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Model Reduction by Chebyshev Polynomial Techniques

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***Abstract*—The problem of reduced-order modeling of high-order, linear, time-invariant, single-input, single-output systems is considered. A method is proposed, based on manipulating two Chebyshev polynomial series, one representing the frequency response characteristics of the high-order system and the other representing the approximating low-order model. The method can be viewed as generalizing the classical Padé approximation problem, with the Chebyshev polynomial series expansion being over a desired frequency interval instead of a power series about a single frequency point. Two different approaches to the problem are considered. Firstly, approximation is carried out in the *s*-plane by a Chebyshev polynomial series. Then, modified Chebyshev polynomials are introduced and a mapping to a new plane is defined. It turns out that in the new plane the advantages of the generalized Chebyshev–Padé approximations are retained while what is actually being solved is the classical Padé problem.**

I. INTRODUCTION

Order reduction of dynamic systems is an area of research that receives considerable attention in the literature [1]–[4]. Among the various model reduction methods, those of the algebraic type are computationally more attractive. These include, for example, the Padé class of methods like the continued fraction expansion [5], time moments [6], and Padé approximations [7]. It was shown [8] that, under certain mild conditions, these methods yield the same Padé approximants, the direct Padé approximation being the more general one. Moment approximants [8] involve a time-domain criterion of approximation and Padé approximation can therefore be seen as a computational procedure for obtaining moment approximants in their Laplace transforms.

However, being mostly approximations about a single frequency point ($s\to 0$), the algebraic methods may yield poor whole frequency response characteristics. Modified Padé approximations at two frequency points ($s\to 0$, $s\to\infty$) were suggested to deal with this problem [9], [10]. Furthermore, some of the Padé methods may produce an unstable reduced-order model even though the high-order system is stable. The Routh approximation method [11], [12] is a procedure for dealing with this problem.

In this paper, reduced-order modeling of linear, time-invariant, single-input, single-output systems is obtained over a desired *frequency interval.* The method also enables the *selection* of *stable* and minimal phase models (low-order models can also be designed with prescribed poles and zeros; details may be found in [13]). All this is being accomplished by manipulating two Chebyshev polynomial series, one representing the frequency characteristics of the high-order system and the other representing the approximating low-order model. The underlying idea can be regarded as a generalization of the classical Padé approximations, with the Chebyshev polynomial series expansion being over a desired *frequency interval* instead of a power series about a single frequency point.

Two different approaches to the problem are considered. In the first (Section II), the problem is dealt with in the *s*-plane and the squared amplitude of the transfer function is expanded by a Chebyshev polynomial series and approximated to the desired order by Maehly's method [14]. In the second approach (Section III), modified Chebyshev polynomials are introduced, labeled Darlington polynomials [15], and used to expand the gain of the high-order transfer function and its reduced-order model. By a special transformation to a new plane, the original problem of generalized Padé approximation becomes the (simple) classical Padé problem.

Two characteristics of the Chebyshev polynomial expansion make the proposed method very attractive. The first is that a truncated Chebyshev series has better convergence than a truncated power series of the same length. The second is that a Chebyshev series has very good (near minimax) accuracy over its interval of expansion in comparison with other possible choices of orthogonal polynomial series expansions over the same interval, and with the accuracy at a single point of the Taylor series expansion.

Thus, the two approaches proposed in this paper, which can be viewed as generalized Chebyshev–Padé approximations, are very attractive model reduction procedures in the frequency domain. Their advantage over existing methods is that they produce reduced-order models over a desired frequency interval which can be selected to be also stable and of minimal phase. Nevertheless, the second approach turns out to be superior to the first since it combines the superiority of Chebyshev polynomial expansion over a desired frequency interval with the simplicity and flexibility of the classical Padé approximation.

Only low-pass amplitude approximations are considered in this paper. However, the method allows extensions to bandpass and high-pass approximations [16], with the second approach having the advantage since it requires only minor changes in the computational algorithm.

II. APPROXIMATIONS IN THE *s*-PLANE

A. Chebyshev Polynomials

Consider the following nth order Chebyshev polynomials [14]

$$T_n(x)=\cos(n\cos^{-1}x) \qquad x\in[-1,1] \tag{2.1}$$

which are orthogonal on the interval $[-1,1]$ with weight function $(1-x^2)^{-1/2}$. These polynomials are recursively related by

$$T_{n+1}(x)=2xT_n(x)+T_{n-1}(x), \qquad n=1,2\cdots \tag{2.2}$$

where $T_0(x)=1$ and $T_1(x)=x$. $T_n(x)$ is even if n is even,

$$T_{2n}(x)=\sum_{i=0}^{n}\alpha_{2i}x^{2i} \tag{2.3a}$$

and is odd if n is odd,

$$T_{2n+1}(x)=\sum_{i=0}^{n}\alpha_{2i+1}x^{2i+1}. \tag{2.3b}$$

Finally, it can be shown that

$$T_m(x)T_n(x)=\frac{1}{2}[T_{m+n}(x)+T_{|m-n|}(x)], \qquad \forall m,n. \tag{2.4}$$

Manuscript received July 10, 1978; revised April 16, 1979. Paper recommended by P. E. Caines, Chairman of the Identification Committee. This work was done while Y. Bistritz was with the Nuclear Research Center Negev, Beer-Sheva, Israel.

