,\phi}(\hat{x}\Delta) \triangleq \int_{-\infty}^{\infty} r(t) \phi(t - \hat{x}\Delta) dt
\]

\[
= \sqrt{\mathbb{E}R_{\phi}} ((x - \hat{x})\Delta) + \sqrt{\frac{\beta}{\Delta}} \int_{\hat{x}\Delta - \Delta/2}^{\hat{x}\Delta + \Delta/2} n(t) dt\]

is the (empirical) cross-correlation function between \( r \) and \( \phi \) with lag (displacement) \( \hat{x}\Delta \), and

\[
R_{\phi}(\tau) = \int_{-\infty}^{\infty} \phi(t) \phi(t - \tau) dt
\]

\[
= \begin{cases} 1 - \frac{\tau^2}{\beta}, & |\tau| \leq \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}
\]

\(^3\)Clearly, the bandwidth of this pulse is infinite. By taking a large enough bandwidth \( W \), one may approximate this pulse to an arbitrarily high precision and attain its performance within an arbitrarily small loss.
is the autocorrelation function of \( \phi \) with lag \( \tau \).

The next proposition bounds from above the distortion of this receiver.

**Proposition 1.** The distortion of the MAP estimator \( \hat{x} \) of a scalar source uniformly distributed over a unit interval transmitted using analog PPM with a rectangular pulse is bounded from above by

\[
D \leq D_S + P_L D_L, \tag{11}
\]

where

\[
D_S = \tilde{D}_S \cdot \left( 1 + 16 \frac{\sqrt{\beta \text{ENR} \cdot e^{-\text{ENR}}}}{13} \right),
\]

\[
D_L = \frac{1}{6} \left( 1 + \frac{2}{\beta} + \frac{4}{\beta^2} \right),
\]

\[
P_L = \tilde{P}_L \cdot \left( 1 + \sqrt{3} \frac{e^{-\text{ENR}}}{4 \sqrt{\text{ENR}}} + 4 \sqrt{\frac{\pi}{\text{ENR}}} \right),
\]

are upper bounds on the “small-error” distortion (when the error is less than or equal to \( 1/\beta \)), “large-error” distortion, and the probability of a large error, respectively, and

\[
\tilde{D}_S \triangleq \frac{13}{8} \frac{1}{(\beta \text{ENR})^2},
\]

\[
\tilde{P}_L \triangleq \frac{\beta \sqrt{\text{ENR}} e^{-\text{ENR}}}{16 \sqrt{\pi}}.
\]

In particular, in the limit of large ENR, and \( \beta \) that increases monotonically with ENR,

\[
D \leq \left( \tilde{D}_S + \tilde{P}_L D_L \right) \{ 1 + o(1) \} \tag{12}
\]

where \( o(1) \rightarrow 0 \) in the limit of \( \text{ENR} \rightarrow \infty \).

**Proof:** Denote the estimation error by \( \epsilon \triangleq x - \hat{x} \). Then, by the law of total expectation:

\[
\mathbb{E} [\epsilon^2] = P \left( |\epsilon| \leq \frac{1}{\beta} \right) \mathbb{E} [\epsilon^2 | \epsilon | \leq \frac{1}{\beta}] + P \left( |\epsilon| > \frac{1}{\beta} \right) \mathbb{E} [\epsilon^2 | |\epsilon| > \frac{1}{\beta}] \tag{13a}
\]

\[
\leq \mathbb{E} [\epsilon^2 | |\epsilon| \leq \frac{1}{\beta}] + P \left( |\epsilon| > \frac{1}{\beta} \right) \mathbb{E} [\epsilon^2 | |\epsilon| > \frac{1}{\beta}] \tag{13b}
\]

We now bound the terms in (13b) by \( D_S, P_L \) and \( D_L \).

