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NESTED BASES OF INVARIANTS FOR MINIMAL REALIZATIONS OF FINITE MATRIX SEQUENCES*

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Abstract. The problem of finding minimal realizations of linear constant systems from finite order input-output Markov matrix sequences is considered. The paper identifies from the sequences sets of independent structural and numerical quantities which are invariants of equivalent state space representations and completely characterize any minimal realization of the sequence. These sets, termed bases of invariants, acquire a "nesting" property by which a subsequent basis of a higher order finite sequence is obtained from the previous basis by addition of some new invariants. Two canonical state space representations of special forms that reflect the input and output structural properties of the underlying systems are presented and readily derived from these bases by a simple algorithm which is provided. Necessary and sufficient conditions for the existence of a unique minimal partial realization to a given finite Markov sequence are given in terms of the invariants of its nested basis. The set of all minimal partial realizations $S^r_{n_r}$ that, in the case of existence of more than one solution, corresponds to many distinct systems, is thoroughly investigated. A minimal set of undetermined quantities that parametrize $S^r_{n_r}$ is obtained. These parameters are used to characterize $S^r_{n_r}$ either in the form of bases of invariants or in the form of the canonical representations, and it is also shown that an arbitrary assignment of values to these parameters leads to a minimal realization of the given finite sequence. Additional properties of these parameters that may be desirable in certain identification problems are also discussed.

Key words. minimal partial realization, system invariants, canonical forms

1. Introduction. The problem of minimal realization of a finite sequence of Markov matrices of a multivariable linear constant system has been considered by various authors [1]–[8]. The early results of Kalman and Tether [1]–[3] showed that a minimal realization, or equivalently, a minimal extension sequence for a finite Markov sequence, always exists but may not be unique. Necessary and sufficient conditions on the incomplete Hankel matrix for the existence of a unique extension sequence as well as the derivation of a corresponding realization have also been described in these papers. The approach of Dickinson, Kailath and Morf in [4] is different in that they derive a matrix fraction representation by direct operation on the matrices of the sequence. References [5]–[8] also consider the incomplete Hankel matrix. Roman and Bullock [7] represent an invariant approach to the problem which is further developed by Candy, Warren and Bullock in [8] by deriving the partial realization from a set of Popov invariants [9].

The present paper provides a comprehensive treatment of the minimal partial realization (m.p.r.) problem of a finite sequence of r Markov matrices, using an invariant description. It puts a special emphasis on the common situation where a unique solution to the problem does not exist. It obtains a characterization of the set $S^r_{n_r}$ of all partial realizations of minimal dimension n_r for the sequence of r Markov matrices.

We show the existence of bases of invariants [10] for the description of equivalent classes of m.p.r.'s which have the property that a basis for a Markov sequence of a subsequent order is obtained from the basis of the former order by the addition of a few new invariants without altering the previous set of invariants. A basis acquiring this property is termed a nested basis. The nested bases are constructed from a set of entries of specified locations in the Markov sequence that were recently suggested

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by Bosgra and Van der Weiden [11] and from a modified set of integer invariants that describe the structure of the underlying system. An important feature of the present approach is that the invariants that compose these bases are not dependent on the choice of some specific canonical representation. This differs from the system descriptions by canonical invariants suggested by Popov and refined and studied by Rissanen [10] and Denham [12]. Instead of the nonuniform descriptions of the set of all m.p.r.'s obtained by methods which adopt canonical invariants [7], [8], we obtain an intrinsic set of parameters $\mathcal{P}_r$ which completely characterizes $S^r_{n_r}$. By applying additional properties of the nested bases of invariants, it is also shown that $\mathcal{P}_r$ is a minimal set of independent parameters for a complete characterization of $S^r_{n_r}$ and that the mapping from the set of equivalent realizations in $S^r_{n_r}$ to $\mathcal{P}_r$ is one-to-one and onto.