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Consider the following Chebyshev polynomial series expansion:

$$f(x)=a_0/2+\sum_{n=1}^{\infty} a_n T_n(x), \qquad x\in[-1,1]. \tag{2.5}$$

Since the polynomials are orthogonal

$$a_n=\frac{2}{\pi}\int_{-1}^{1}\frac{f(x)T_n(x)}{\sqrt{1-x^2}}dx. \tag{2.6}$$

If $f(x)$ is an odd function, then

$$f(x)=\sum_{n=1}^{\infty} a_{2n+1}T_{2n+1}(x) \tag{2.7}$$

and if $f(x)$ is an even function, then

$$f(x)=a_0/2+\sum_{n=1}^{\infty} a_{2n}T_{2n}(x). \tag{2.8}$$

B. Model Reduction in the s-Plane

Consider the linear, time-invariant system represented by its transfer function

$$G(s)=K\frac{\prod_{i=1}^{m}(s-s_i')}{\prod_{i=1}^{n}(s-s_i'')} \qquad m<n \tag{2.9}$$

where s_i' and s_i'' are the ith zero and pole, respectively. In the sequel, the symbol $\{m/n\}$ will be used to denote a transfer function of the form (2.9).

The following representative quantities related to (2.9) will be referred to in the sequel; let $s=j\omega$.

1) The amplitude of $G(s)$ is denoted by $|G(j\omega)|$.
2) The squared amplitude of $G(s)$ is

$$H(\omega^2)\triangleq|G(j\omega)|^2 \tag{2.10}$$

$$=K^2\frac{\prod_{i=1}^{m}(s_i'^2-s^2)}{\prod_{i=1}^{n}(s_i''^2-s^2)}\Bigg|_{s=j\omega}. \tag{2.11}$$

3) Define the gain and phase of $G(s)$ by

$$g+j\psi\triangleq\ln G(s). \tag{2.12}$$

Hence, the gain of $G(s)$ is

$$g=\ln|G(s)|=\frac{1}{2}\ln[G(s)\overline{G(s)}] \tag{2.13a}$$

in Neppers (or in dB if converted to log of base 10), and the phase of $G(s)$ is

$$j\psi=\frac{1}{2}\ln[G(s)/\overline{G(s)}] \tag{2.13b}$$

where $\overline{G(s)}$ in (2.13) denotes the complex conjugate of $G(s)$. For $s=j\omega$, (2.13) yields, respectively,

$$g=\frac{1}{2}\ln[G(s)G(-s)]|_{s=j\omega}=\frac{1}{2}\ln H(\omega^2) \tag{2.14a}$$

$$j\psi=\frac{1}{2}\ln[G(s)/G(-s)]|_{s=j\omega}. \tag{2.14b}$$

Assume that for a desired low-pass approximation the squared amplitude of $G(s)$ is approximated over the frequency interval $[0,\omega_0]$. Expanding the even function $H(\omega^2)$ of (2.10) by a Chebyshev polynomial series yields

$$H(\omega^2)=\sum_{i=0}^{\infty} c_{2i}T_{2i}\left(\frac{\omega}{\omega_0}\right). \tag{2.15}$$

Let the desired low-order model be $\{k/p\}$, $p<n$, namely,

$$\hat{G}(s)=\hat{K}\frac{\prod_{i=1}^{k}(s-\hat{s}_i')}{\prod_{i=1}^{p}(s-\hat{s}_i'')} \tag{2.16}$$

and let $\hat{H}(\omega^2)$ be its squared amplitude function whose Chebyshev polynomial series expansion is

$$\hat{H}(\omega^2)=\sum_{i=0}^{\infty}\hat{c}_{2i}T_{2i}\left(\frac{\omega}{\omega_0}\right). \tag{2.17}$$

Thus, our problem is to find $\hat{H}(\omega^2)$ such that

$$\hat{H}(\omega^2)\overset{r}{\equiv}H(\omega^2) \tag{2.18}$$

where $\equiv$ denotes the requirement that the first r terms in the series expansion of $\hat{H}(\omega^2)$ should coincide with those of the series expansion of $H(\omega^2)$, i.e., $\hat{c}_{2i}=c_{2i}$ $i=0,1,\cdots,r-1$. r is related to the order of the desired model, and will be determined in the sequel.

Representing both numerator and denominator of $\hat{H}(\omega^2)$ by linear combinations of Chebyshev polynomials, (2.18) is replaced by

$$\frac{\sum_{i=0}^{k}\alpha_i T_{2i}\left(\frac{\omega}{\omega_0}\right)}{\sum_{i=0}^{p}\beta_i T_{2i}\left(\frac{\omega}{\omega_0}\right)}\overset{r}{\equiv}\sum_{i=0}^{\infty}c_i T_{2i}\left(\frac{\omega}{\omega_0}\right) \tag{2.19}$$

where α_i, β_i, and c_i are understood to stand for α_{2i}, β_{2i}, and c_{2i} of (2.3) and (2.15). It is assumed that $\beta_0=1$ for unique determination of the coefficients. Thus, (2.19) yields

$$\sum_{i=0}^{k}\alpha_i T_{2i}=\sum_{j=0}^{p}\sum_{r=0}^{\infty}b_j c_r T_{2j}T_{2r}.$$

Using the relation (2.4), substituting $\beta_0=1$, and rearranging terms we have

$$\sum_{i=0}^{k}\alpha_i T_{2i}=\sum_{r=0}^{\infty}c_r T_{2r}+\frac{1}{2}\sum_{j=1}^{p}c_0\beta_j T_{2j}+\frac{1}{2}\sum_{j=1}^{p}c_j\beta_j T_0+\frac{1}{2}\sum_{j=1}^{p}\beta_j\sum_{i=1}^{\infty}[c_{|j-i|}+c_{j+i}]T_{2i}.$$

Comparing of the coefficients of the T_{2i} polynomials yields

$$\alpha_0=c_0+\frac{1}{2}\sum_{r=1}^{p}c_r\beta_r \tag{2.20a}$$

$$\alpha_i=c_i+\frac{1}{2}c_0\beta_i+\frac{1}{2}\sum_{r=1}^{p}[c_{|r-i|}+c_{r+i}]\beta_r, \qquad i=1,2,\cdots,k. \tag{2.20b}$$

Comparing of the coefficient of p further polynomials yields

$$0=c_{k+r}+\frac{1}{2}c_0\beta_{k+r}+\frac{1}{2}\sum_{i=1}^{p}[c_{|p+r-i|}+c_{p+r+i}]\beta_i, \qquad r=1,2,\cdots,n \tag{2.21}$$

where $\beta_{k+r}=0$ for $k+r>p$.