\[
\mathbb{E} [\epsilon^2 | |\epsilon| \leq \frac{1}{\beta}] \leq D_S, P \left( |\epsilon| > \frac{1}{\beta} \right) \leq P_L.
\]

follows from \( [16, \text{Eq. 6}] \) and \( [17, \text{Eq. 15}] \), respectively. To bound the remaining term in (13b), note that for \( |\epsilon| > \frac{1}{\beta} \), or equivalently \( |(x - \hat{x})| > \frac{1}{2\beta} \), the autocorrelation term

\[
R_{\phi}(x - \hat{x}) \Delta
\]

in (10a) is nullified and only a noise term remains. Thus, using (13), (10), the stationarity of the noise process \( n \) and symmetry, we arrive at

\[
\mathbb{E} [\epsilon^2 | |\epsilon| > \frac{1}{\beta}] = \mathbb{E} \left[ \left( x - \arg\max_{\hat{x}} R_{\phi}(\hat{x}) \Delta \right)^2 | |\epsilon| > \frac{1}{\beta} \right].
\]

Fig. 1: The optimized empirical SDR and the lower bound on the SDR of Prop. 1 for analog PPM for a uniform source, and the corresponding optimal \( \beta \) values.

\[
= \mathbb{E} \left[ \left( x - \arg\max_{\hat{x}} R_{\phi}(\hat{x}) \Delta \right)^2 | |\epsilon| > \frac{1}{\beta} \right].
\]

By optimizing over \( \beta \), we are able to derive the following upper bound on the achievable distortion.

**Theorem 1.** The achievable distortion of a uniform scalar source transmitted over an energy-limited channel is bounded from above as

\[
D \leq K \cdot e^{-\text{ENR}} \cdot (\text{ENR})^{-\frac{2}{\beta}} \cdot \{ 1 + o(1) \}, \tag{14}
\]

where \( K \leq 0.058 \), and \( o(1) \rightarrow 0 \) as \( \text{ENR} \rightarrow \infty \).

**Proof:** Setting \( \beta = (312 \sqrt{\pi})^\frac{1}{3} (\text{ENR})^{-\frac{2}{3}} e^{\frac{\text{ENR}}{3}} \) in (12) of Prop. 1 yields (14).

Thus, using the analog PPM scheme, we are able to meet Burnashev’s asymptotic outer bound with \( K_1 = 1/K \approx 17.2 \) and \( K_2 = 1/3 \). Therefore, in addition to attaining the optimal exponential decay with the ENR, it achieves also the next-order polynomial decay with the ENR. This offers an improvement over the performance of the separation-based scheme of Seviç and Tuncel (3), which, despite attaining the best exponential decay of 1/3, does not attain the additional polynomial decay with the ENR.

We further simulated the analog PPM scheme for finite (non-asymptotic) ENR values, and optimized its performance numerically over \( \beta \). The optimized performance of the scheme along with the optimized bound of Prop. 1 over \( \beta \) are depicted in Fig. 1. Evidently, there is a slack in the bound which is discussed in Sec. IV.

**B. Upper Bound on the Distortion for a Gaussian Source**

We now construct an upper bound on the achievable distortion of the proposed analog PPM scheme for a standard Gaussian source.
Again, for the sake of analysis, we examine the performance of the (suboptimal) MAP estimator in lieu of the (optimal) MMSE estimator (7). Using a similar set of steps to that in (8) and (11) for a Gaussian source, we arrive at
\[ x^{\text{MAP}} = \arg\max_{a \in \mathbb{R}} \{ \lambda(a) \} , \quad (15a) \]
where
\[ \lambda(a) = R_{r,\phi}(a\Delta) - \frac{N_0}{4\sqrt{\beta}} a^2 , \quad (15b) \]
and \( R_{r,\phi} \) was defined in (10a).

**Remark 1.** Since a Gaussian source has infinite support, the required overall transmission time \( T \) is infinite. Of course this is not possible in practice. Instead, one may limit the transmission time \( T \) to a very large—yet finite—value. This will incur a loss compared to the bound that will be stated next; this loss can be made arbitrarily small by taking \( T \) to be large enough.