Descriptions for m.p.r.'s other than the nested bases are also presented. In fact, any equivalent canonical representation can be derived from a nested basis. Two canonical state space representations of a special form that reflects the input and output invariant structure of the underlying system are presented and a simple algorithm for their derivation from a nested basis is provided. The two canonical forms tie together, in the special case of an infinite order Markov sequence, the realizations of Rissanen [10] and Silverman [14]. They also supply a simplified algorithm for the derivation of the invariants of Rissanen and provide a system invariant description for the realizations of Silverman.

It is desirable in general, to have a system description by a minimal set of parameters [9], [11], [5] and [7]. This is advantageous, for example, in solving the problem of system identification from statistical data which is possibly the most important practical implication of the present study. The stochastic interpretation of a deterministic partial realization is discussed by Akaike [15]. The problem of selection of free parameters for the description of all possible minimal realizations of a finite Markov sequence, which Ledwich and Fortman [6] recognized as a difficult one, is solved by the above-mentioned set $\mathcal{P}_r$. The set $\mathcal{P}_r$ is not only a minimal set of independent parameters but is composed of entries of specified locations of the input-output data, which become available in further measurements.

The paper is written in continuous-time formulation but all the results apply also to discrete-time systems with some obvious redefinition of concepts. Section 2 contains the necessary definitions for the representation of the results, including the definition of a nested basis of system invariants. Section 3 represents bases of system invariants and suggests the above two canonical representations. Section 4 deals with the partial realization problem. The background of §§ 2 and 3 is used to derive nested bases of invariants for the descriptions of m.p.r.'s. The existence of a unique m.p.r. can be tested by its invariants. In the case where there exists more than one solution, the set of all m.p.r.'s is described by nested bases of invariants. These bases are expressed in terms of the minimal set of independent parameters $\mathcal{P}_r$. The m.p.r.'s can also be presented in the canonical forms described in § 3. These results are illustrated by a demonstrative example taken from [2]. This example appears also in [5], [7], [8] and allows a convenient comparison with former results.

2. System invariants of equivalent realizations. Let Σ_n (A, B, C) denote the set of all matrix triples A, B, C, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{l\times n}$ with (A, B) controllable and (A, C) observable. The elements $(A, B, C) \in \Sigma_n$ are state space representations of a linear system and each defines a transfer function matrix

$$G(s) = C(sI - A)^{-1}B. \tag{2.1}$$

The transfer function matrix can be expressed in a Laurent series about infinity

$$G(s)=G_1s^{-1}+G_2s^{-2}+\cdots, \tag{2.2}$$

where $G_i \in R^{l\times m}$ are called the Markov matrices of the system represented by (A, B, C) and are related to the representation by

$$G_i=CA^{i-1}B, \qquad i=1,2,\cdots. \tag{2.3}$$

Let E_n denote the equivalence of state space coordinate transformations, defined on elements of Σ_n, (A, B, C), $(\tilde{A}, \tilde{B}, \tilde{C})\in\Sigma_n$, by

$$(A,B,C)E_n(\tilde{A},\tilde{B},\tilde{C})\leftrightarrow CA^{i-1}B=\tilde{C}\tilde{A}^{i-1}\tilde{B}, \qquad i=1,2,\cdots. \tag{2.4}$$

The relation E_n partitions Σ_n into equivalence classes

$$E_n(\tilde{A},\tilde{B},\tilde{C})=\{(A,B,C)\,|\,(A,B,C)\in\Sigma_n,\ (A,B,C)E_n(\tilde{A},\tilde{B},\tilde{C})\}. \tag{2.5}$$

The set of all such equivalence classes is called the quotient set and is denoted by Σ_n/E_n. Given an infinite sequence of Markov matrices G_i, $i=1,2,\cdots$, a representation $(A, B, C)\in\Sigma_n$ is called a minimal realization if (2.3) is satisfied. Given a finite sequence of only r Markov matrices $\{G_1, G_2,\cdots, G_r\}$ the representation (A, B, C) is called an rth order partial realization if

$$CA^{i-1}B=G_i, \qquad i=1,2,\cdots,r \tag{2.6}$$

and it is called a minimal partial realization (m.p.r.) of r if n is the minimal dimension of a system which satisfies (2.6).