Equation (2.21) is a system of p equations in p unknowns, $\beta_1,\beta_2,\cdots,\beta_p$. Its solution enables the coefficients $\alpha_0,\alpha_1,\cdots,\alpha_k$ to be determined by (2.20). To find a reduced-order model $\{k/p\}$, the c_i coefficients of (2.15) are required up to and including c_{k+2p}. Hence, r of (2.18) is related to k and p by

$$r=k+2p+1. \tag{2.22}$$

C. Stable Reduced-Order Models

If the given system (2.9) has poles and zeros in the left-hand side of the s-plane, the method described so far enables the selection of the reduced-order model such that it too would have poles and zeros in the left-hand side of the s-plane.

After determining the β_i and α_j coefficients, $i=1,2,\cdots,p$; $j=1,2,\cdots,k$, we have $\hat{H}(\omega^2)$ which could be written in s^2 as

$$\hat{H}(s^2)\Big|_{s=j\omega}=\frac{N(s^2)}{D(s^2)}\Big|_{s=j\omega}$$

where $N(s^2)$ and $D(s^2)$ are k and p order polynomials, respectively. Let $P(s)$ and $Q(s)$ be k and p order polynomials, respectively, whose zeros are the square roots of the zeros with negative real parts of $N(s^2)$ and $D(s^2)$, respectively.

Hence, $N(s^2)=P(s)P(-s)$ and $D(s^2)=Q(s)Q(-s)$ and the low-order $\{k/p\}$ approximation of $G(s)$ can be selected to be stable and minimal phase, namely

$$\hat{G}(s)=P(s)/Q(s).$$

III. Approximations in the z-Plane

A. Darlington Polynomials

Consider the following nth order modified Chebyshev polynomials, to be referred to in the sequel as Darlington polynomials [15]:

$$D_n(x)=\cos(n\sin^{-1}x),\qquad n \text{ even} \tag{3.1a}$$

$$=j\sin(n\sin^{-1}x),\qquad n \text{ odd} \tag{3.1b}$$

which are orthogonal on $x\in[-1,1]$ with weight function $(1-x^2)^{-1/2}$ and can be shown to be related to Chebyshev polynomials by [17]

$$D_n(x)=j^nT_n(x),\qquad n\geqslant 0,\ x\in[-1,1]. \tag{3.2}$$

Darlington polynomials are recursively related, are even for even n, and are odd for odd n, in complete analogy to the Chebyshev polynomial equations (2.2) and (2.3).

If (2.5) is a Chebyshev series expansion of a given function $f(x)$, then

$$f(x)=a_0/2+\sum_{i=0}^{\infty}b_iD_i(x) \tag{3.3}$$

is the function's Darlington polynomial series expansion where, by (3.2),

$$b_i=a_i/j^i\qquad \forall i. \tag{3.4}$$

Thus, in particular, if $f(x)$ is an even function and

$$f(x)=a_0/2+\sum_{i=0}^{\infty}a_{2i}T_{2i}(x),$$

then its corresponding Darlington series expansion is

$$f(x)=a_0/2+\sum_{i=0}^{\infty}(-1)^ia_{2i}D_{2i}(x). \tag{3.5}$$

Let $[0,\omega_0]$ be a given (low-pass) frequency interval and let $x=\omega/\omega_0$ in (3.1). Thus,

$$D_n(\omega/\omega_0)=\cos(n\phi),\qquad n \text{ even} \tag{3.6a}$$

$$D_n(\omega/\omega_0)=j\sin(n\phi)\qquad n \text{ odd} \tag{3.6b}$$

where

$$\phi=\sin^{-1}(\omega/\omega_0). \tag{3.6c}$$

Let z be a new variable

$$z\triangleq\exp(j\omega). \tag{3.7}$$

Then, with $s=j\omega$, the relationship between s and z is obtained using (3.6c) as follows:

$$\phi=\sin^{-1}(\omega/\omega_0)=\sin^{-1}(s/j\omega_0)$$
$$\Rightarrow \sin\phi=s/j\omega_0=[\exp(j\phi)-\exp(-j\phi)]/2j;$$

hence,

$$s=\frac{\omega_0}{2}\left[z+\left(-\frac{1}{z}\right)\right]. \tag{3.8}$$

Thus, expressed in z, Darlington polynomials obtain a simpler form

$$D_n(z)=\frac{1}{2}\left[z^n+\left(\frac{1}{z}\right)^n\right],\qquad n \text{ even}$$

$$D_n(z)=\frac{1}{2}\left[z^n-\left(\frac{1}{z}\right)^n\right],\qquad n \text{ odd}$$

or, combined together

$$D_n(z)=\frac{1}{2}\left[z^n+\left(-\frac{1}{z}\right)^n\right]\qquad \forall n. \tag{3.9}$$

B. Relationship Between s-Plane and z-Plane

Referring to Fig. 1, let $s\in[-j\omega_0,j\omega_0]$, then ϕ is real and therefore $|z|=1$. As s changes from $-j\omega_0$ to $j\omega_0$ passing through the origin in the s-plane, ϕ changes from $-\pi/2$ to $\pi/2$ through 0 and, therefore, z changes from $-j$ to j passing through $z=1$. With s changing from $j\omega_0$ to $-j\omega_0$, z completes the unit circle. Circles in the z-plane whose radii is not unity map into ellipses in the s-plane with foci in $\pm j\omega_0$.