**Proposition 2.** The distortion of the MAP estimator (15a) of a standard Gaussian scalar source transmitted using analog PPM with a rectangular pulse is bounded from above as in (11), with\(^4\)
\[ P_L D_L \leq 2\beta \sqrt{\text{ENR}} e^{-\frac{\text{ENR}}{2}} \left( 1 + 3\sqrt{\frac{2\pi}{\text{ENR}}} + \frac{12e^{-1}}{\beta \sqrt{\text{ENR}}} \right) + \frac{8e^{-1}}{\beta \sqrt{8\pi} \beta} + \frac{12e^{-2}}{\beta \sqrt{32\pi \text{ENR}}} \right) \]
\[ + \beta \sqrt{8 \pi} e^{-\text{ENR}} \left( 1 + \frac{4e^{-1}}{\beta \sqrt{2\pi}} \right) , \]
\[ D_S \leq \frac{13}{8} + 2 \left( 2\beta \sqrt{\text{ENR}} - 1 \right) \cdot e^{\left( \sqrt{\text{ENR}} - \frac{1}{\sqrt{2\pi}} \right)^2} + e^{-\beta \text{ENR}} \frac{1}{\beta^2} , \]
assuming \( \beta \text{ENR} > 1/2 \). In particular, in the limit of large ENR, and \( \beta \) that increases monotonically with ENR,
\[ D \leq \left( \tilde{D}_S + \tilde{D}_L \right) \{ 1 + o(1) \} \quad (16) \]
where
\[ \tilde{D}_S \triangleq \frac{13/8}{(\beta \text{ENR})^2} , \]
\[ \tilde{D}_L \triangleq 2\beta \sqrt{\text{ENR}} e^{-\frac{\text{ENR}}{2}} , \]
and \( o(1) \to 0 \) in the limit of \( \text{ENR} \to \infty \).

**Proof:** The proof follows the same lines as that of Prop. 11. Following Ziv and Zakai (17), we divide the real line into consecutive intervals of length \( \frac{1}{\beta} \):
\[ B_i \triangleq \left\{ \frac{i-1}{\beta} < \epsilon \leq \frac{i}{\beta} \right\} , \quad i \in \mathbb{Z} . \]
and define
\[ A_i \triangleq \lambda(x) = \max_{a : \frac{1}{\beta} < a \leq \frac{i}{\beta}} \lambda(a) , \quad i \in \mathbb{Z} . \quad (17) \]

\(^4\)The notation \( P_L D_L \) is used for consistency with Prop. 11.

where \( \lambda \) was defined in (15b).

By using the law of total probability, we have
\[ E\left[ e^2 \right] = \sum_{i=0}^{\infty} E\left[ e^2 | \epsilon \in B_i \right] Pr \left( \epsilon \in B_i \right) \quad (18a) \]
\[ \leq E\left[ e^2 | \epsilon \leq \frac{1}{\beta} \right] + 2 \sum_{i=1}^{\infty} E\left[ e^2 | \epsilon \in B_i \right] Pr \left( \epsilon \in B_i \right) \quad (18b) \]
\[ \leq E\left[ e^2 | \epsilon \leq \frac{1}{\beta} \right] + 2 \sum_{i=2}^{\infty} \left( \frac{1}{\beta} \right)^2 E \left[ Pr \left( \epsilon \in B_i | x \right) \right] \quad (18c) \]
\[ \leq E\left[ e^2 | \epsilon \leq \frac{1}{\beta} \right] + 2 \sum_{i=2}^{\infty} \left( \frac{1}{\beta} \right)^2 E \left[ Pr \left( A_i | x \right) \right] \quad (18d) \]
\[ \leq D_S + P_L D_L \quad (18e) \]
where (18d) holds since \( Pr \left( A_i | x \right) \geq Pr \left( B_i | x \right) \) for all \( x \), and the two terms in (18d) are proved to be bounded from above by the two terms in (18e) in Apps. 11 and 14 respectively. ■

**Theorem 2.** The achievable distortion of a standard Gaussian scalar source transmitted over an energy-limited channel with a known ENR is bounded from above as
\[ D \leq K \cdot e^{-\frac{\text{ENR}}{2}} \cdot \left( \text{ENR} \right)^{-\frac{1}{2}} \cdot \{ 1 + o(1) \} , \quad (19) \]
where \( K \leq 3 \cdot \left( \frac{13}{8} \right)^2 \) and \( o(1) \to 0 \) as \( \text{ENR} \to \infty \).