An rth order m.p.r. is said to be unique if there exists only one infinite extension sequence G_{r+i}, $i=1,2,\cdots$ such that a m.p.r. is also a (complete) minimal realization of the infinite sequence $\{G_1, G_2,\cdots, G_r, G_{r+1}, G_{r+2},\cdots\}$. If it is not unique, other triples of matrices exist that also minimally realize the rth order sequence but determine different extension sequences.

Let S_n^r be the set of all representations of m.p.r.'s of $\{G_1, G_2,\cdots, G_r\}$

$$S_n^r=\{(A,B,C)\ CA^{i-1}B=G_i,\ i=1,\cdots,r,\ E_n(A,B,C)\subset\Sigma_n\}. \tag{2.7}$$

The m.p.r. of $\{G_1, G_2,\cdots, G_r\}$ is called unique if S_n^r consists of a single equivalence class. If it is not unique, the equivalence relation E_n partitions S_n^r into distinct classes that represent different systems whose first r Markov matrices are $\{G_1, G_2,\cdots, G_r\}$. The set of all these classes is denoted by S_n^r/E_n and is a subset of Σ_n/E_n. The set of all m.p.r.'s S_n^r is discussed in § 4, the main section of this paper. The characterization and derivation of S_n^r uses system invariant descriptions and canonical representations. The required concepts are defined below and elaborated in § 3. Many of the following definitions can be found elsewhere [16], [10]–[12].

DEFINITION 2.1. An *invariant* of the equivalence relation E_n is a function $f:\Sigma_n\to R$ for which $(A,B,C)E_n(\tilde{A},\tilde{B},\tilde{C})$ implies $f(A,B,C)=f(\tilde{A},\tilde{B},\tilde{C})$.

DEFINITION 2.2. An invariant $f:\Sigma_n\to R$ is a *complete invariant of* E_n if $f(A,B.C)=f(\tilde{A},\tilde{B},\tilde{C})$ implies $(A,\tilde{B},C)E_n(\tilde{A},\tilde{B},\tilde{C})$.

A set of invariants $f,\cdots,f_N$ is called complete if Definition 2.2 is satisfied for $F=(f_1,\cdots,f_N):\Sigma_n\to R^N$.

DEFINITION 2.3. The set of invariants $f_i:\Sigma_n\to R$ $i=1,\cdots,N$ is said to be *independent* if the complement of the range of $F=(f_1,\cdots,f_N)$ in its codomain is a finite union of sets V_i

$$V_i=\{x\,|\,x\in R^N, P_{ij}(x)=0, j=1,\cdots,L;\text{ finite } L\}, \tag{2.8}$$

where P_{ij} are polynomials.

Definition 2.3 implies that F is surjective on its codomain except possibly on a subset of "measure zero" and that no f_i can be expressed as a function of any f_j, $j \neq i$. This definition is a refinement due to [10] of the definition in [9] for independence of invariants. A complete set of invariants for the equivalence relation E_n of (2.4) may be divided into two sets $F = (F_\sigma; F_\alpha)$ where F_σ is a set of integers called arithmetic invariants that correspond to the structure of $E_n(A, B, C)$ and F_α is a complementary set of numerical values called algebraic invariants for the description of $E_n(A, B, C)$ [11], [10]. We adopt the term basis of invariants [10] for the following intuitive notion of a complete set of independent invariants.

DEFINITION 2.4. A set of invariants F is called a *basis of invariants* for the equivalence class $E_n(A, B, C)$ if $F = (F_\sigma; F_\alpha)$ is complete, the set of arithmetic invariants $F = (\sigma_1, \cdots, \sigma_{N_1})$ is surjective and the complementary set of algebraic invariants $F_\alpha = (\alpha_1, \cdots, \alpha_{N_2})$ is independent.