By (3.8), s is invariant to replacing z by $-1/z$; hence, every point s_a in the s-plane maps into two points in the z-plane, z_a and $-1/z_a$, such that one of them is inside and the other is outside the unit circle.

To obtain a one-to-one mapping, let

$$s\triangleq\frac{\omega_0}{2}\left[z-\frac{1}{z}\right],\qquad |z|>1 \tag{3.10}$$

which is defined for every point in the s-plane outside the interval $[-j\omega_0,j\omega_0]$. Note that (3.10) maintains the correspondences

$$s\leftrightarrow z\Rightarrow\bar{s}\leftrightarrow\bar{z}$$
$$\operatorname{sgn}\operatorname{Re}\{s\}=\operatorname{sgn}\operatorname{Re}\{z\} \tag{3.11}$$

where the overbar denotes complex conjugate.

C. Representing the System in the z-Plane

Given the $\{m/n\}$ system (2.9) and using the mapping (3.10) it can be shown that every

$$s-s_i\Rightarrow-\frac{\omega_0}{2z_i}(z-z_i)\left(-\frac{1}{z}-z_i\right).$$

Hence, $G(s)$ of (2.9) maps into

$$D(z)=R(z)R\left(-\frac{1}{z}\right) \tag{3.12}$$

where

$$R(z)\triangleq K_z^{1/2}\frac{\prod_{i=1}^{m}(z-z_i')}{\prod_{i=1}^{n}(z-z_i'')} \tag{3.13}$$

$$K_z=K\frac{\prod_{i=1}^{m}(-\omega_0/2z_i')}{\prod_{i=1}^{n}(-\omega_0/2z_i'')}. \tag{3.14}$$

The following representative quantities, related to (3.12), are of inter-

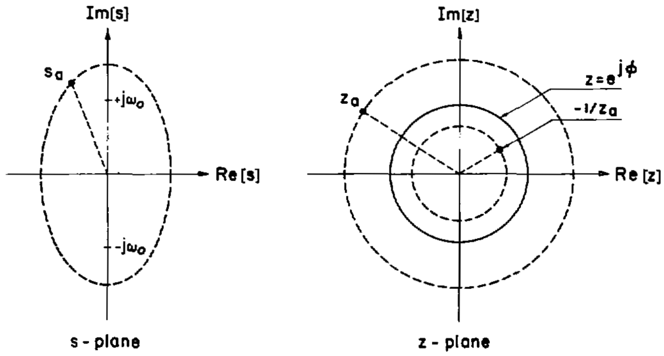

Fig. 1. The relationship between s- and z-planes.

est and in analogy with those defined for $G(s)$ by (2.10)–(2.14); let $|z|=1$.

1) The quantity

$$W(z^2)=R(z)R(-z)=K_z\frac{\prod_{i=1}^{m}(z_i'^2-z^2)}{\prod_{i=1}^{n}(z_i''^2-z^2)} \tag{3.15}$$

has an analogous form to $H(\omega^2)$ of (2.11).

2) By (3.12), the gain and phase of $G(s)$ in the z-plane have the form

$$g+j\psi=\ln R(z)+\ln R(-1/z) \tag{3.16}$$

where the gain of $G(s)$ is

$$g=\frac{1}{2}\ln W(z^2)+\frac{1}{2}\ln W[(-1/z)^2] \tag{3.17a}$$

and the phase is

$$j\psi=\frac{1}{2}\ln R(z)/R(-z)+\frac{1}{2}\ln R(-1/z)/R(1/z). \tag{3.17b}$$

For $|z|=1$, $\bar{z}=1/z$ and therefore (3.17) yields, respectively,

$$g=\ln|W(z^2)| =\ln\left|K_z\frac{\prod_{i=1}^{m}(z_i'^2-z^2)}{\prod_{i=1}^{n}(z_i''^2-z^2)}\right| \tag{3.18a}$$

$$j\psi=\frac{1}{2}\ln R(z)/R(-z) =\frac{1}{2}\ln\frac{\prod^{m}(z-z_i')\prod^{n}(-z-z_i'')}{\prod^{n}(z-z_i'')\prod^{m}(-z-z_i')}. \tag{3.18b}$$

D. Darlington Polynomial Series Expansion of Transfer Functions

1) Expanding in the z-plane: Given the $\{m/n\}$ system (2.9), let g and ψ be its gain and phase, respectively [see (2.12)],

$$g+j\psi=\ln G(s).$$

A Darlington polynomial series expansion is desired over the low-pass frequency interval $[0,\omega_0]$ such that

$$g+j\psi=\sum_{i=0}^{\infty}d_iD_i(\omega/\omega_0), \tag{3.19}$$

$D_i(\omega/\omega_0)$ being defined by (3.6).

Mapping into the z-plane, (3.19) yields, using (3.16) for its left-hand side and (3.9) for its right-hand side,

$$\ln R(z)+\ln R(-1/z)=\frac{1}{2}\sum_{i=0}^{\infty}d_iz^i+\frac{1}{2}\sum_{i=0}^{\infty}d_i(-1/z)^i \tag{3.20}$$

where $R(z)$ is given by (3.13).