**Proof:** Setting \( \beta = \left( \frac{13}{8} \right)^2 (\text{ENR})^{-\frac{1}{2}} e^{-\frac{\text{ENR}}{2}} \) in (16) of Prop. 2 yields (19). ■

**IV. DISCUSSION AND FUTURE RESEARCH**

In this work, we examined the problem of JSCC over an energy-limited channel with unlimited bandwidth and transmission time. We showed that analog PPM that maps directly the source sample to a time shift of the transmission pulse, offers improvement over separation-based schemes due to the low-delay nature of the problem on the source side.

Although we assumed that both the bandwidth and the time are unlimited, the scheme and analysis presented in this work carry over to the setting where one of the two is bounded as long as the other one is unlimited. In fact, the parameter \( T \) was inconsequential in the analysis and performance of the scheme. Clearly, the presented scheme may be adjusted for the setting of limited bandwidth and unlimited transmission time to achieve the same performance guarantees.

For a uniform source, Burnashev’s impossibility bound (2) holds with \( K_1 = 1/K_2 \). Comparing it to the derived achievability bound of Th. 1 in this work, suggests that there is still a gap between the two bounds, since \( K > 1/3 \). In fact, Ibragimov and Khas’minskii (19) (see also the nice summary in (16)) derived the asymptotic performance of the MMSE estimator (7) and that of the MAP/ML estimator (8), and showed that the latter is strictly suboptimal. Thus, careful analysis of the MMSE estimator in lieu of the MAP one analyzed in this work should yield better performance. Another interesting future research direction is deriving the optimal coefficients in Burnashev’s impossibility bound.

In this work we concentrated on the setting of transmitting a single source sample over the channel. An interesting research
direction would be to construct a non-trivial extension of the proposed scheme to vector sources.

When the noise level is not known at the transmitter and for large source blocklengths [for which separation becomes optimal, i.e., achieves (1)], universal schemes that attain a desired distortion profile have been devised in [20], [21]. All of these schemes employ digital and analog linear layers for transmission; replacing the analog linear layers with analog PPM ones offers an improvement.

**APPENDIX A**

**UPPER BOUND ON THE SMALL-ERROR DISTORTION IN PROP. 2**

In this appendix, we analyze the small-error distortion $\mathbb{E}[e^2 | e \leq 1/\beta]$ for a Gaussian source. Our analysis builds on that of Zehavi [16]. We first analyze the distortion given $x$.

When the estimate $\hat{x}$ deviates from $x$ by no more than $1/\beta$, the MAP decoder of $x$ given $r$ may be expressed as

$$\hat{x} = \arg\max_{\hat{x} : |x - \hat{x}| \leq \frac{1}{\beta}} \left\{ \sqrt{\mathbb{E} \left[ \frac{1 - \beta}{\Delta} \hat{x} - x \right]} + \sqrt{\frac{\beta}{\Delta} \int_{|\hat{x} - x|/\beta}^{\hat{x} + x/\beta} n(t) dt} - \frac{N_0}{4\sqrt{E}} \right\}$$  

(20a)

$$= \arg\max_{\hat{x} : |x - \hat{x}| \leq \frac{1}{\beta}} \left\{ -\sqrt{2\beta ENR} |\hat{x} - x| - \frac{\hat{x}^2}{8\beta ENR} + \int_{\hat{x} - x/\beta}^{\hat{x} + x/\beta} \hat{n}(t) dt \right\}$$  

(20b)

$$= \arg\max_{\hat{x} : |x - \hat{x}| \leq \frac{1}{\beta}} \left\{ -\sqrt{2\beta ENR} |\hat{x} - x| + \frac{x^2 - \hat{x}^2}{8\beta ENR} + B \left( \hat{x} + \frac{1}{\beta} \right) - B(\hat{x}) + B(x) - B \left( x + \frac{1}{\beta} \right) \right\}$$  