A subset $\Sigma_c \subset \Sigma_n$ is called a canonical form if for each $(A, B, C) \in \Sigma_n$ there exists one and only one $(A_c, B_c, C_c) \in \Sigma_c$ for which $(A, B, C)E_n(A_c, B_c, C_c)$. A canonical representation induces a (trivial) complete set of invariants for $E_n(A, B, C)$ simply by $F_c(A, B, C) = (A_c, B_c, C_c)$ with the arithmetic and algebraic invariants being the location and content, respectively, of the entries of the matrices A_c, B_c, C_c. It is well understood that such a set is not in general a basis because the entries in the canonical representation satisfy certain constraints (e.g., minimal dimensionality) by which they are dependent. However subsets of independent invariants can be extracted from (A_c, B_c, C_c) and a complete set of independent invariants determined [7], [8], [10]. We call a basis of invariants whose algebraic invariants are a subset of entries of a canonical representation a *canonical basis*. Two canonical bases are presented in § 3 of this paper where a different type of basis is introduced. The new basis of invariants does not depend on any specific canonical form and it is shown later to acquire the additional "nesting" property which is defined below. Nested bases of invariants play a major role in our forthcoming investigation of the set of all minimal partial realizations. Let $G_i \in R^{l \times m}$ $i = 1, 2, \cdots$ be a sequence of matrices and let $S^r_{n_r}$ be the set of all rth order partial realizations of minimal dimension n_r (2.7). Let $F^r = (F^r_\sigma; F^r_\alpha)$ be a basis of invariants for E_n (A, B, C) where $(A, B, C) \in S^r_{n_r}$.

DEFINITION 2.5. The basis $F^r = (F^r_\sigma; F^r_\alpha)$ of E_n (A, B, C) is said to be a *nested basis of invariants* if for $j < r$ there exist subsets $F^j_\sigma \subset F^r_\sigma$ and $F^j_\alpha \subset F^r_\alpha$ such that $F^j = (F^j_\sigma; F^j_\alpha)$ is a basis of invariants for some equivalence class in $S^j_{n_j}$ $(n_j \leqq n_r)$, the set of all m.p.r.'s of the jth order sequence of the same Markov matrices $(j = r-1, r-2, \cdots \geqq m, l)$.

Nested bases of invariants can be considered as a natural elaboration on concepts of the previous system invariants for the descriptions of partial realizations. Subsets of nested bases of invariants form bases of invariants for lower order m.p.r.'s in a manner reminiscent of that by which subspaces of projections of linear spaces are spanned by subsets of their bases. Thus, the nested bases add, to the previous notion of independence and completeness of the bases of invariants, an additional notion of familiarity with bases in linear algebra. These bases provide a useful tool for the investigation of the partial realization problem and also have important consequences for efficient sequential realization algorithms of partial realizations of successive orders.

3. Bases of invariants and canonical forms for minimal realizations. Consider the infinite sequence of Markov matrices G_i, $i = 1, 2, \cdots$, and define the infinite Hankel block matrix H whose (i, j) block is G_{i+j-1}. Denote by $H_{i,j}$ the finite submatrix of the first i block rows and j block columns of H. It is well known that if the infinite Markov sequence has a minimal realization $(A, B, C) \in \Sigma_n$ than this realization is completely determined by a submatrix $H_{i,j}$ not larger than $H_{n,n}$, where n is the rank of H [1], [14]. The matrix $H_{n,n}$ satisfies

$$(3.1)\quad H_{n,n} = \begin{bmatrix} G_1 & G_2 & \cdots & G_n \\ G_2 & & & \\ \vdots & & & \\ G_n & & \cdots & G_{2n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} [B \quad AB \quad \cdots \quad A^{n-1}B] = H_C H_B,$$

where H_C and H_B are the observability and controllability matrices for $(A, B, C) \in \Sigma_n$. The row and column dependencies of $H_{n,n}$ are equivalent to row dependencies of H_C and the column dependencies of H_B, respectively.