Proposition: Let z_i' and z_j'' be the z-plane zeros and poles corresponding to s_i' and s_j'', the s-plane zeros and poles, respectively, $i=1,2,\cdots,m$; $j=1,2,\cdots,n$, under the mapping (3.10). Assume that s_i', s_j'' are all in the left-hand side of the s-plane and none on the $[-j\omega_0,j\omega_0]$ part of the imaginary axis. Then, for $|z|=1$, (3.20) implies

$$\ln R(z)=\frac{1}{2}\sum_{i=0}^{\infty}d_iz^i \tag{3.21a}$$

$$\ln R(-1/z)=\frac{1}{2}\sum_{i=0}^{\infty}d_i(-1/z)^i. \tag{3.21b}$$

Proving this proposition is omitted for brevity [17].

Thus, the problem of Darlington polynomial series expansion of $g+j\psi$ in the s-plane

$$\ln G(s)=\sum_{i=0}^{\infty}d_iD_i(\omega/\omega_0) \tag{3.22a}$$

is transformed in the z-plane to a problem of power series expansion

$$\ln R(z)=\frac{1}{2}\sum_{i=0}^{\infty}d_iz^i \tag{3.22b}$$

with $R(z)$ structured analogously to $G(s)$. Similarly, a Darlington series expansion of the gain g of $G(s)$,

$$\frac{1}{2}\ln G(s)G(-s)=\sum_{i=0}^{\infty}d_{2i}D_{2i}(\omega/\omega_0) \tag{3.23a}$$

is a power series in the z-plane

$$\ln R(z)R(-z)=\sum_{i=0}^{\infty}d_{2i}z^{2i}. \tag{3.23b}$$

And finally, a Darlington series expansion of the phase ψ in the s-plane

$$\frac{1}{2}\ln G(s)/G(-s)=\sum_{i=0}^{\infty}d_{2i+1}D_{2i+1}(\omega/\omega_0) \tag{3.24a}$$

is a power series expansion in the z-plane

$$\ln R(z)/R(-z)=\sum_{i=0}^{\infty}d_{2i+1}z^{2i+1}. \tag{3.24b}$$

Remarks: i) Whilst (3.19) and (3.20) hold for every s and z, (3.21) is valid for $|z|=1$ only. Equations (3.22a), (3.23a), and (3.24a) hold for $s=j\omega$ while (3.22b), (3.23b), and (3.24b) hold for $|z|=1$.

ii) d_{2i} in (3.23) and d_{2i+1} in (3.24) are the even and odd terms, respectively, of the coefficients series $\{d_i\}$ of (3.19)–(3.21).

2) Algorithm for series expansion of transfer functions: Given the $\{m/n\}$ system (2.9), assume that its zeros and poles, s_i' and s_i'', respectively, are known. Map them into the z-plane by the inverse transformation of (3.10)

$$z_i=\frac{s_i}{\omega_0}\pm\left[1+\left(\frac{s_i}{\omega_0}\right)^2\right]^{1/2}, \tag{3.25}$$

the sign (+) or (−) being selected such that $|z_i|>1$. Equation (3.25) maps real, $s_i=\sigma_i$ (complex conjugate pair, $s_i=\sigma_i\pm j\omega_i$) s-plane points into real, $z_i=x_i$ (complex conjugate pair, $z_i=x_i\pm jy_i$) z-plane points.

Consider (3.21a) and substitute (3.13) and (3.14) into it to yield

$$\ln K^{1/2}+\sum_{i=1}^{m}\ln(-\omega_0/2z')^{1/2}(z-z')-\sum_{i=1}^{n}\ln(-\omega_0/2z'')(z-z'') =\frac{1}{2}\sum_{i=0}^{\infty}d_iz^i. \tag{3.26}$$

Since the terms on the left-hand side form a sum of logarithms, we can consider expansion of single terms of the form

$$\ln(-\omega_0/2z_i)^{1/2}(z-z_i) \tag{3.27}$$

and add the results of the expansion of all terms.

A typical term for a real pole (or zero) $z_i = x$ yields an expansion

$$\ln(-\omega_0/2x)^{1/2} + \ln(z-x) = \frac{1}{2}\sum_{i=0}^{\infty} p_i z^i. \tag{3.28}$$

To find the sequence $\{p_i\}_0^\infty$ using (3.28):

i) $z=0 \Rightarrow p_0 = \ln(-\omega_0 x/2)$ (3.29a)

(note that $-\omega_0 x/2 > 0$);

ii) differentiating (3.28) with respect to z yields after some manipulations

$$p_1 = -2/x \tag{3.29b}$$

$$p_{i+1} = ip_i/(i+1)x, \qquad i=1,2\cdots. \tag{3.29c}$$

Assume now that $z_i = x + jy$ and its complex conjugate is $\bar{z}_i = z_{i+1}$. A typical term of (3.26) corresponding to this pair is

$$\ln(\omega_0^2/4A)^{1/2} + \ln(z^2 - 2xz + A) = \frac{1}{2}\sum_{i=0}^{\infty} q_i z^i \tag{3.30}$$

where $A = x^2 + y^2$. To find the sequence $\{q_i\}_0^\infty$ using (3.30):

i) $z=0 \Rightarrow q_0 = \ln\omega_0^2 A/2;$ (3.31a)

ii) differentiating (3.30) with respect to z yields after some manipulations

$$q_1 = -4x/A \tag{3.31b}$$

$$q_2 = (2 + b_1 x)A \tag{3.31c}$$

$$q_{i+1} = [2ib_i x - (i-1)b_{i-1}]/(i+1)A. \tag{3.31d}$$

Using the relations (3.29) and (3.31) yields the sequence $\{d_i\}_0^\infty$ and, therefore, the desired Darlington polynomial series expansion of transfer function.