(20c)

where (20a) follows from (15) and (10); in (20b) and in (20c), we define the white Gaussian noise $\hat{n}(t) = \sqrt{\frac{2}{N_0}} n(t)$ with unit level and normalize the integration interval by factor $\Delta$, we define the two-sided Wiener process $B(x) = \int_{x}^{\hat{x}} \hat{n}(t) dt$, that is, $B(x) = W_1(x)$ for $x \geq 0$ and $B(x) = W_2(-x)$ for $x < 0$, such that $W_1$ and $W_2$ are independent standard Wiener processes. We next substitute the upper bound

$$\frac{x^2 - \hat{x}^2}{8\beta ENR} \leq |x (\hat{x} - x)|$$

in (20c). One may verify that the difference function

$$|x (\hat{x} - x)| - \frac{x^2 - \hat{x}^2}{2}$$

is monotonically increasing in the absolute error $|\hat{x} - x|$ for a given $x$. Consequently, the probability that the $\arg\max$ in (20c) will produce larger absolute errors is higher after this change, meaning that the mean quadratic error (i.e., the distortion) will increase as well.

Therefore, the virtual decoder

$$\arg\max_{\hat{x} : |x - \hat{x}| \leq \frac{1}{\beta}} \left\{ -\sqrt{2\beta ENR} \left( 1 - \frac{|x|}{2\beta ENR} \right) |\hat{x} - x| + B \left( \hat{x} + \frac{1}{\beta} \right) - B(\hat{x}) + B(x) - B \left( x + \frac{1}{\beta} \right) \right\}$$

incurs a larger distortion than the MAP one (20), and is equivalent to the decoder in [16] Eq. (10)] with

$$C = \sqrt{2\beta ENR} \left( 1 - \frac{|x|}{2\beta ENR} \right).$$

Thus, using the analysis of [16] gives rise to the following upper bound on the conditional distortion of the MAP decoder given $x$:

$$\mathbb{E} [e^2 | x] \leq D_S(x) \leq \frac{13}{\pi} + 2 \left( \frac{2\beta ENR}{\sqrt{2\beta ENR}} \right)^4 \frac{D_S(a) da}{\sqrt{2\beta ENR}}$$

$$\leq D_S \left( \frac{2\beta ENR}{\sqrt{2\beta ENR}} \right) + \frac{2}{\beta^2} \frac{\mathbb{E} \left[ |x - \hat{x}|^2 | x > \sqrt{2\beta ENR} \right]}{\beta^2}$$

where we recognize $D_S$ from Prop. 2 in the last step.

**APPENDIX B**

**UPPER BOUND ON THE LARGE-ERROR DISTORTION IN PROB. 2**

In this appendix, we will make frequent use of the following standard upper and lower bounds on the $Q$ function (22):

$$\frac{a}{1 + a^2} \cdot e^{-a^2/2} < Q(a) < \frac{1}{2} e^{-a^2/2}$$  

(21)

We prove here that

$$2 \sum_{i=2}^{\infty} \left( \frac{i}{\beta} \right)^2 \mathbb{E} \left[ \Pr (A_i | x) \right] \leq P_L D_L.$$  

(22)

To that end, we first bound from above the probability of $A_i$ given $x$ for $i \geq 2$. In the following set of inequalities, we do not write explicitly the conditioning on $x$, to simplify notation.

$$\Pr (A_i) = \Pr \left( \lambda(x) \leq a : \max_{\beta^2 < a \leq \beta} \lambda(a) \right)$$  

(23a)
\[
\Pr \left( \sqrt{E} + \sqrt{\frac{\beta}{\Delta}} \int_{x_0}^{x_0 + \frac{\Delta}{\beta}} n(t)dt - \frac{N_0}{4\sqrt{E}} \leq a \right) \leq \Pr \left( \eta \leq \max_{a: \frac{1}{\beta} - x_0 < a < \frac{1}{\beta}} \sqrt{\frac{\Delta}{\beta}} \int_{x_0}^{x_0 + \frac{\Delta}{\beta}} n(t)dt - \frac{N_0}{4\sqrt{E}} \right)
\]