Let $I_n = \{i_1, \cdots, i_n\}$ and $J_n = \{j_1, \cdots, j_n\}$ denote the indices of the first independent rows and columns of H or $H_{n,n}$. A selection of rows I_n and columns J_n is called a nice selection [12] if they satisfy

$$(3.2)\qquad l < i_k \in I_n \rightarrow i_k - l \in I_n, \qquad m < j_k \in J_n \rightarrow j_k - m \in J_n.$$

The choice of the first n independent rows and columns is recognized as a nice selection by the decomposition of $H_{n,n}$ in (3.1) into H_C and H_B. The sets of integers I_n and J_n thus defined on H are invariants of the equivalence relation E_n. They are closely related to the observability and controllability indices

$$(3.3)\qquad \nu = \{\nu_1, \cdots, \nu_l\} \quad \text{and} \quad \mu = \{\mu_1, \cdots, \mu_m\}$$

of the underlying system. The observability index $\nu_i \in \nu$ is the highest integer k for which the row $c_i^t A^{k-1}$ (c_i^t is the ith row of C) still appears in the selection of rows I_n in H_C. Similarly, the controllability index $\mu_j \in \mu$ is the highest integer k for which column $A^{k-1}b_j$ (b_j is the jth column of B) is in the selection of columns J_n in H_B. It is therefore obvious from the decomposition of $H_{n,n}$ in (3.1) that ν and μ are related to I_n and J_n by

$$(3.4)\qquad \begin{aligned} I_n &\leftrightarrow \nu, \quad \nu_i = \#I_n/i, \; i \in \mathbf{l}, \\ J_n &\leftrightarrow \mu, \quad \mu_j = \#J_n/j, \; j \in \mathbf{m}, \end{aligned}$$

where $\#S$ denotes the number of elements in the set S, $\mathbf{n} = \{1, 2, \cdots, n\}$ and I_n/i and J_n/j denote the subsets of the arithmetic series $\{i, i+l, i+2l, \cdots\}$ and $\{j, j+m, j+2m, \cdots\}$ that are included in the sets I_n and J_n, respectively. The relation between the sets I_n and J_n and the sets ν and μ is bijective and an alternative way to derive them is to use the crate diagram [11], [17], [18]. Assume for example that $J_6 = \{1, 2, 3, 4, 5, 7\}$ and $m = 2$ then $J_6/1 = \{1, 3, 5, 7\} \rightarrow \mu_1 = 4$ and $J_6/2 = \{2, 4\} \rightarrow \mu_2 = 2$ and therefore $\mu = \{4, 2\}$. The integers β and α defined on ν and μ by

$$(3.5)\qquad \beta = \max_{i \in \mathbf{l}} \nu_i, \qquad \alpha = \max_{j \in \mathbf{m}} \mu_j$$

are the first integers for which the realizability condition, $n = \rho H_{\beta,\alpha} = \rho H_{\beta+1,\alpha} = \rho H_{\beta,\alpha+1}$, is satisfied [14]. The submatrices of H in the following definition are uniquely determined by I_n and J_n and can be recognized as the submatrices defined also by Silverman for his realization algorithm [14].

DEFINITION 3.1. The following submatrices of the Hankel matrix H are defined for the sets I_n and J_n:

Q: The nonsingular $n \times n$ matrix formed from $H_{\beta,\alpha}$ by the intersection of the columns J_n and the rows I_n.

$\hat{A}$: The $n \times n$ matrix whose entries in $H_{\beta,\alpha+1}$ are positioned m columns to the right of the positions of corresponding entries of Q.

$\hat{B}$: The $n \times m$ matrix formed from $H_{\beta,\alpha}$ by the intersection of the columns $\mathbf{m}$ with the rows I_n.