E. Model Reduction in the z-Plane

Given the $\{m/n\}$ system

$$G(s) = K\frac{\prod_{i=1}^{m}(s-s_i')}{\prod_{i=1}^{n}(s-s_i'')} \qquad m<n \tag{3.32}$$

assume that it is stable and minimum phase. For a desired low-pass approximation, its gain

$$g = \frac{1}{2}\ln G(s)G(-s)|_{s=j\omega} = \frac{1}{2}\ln H(\omega^2) \tag{3.33}$$

is approximated over the frequency interval $[0,\omega_0]$.

Expanding the gain using Darlington polynomial series expansion (3.23a) we have

$$g = \sum_{i=0}^{\infty} d_{2i} D_{2i}(\omega/\omega_0) \tag{3.34}$$

where the coefficients d_{2i} can be derived using the z-plane algorithm of Section III-D.2).

In the z-plane, (3.34) is of the form (3.23b)

$$\ln R(z)R(-z) = \sum_{i=0}^{\infty} d_{2i} z^{2i} \tag{3.35}$$

where $R(z)$ is defined by (3.13) and (3.14).

Let $\hat{G}(s)$ be the $\{k/p\}$, $p<n$, required reduced-order model of $G(s)$ over the frequency interval $[0,\omega_0]$ and assume that it is stable and minimum phase. Let $\hat{g}$ be the gain of $\hat{G}(s)$

$$\hat{g} = \ln\hat{G}(s)\hat{G}(-s)|_{s=j\omega} = \frac{1}{2}\ln H(\omega^2)$$

whose Darlington polynomial series expansion over $[0,\omega_0]$ is

$$\hat{g} = \sum_{i=0}^{\infty}\hat{d}_{2i} D_{2i}(\omega/\omega_0)$$

or, in the z-plane

$$\ln\hat{R}(z)\hat{R}(-z) = \sum_{i=0}^{\infty}\hat{d}_{2i} z^{2i},$$

$\hat{R}(z)$ being a $\{k/p\}$ rational function, structured analogously to $\hat{G}(s)$.

Hence, we have to find a $\{k/p\}$ rational $\hat{R}(z)$ such that

$$\ln\hat{R}(z)\hat{R}(-z) \overset{r}{\equiv} \ln R(z)R(-z) \tag{3.36}$$

where $\overset{r}{\equiv}$ denotes the requirement that the first r terms in the series expansions of the left-hand and right-hand sides of (3.36) be identical. r is related to the order of the desired model and will be determined in the sequel. Substituting (3.34) into (3.36) yields the equivalent problem

$$\ln\hat{R}(z)\hat{R}(-z) \overset{r}{\equiv} \sum_{i=0}^{\infty} d_{2i} z^{2i}. \tag{3.37}$$

Let $\{f_{2i}\}_0^\infty$ be a set of coefficients such that

$$\sum_{i=0}^{\infty} d_{2i} z^{2i} = \ln\sum_{i=0}^{\infty} f_{2i} a^{2i}. \tag{3.38}$$

Letting $z=0$ in (3.38) yields

$$f_0 = \exp(d_0). \tag{3.39a}$$

Differentiating (3.38) with respect to z yields after some manipulations

$$f_{2i} = \frac{1}{i}\sum_{l=1}^{i} l d_{2l} f_{2(1-l)} \qquad i=1,2\cdots. \tag{3.39b}$$

Using (3.39), the knowledge of p coefficients d_{2i}, $i=0,1,\cdots,p-1$, enables the determination of the p coefficients f_{2i}.

Substituting (3.38) into (3.37) yields the requirement

$$\hat{R}(z)\hat{R}(-z) \overset{r}{\equiv} \sum_{i=0}^{\infty} f_{2i} z^{2i}. \tag{3.40}$$

Or, expressing the left-hand side of (3.40) as a ratio of two power series in z, we have

$$\frac{\sum_{i=0}^{k}\gamma_i z^{2i}}{\sum_{i=0}^{p}\delta_i z^{2i}} \overset{r}{\equiv} \sum_{i=0}^{\infty} f_{2i} z^{2i} \tag{3.41}$$

assuming $\delta_0 = 1$ for unique determination of the coefficients.

Notice that (3.41) represents the *classical Padé approximation problem in z^2*.

Equating the first $k+1$ powers in (3.41) yields

$$\gamma_i = \sum_{j=0}^{i} f_{2j}\delta_{i-j} \qquad i=0,1,\cdots,k. \tag{3.42a}$$

To find the p coefficients δ_i, $i=1,2,\cdots,p$ (recall that $\delta_0=1$), further p powers are equated to yield

$$\sum_{i=1}^{p} f_{2(k+j-i)}\delta_i = f_{2(k+j)} \qquad j=1,2,\cdots,p. \tag{3.42b}$$

Equation (3.42b) is a system of p equations in p unknowns, δ_1, $\delta_2,\cdots,\delta_p$. Solving for these unknowns, the $k+1$ coefficients, $\gamma_0,\gamma_1,\cdots,\gamma_k$ can be calculated using (3.42a). Hence, to determine the $\{k/p\}$ Padé approximation (3.41) we see that

$$r = k+p+1 \tag{3.43}$$

coefficients f_{2i} are required, or alternatively, d_{2i}, $i=0,1,\cdots,k+p$, Darlington coefficients are needed.