\[
\Pr \left( \eta \leq \max_{a: \frac{1}{\beta} - x_0 < a < \frac{1}{\beta}} \sqrt{\frac{\Delta}{\beta}} \int_{x_0}^{x_0 + \frac{\Delta}{\beta}} n(t)dt - \frac{N_0}{4\sqrt{E}} \right) \leq \Pr \left( \eta \leq \max_{a: \frac{1}{\beta} - x_0 < a < \frac{1}{\beta}} \sqrt{\frac{a \Delta + \Delta}{\beta}} \int_{x_0}^{x_0 + \frac{\Delta}{\beta}} n(t)dt - \frac{N_0}{4\sqrt{E}} \right)
\]

\[
\eta \leq \max_{a: \frac{1}{\beta} - x_0 < a < \frac{1}{\beta}} \sqrt{\frac{a \Delta + \Delta}{\beta}} \int_{x_0}^{x_0 + \frac{\Delta}{\beta}} n(t)dt - \frac{N_0}{4\sqrt{E}} \]

\[
\Pr \left( \eta \leq \max_{a: \frac{1}{\beta} - x_0 < a < \frac{1}{\beta}} \sqrt{\frac{a \Delta + \Delta}{\beta}} \int_{x_0}^{x_0 + \frac{\Delta}{\beta}} n(t)dt - \frac{N_0}{4\sqrt{E}} \right) \leq \Pr \left( \eta \leq \max_{a: \frac{1}{\beta} - x_0 < a < \frac{1}{\beta}} \sqrt{\frac{\Delta}{\beta}} \int_{x_0}^{x_0 + \frac{\Delta}{\beta}} n(t)dt - \frac{N_0}{4\sqrt{E}} \right)
\]

\[
\Delta \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q(x) e^{-\frac{(x-\mu)^2}{2}} dx
\]

\[
\leq \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{0} e^{-\frac{(x-\mu)^2}{2}} dx + e^{-\frac{\mu^2}{2}} \int_{0}^{\infty} e^{-\frac{(x-\mu)^2}{2}} dx \right)
\]

\[
= Q(\mu) + \frac{e^{-\mu^2/2}}{2\sqrt{\pi}} \left( 1 - Q\left( \frac{\mu}{\sqrt{2}} \right) \right),
\]

and by defining

\[
q(x) = \sqrt{2ENR} \left( 1 - \frac{x^2}{4ENR} \right)
\]

\[
= \sqrt{2ENR} \left( 1 + \frac{1}{4ENR} \left( \frac{i}{\beta} + x \right)^2 \left( \frac{i - 1}{\beta} + x \right)^2 \right) + \frac{1}{4ENR} \min \left\{ 0, \frac{i}{\beta^2} - \frac{2}{\beta} \left( x + \frac{i}{\beta} \right) \right\}
\]

\[
\ell(x) = \sqrt{2ENR} \left( 1 + \frac{1}{4ENR} \left( \frac{i}{\beta} \right)^2 + \frac{1}{2ENR} \frac{i}{\beta} x \right),
\]

which by definition is always greater than or equal to \( q \) and since the probability is always bounded from above by 1; (23g) follows from (17, 23) for all \( q \) using (23f) by defining

\[
\ell(x) = \sqrt{2ENR} \left( 1 + \frac{1}{4ENR} \left( \frac{i}{\beta} \right)^2 + \frac{1}{2ENR} \frac{i}{\beta} x \right),
\]

where (23a) and (23b) hold by the definitions of \( A_i, (17) \) and \( \lambda (15b) \), respectively, and since \( R_{r,\phi}(a\Delta) = 0 \) for \( i \geq 2 \); (23c) holds by defining

\[
\eta \leq \sqrt{\frac{\Delta}{\beta}} \int_{x_0}^{x_0 + \frac{\Delta}{\beta}} \sqrt{\frac{2}{N_0}} n(t)dt + \frac{1}{\sqrt{2ENR}} \left[ \min \left\{ \left( \frac{i}{\beta} + x \right)^2, \left( \frac{i - 1}{\beta} + x \right)^2 \right\} - x^2 \right],
\]

by taking the minimum value of \( a^2 \) inside the interval \( \{ a: \frac{i - 1}{\beta} < a < \frac{i}{\beta} \} \); (23d) holds by defining the process

\[
w(a) \leq \max_{\frac{i - 1}{\beta} < a < \frac{i}{\beta}} \sqrt{\frac{\Delta}{\beta}} \int_{x_0}^{x_0 + \frac{\Delta}{\beta}} \sqrt{\frac{2}{N_0}} n(t)dt \]

\[
\Pr (A_i) = \mathbb{E} \left[ \Pr (A_i | x) \right] \]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Pr (A_i | x) e^{-\frac{x^2}{2}} dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{x_0}^{\infty} \Pr (A_i | x) e^{-\frac{x^2}{2}} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} \left( \sqrt{\frac{3}{4\pi}} e^{-\frac{\ell^2(x)}{4\pi}} + \frac{1}{2\sqrt{\pi}} + \frac{\ell(x)}{4\sqrt{\pi}} e^{-\frac{\ell^2(x)}{4\pi}} + e^{-\frac{\ell^2(x)}{4\pi}} \right) dx
\]
\[
\frac{1}{2\sqrt{2}} + \sqrt{\frac{\text{ENR}}{8\pi} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \cdot e^{-\text{ENR} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \\
\quad + \frac{1}{\sqrt{1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2}} \cdot e^{-\text{ENR} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \\
\quad + Q \left(\frac{\text{ENR} \beta}{i} + \frac{i}{2\beta}\right) \\
\leq \frac{1}{2\sqrt{2}} + \sqrt{\frac{\text{ENR}}{8\pi} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \cdot e^{-\text{ENR} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \\
\quad + \frac{1}{\sqrt{1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2}} \cdot e^{-\text{ENR} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \\
\quad + Q \left(\frac{\text{ENR} \beta}{i} + \frac{i}{2\beta}\right) \\
\leq \left(\frac{1}{2\sqrt{2}} + \sqrt{\frac{\text{ENR}}{8\pi} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \cdot e^{-\text{ENR} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \\
\quad + \frac{1}{\sqrt{1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2}} \cdot e^{-\text{ENR} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \\
\quad + Q \left(\frac{\text{ENR} \beta}{i} + \frac{i}{2\beta}\right) \right) \\
\leq \left(\frac{1}{2\sqrt{2}} + \sqrt{\frac{\text{ENR}}{8\pi} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \cdot e^{-\text{ENR} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \\
\quad + \frac{1}{\sqrt{1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2}} \cdot e^{-\text{ENR} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \\
\quad + Q \left(\frac{\text{ENR} \beta}{i} + \frac{i}{2\beta}\right) \right) \\
\quad \text{where (25c) follows from (23), (25d) follows from the definition of } \ell (24) \text{ and the following integral identity (and upper bound) which holds for any positive reals } a, b, c, k_2 \text{ and any real } k_1:\n\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(k_1 + k_2x\right)e^{-\frac{1}{2} \left(a(6+cx)^2+x^2\right)}dx \\
= \frac{k_1}{\sqrt{1 + ac^2}} e^{-\frac{ac^2}{2(\pi + \pi x^2)}} - \frac{abck_2}{\sqrt{1 + ac^2}} e^{-\frac{ac^2}{2(\pi + \pi x^2)}} \\
\leq \frac{k_1}{\sqrt{1 + ac^2}} e^{-\frac{ac^2}{2(\pi + \pi x^2)}} \\
\text{Equipped with an upper bound (25) on } \text{Pr} \left(A_1\right) \text{ and using (22), we will derive next the desired upper bound on } D_L:\nP_L D_L \leq 2 \sum_{i=2}^{\infty} \left(\frac{i}{\beta}\right)^2 \cdot Q \left(\frac{\text{ENR} \beta}{i} + \frac{i}{2\beta}\right) \\
\quad + 2 \sum_{i=2}^{\infty} \left(\frac{i}{\beta}\right)^2 \left(\frac{3}{2} + \sqrt{\frac{\text{ENR}}{8\pi} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \right) \cdot e^{-\text{ENR} \left(1 + \frac{1}{\text{ENR}} \left(\frac{i}{\beta}\right)^2\right)} \\
\leq 2 (D_{L,1} + D_{L,2}) \\
\leq P_L D_L, \\
\text{where } D_{L,1} \text{ and } D_{L,2} \text{ in (26b) denote the two sums in (26a), and (26c) is proved next.}
REFERENCES