Remark: Operating in the s-plane a $\{k/p\}$ reduced-order model of the $\{m/n\}$ transfer function required $(k+2p+1)$ terms in the Chebyshev polynomial series expansion (2.21). In contrast, using the z-plane method, only $k+p+1$ terms in the Darlington polynomial series expansion are required.

F. Stable Reduced-Order Models

Consider (3.10) squared

$$s^2=\frac{\omega_0^2}{4}\left(z^2+\frac{1}{z^2}-2\right). \tag{3.44}$$

Now, the derived approximation $\hat{R}(z)\hat{R}(-z)$ is a $\{k/p\}$ rational function in z^2. Substituting its (squared) zeros and poles $\hat{z}^2$ in (3.44) yields the corresponding s-plane (squared) zeros and poles $\hat{s}^2$. Thus, $\hat{s}$ can be selected such that $\mathrm{Re}\{\hat{s}\}<0$.

G. Summary of Computational Steps

The computational procedure that has to be followed basically involves two subroutines.

1) For computing Darlington coefficients using (3.36) and (3.28)–(3.31), and from these, using (3.39), the f_{2i} coefficients.

2) For deriving Padé approximations using (3.41).

i) Solve the system of p equations in p unknowns (3.42b) to yield the p coefficients, δ_i, of the denominator of $\hat{R}(z)\hat{R}(-z)$, the desired rational approximant in z^2 (the leading coefficient, δ_0, is assumed 1).

ii) Use the δ's to solve for the $k+1$ coefficients, γ_i, using (3.42a), of the numerator of $\hat{R}(z)\hat{R}(-z)$.

The numerator and denominator of $\hat{R}(z)\hat{R}(-z)$ must be factorized and, using (3.44), the corresponding s-plane squared zeros and poles $\hat{s}^2$ can thus be obtained. These finally yield the zeros and poles of the desired rational approximant in s, $\hat{G}(s)$.

IV. Numerical Example

A rather celebrated example is the following (see [2], [18]):

$$G(s)=\frac{375000(s+0.08333)}{s^7+83.64s^6+4097s^5+70342s^4+853703s^3+2814271s^2+3310875s+281250}. \tag{4.1}$$

Figs. 2 and 3 show time and frequency responses, respectively, of the given high-order $\{1/7\}$ system (4.1) and three of its reduced-order models (by our second approach) over the frequency interval $[0,0.5]$:

$$G_{\{0/2\}}(s)=\frac{0.4924}{(s+0.4193)^2+0.5627} \tag{4.2}$$

$$\hat{G}_{\{1/2\}}(s)=\frac{2.1514(s+0.0854)}{(s+0.0943)(s+1.9497)} \tag{4.3}$$

$$\hat{G}_{\{2/3\}}(s)=\frac{0.3679(s+0.0833)(s+15.0771)}{[(s+2.0244)^2+0.9646](s+0.0919)}. \tag{4.4}$$

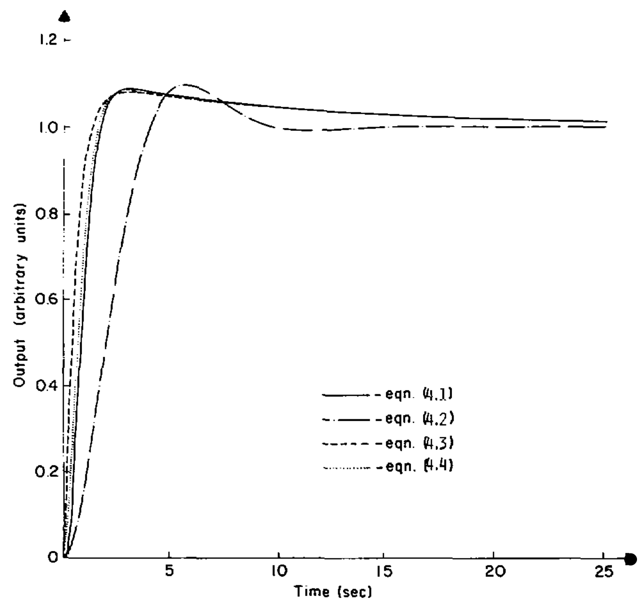

Fig. 2. Time response to a unit step input of the system (4.1) and some of its reduced-order models over the frequency interval [0,0.5].

In order to compare the results of our approximations to order reduction by other methods, Figs. 4 and 5 depict time and frequency responses, respectively, of the high-order $\{1/7\}$ system (4.1), our $\{1/2\}$ model (4.3), and the following second-order approximations.

1) Davison [19]:

$$\hat{G}_{\{1/2\}}(s)=\frac{0.5557(1-0.0909s)}{s^2+4.1176s+5.0296}. \tag{4.5}$$

2) Chen and Shieh [5]:

$$\hat{G}_{\{1/2\}}(s)=\frac{0.1299s+0.01105}{s^2+1.1464s+0.0994}. \tag{4.6}$$

3) Anderson [20]:

$$\hat{G}_{\{0/2\}}(s)=\frac{0.3096}{s^2+1.9026s+2.6879}. \tag{4.7}$$

4) Optimal projection method (in [18]):

$$\hat{G}_{\{1/2\}}(s)=\frac{0.5648(1-0.0282s)}{s^2+4.0488s+5.0277}. \tag{4.8}$$

5) Fellows et al. [18]:

$$\hat{G}_{\{0/2\}}(s)=\frac{0.2098}{s^2+1.6904s+1.8879}. \tag{4.9}$$

As can be seen, the method proposed in this paper favorably compares with various accepted procedures for model reduction. Many other results [17] equally demonstrate the usefulness and wide applicability of the method and enable its comparison with existing model reduction methods.

V. Conclusions

Two methods for reduced-order modeling were presented. Both were based on expanding the transfer function of a given system into orthogonal polynomial series over a given frequency interval $[0,\omega_0]$.

By the first method, the system's squared amplitude was expressed by a Chebyshev polynomial series. The series was then approximated in the frequency domain by a lower order rational expression in $(\omega/\omega_0)^2$. Operating with a squared variable enables the selection of poles and zeros of the reduced-order model to be in the left-hand s-plane. The Chebyshev polynomial series approximation can be viewed as a generalized Padé method.

In the second method, modified Chebyshev polynomials, labeled Darlington polynomials, were used to expand the system's gain. A new variable z was defined and the problem transformed into the resulting z-plane. In this plane, the Darlington series expansion became a power series (in z^2) and the problem of model reduction thus became an ordinary Padé approximation problem in z^2. Transforming back into the z-plane, a degree of freedom was once more obtained enabling the selection of stable and minimal phase reduced-order models.

The present paper reported of approximations over a low-pass frequency band only. Further results [16] show that bandpass, high-pass, and phase approximations can be obtained, with the second approach having the advantage in that it requires only minor changes in the computational algorithm.

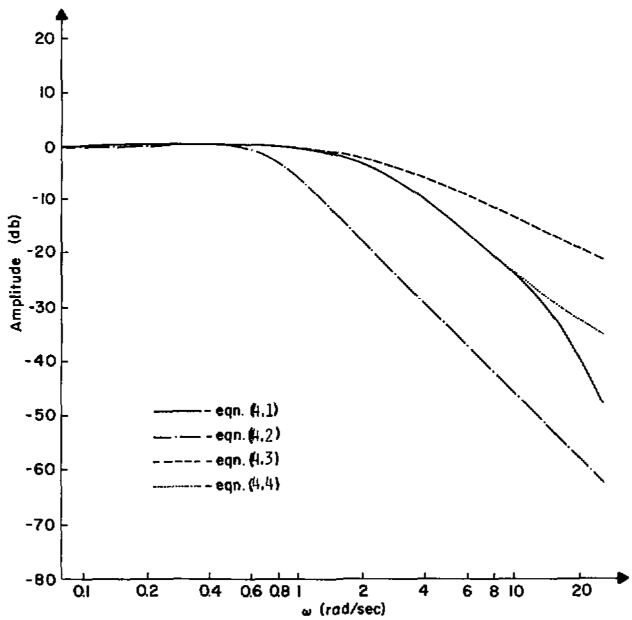

Fig. 3. Frequency response of the system (4.1) and some of its reduced-order models.

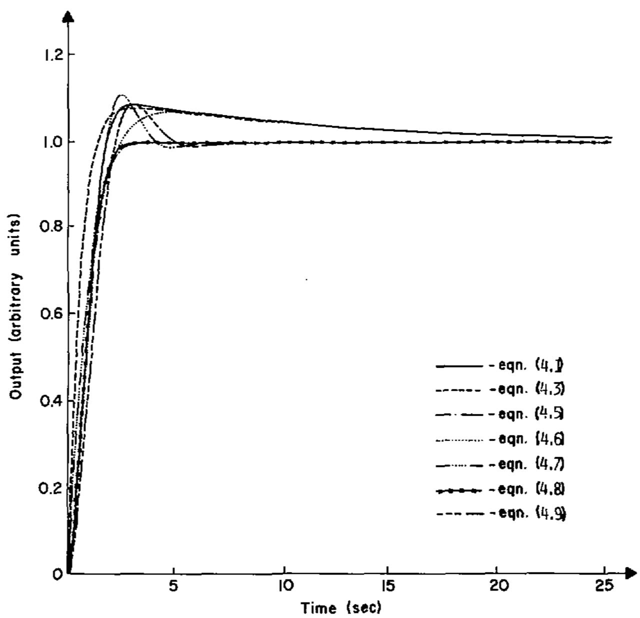

Fig. 4. Comparison of time responses to a unit step input of the system (4.1), its (1/2) reduced-order model (4.3), and various approximations by other methods.

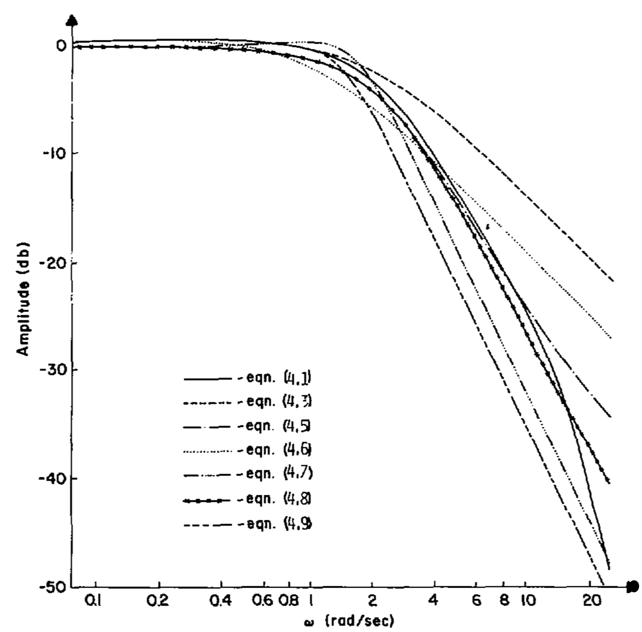

Fig. 5. Comparison of frequency responses of the system (4.1), its (1/2) reduced-order model (4.3), and various other approximations by other methods.

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