$\hat{C}$: The $l \times n$ matrix formed from $H_{\beta,\alpha}$ by the intersection of the rows $\mathbf{l}$ with the columns J_n.

Remark 3.1. $\hat{A}$ is equivalently formed by the $n \times n$ matrix whose entries in $H_{\beta+1,\alpha}$ are positioned l rows below the position of corresponding entries of Q.

Remark 3.2. The columns J_n of $[B, A]$ and the rows I_n of $[{}^{C}_{A}]$, each separately, form Q.

Remark 3.3. The matrix triple $(\hat{A}Q^{-1}, \hat{B}, \hat{C}Q^{-1})$ is a realization of the infinite sequence G_i, $i = 1, 2, \cdots$ [14].

The first two remarks result from the special structure of the Hankel matrix. The triple of matrices $(\hat{A}, \hat{B}, \hat{C})$ involves the following collection of $n(m+l)$ entries of the infinite Markov sequence [11]

$$\mathcal{G} = \{g_{ijk} \mid k = 1, 2, \cdots, \nu_i + \mu_j, \ i \in \mathbf{l}, j \in \mathbf{m}\}, \tag{3.6}$$

where $g_{ijk} = (G_k)_{ij}$. It follows from [11] and the bijective relation between I_n, J_n and ν and μ that I_n, J_n and $\mathcal{G}$ define a complete set of independent invariants in the sense of Definitions 2.3 and 2.4.

THEOREM 3.1. *$\mathcal{B} = (I_n, J_n; \mathcal{G})$ is a basis of invariants for $E_n(A, B, C)$, the equivalence class of minimal realizations of the infinite Markov sequence G_i, $i = 1, 2, \cdots$, with I_n, J_n the sets of arithmetic invariants and $\mathcal{G}$ the associated set of algebraic invariants.*

The basis $\mathcal{B}$ deserves the name of *Markov basis* to indicate that its set of algebraic invariants are entries of the Markov matrices. This is in contrast to the canonical invariants and bases of [9], [10], in which the algebraic invariants form entries of the canonical representations. It must be noted that other bases whose algebraic invariants are Markov entries may be defined in association with nice selections other than the choice I_n, J_n of first independent rows and columns [11]. The advantage of the above basis $\mathcal{B}$ over these other bases for the partial realization problem will be clarified in the next section.

Example 3.1. We shall illustrate the Markov basis for the following Markov sequence

$$G_1, G_2, G_3, G_4, \cdots = \begin{bmatrix} ① & ① \\ ① & ① \end{bmatrix}, \begin{bmatrix} ① & ② \\ ① & ④ \end{bmatrix}, \begin{bmatrix} ③ & ⑤ \\ 7 & ⑨ \end{bmatrix}, \begin{bmatrix} 7 & ⑥ \\ 11 & 4 \end{bmatrix}, \cdots. \tag{3.7}$$

The Hankel matrix is then

$$H = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 5 & \cdots \\ 1 & 1 & 1 & 4 & 7 & 9 & \cdots \\ 1 & 2 & 3 & 5 & 7 & 6 & \cdots \\ 1 & 4 & 7 & 9 & 11 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

The rank of H is $n = 3$ and the first independent rows and columns are $I_3 = \{1, 2, 3\}$ and $J_3 = \{1, 2, 4\}$. A systematic elimination procedure to determine these values will be described later. Thus the observability and controllability indices are $\nu = \{2, 1\}$ and

$\mu = \{1, 2\}$ and the Hankel submatrices of Definition 3.1 are

$$Q = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 4 \\ 1 & 2 & 5 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 4 & 9 \\ 3 & 5 & 6 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

The set of algebraic invariants $\mathcal{G}$ of (3.6) consists of the encircled entries in (3.7). In the rest of this section we shall present two canonical forms and bases of invariants of a special structure and derive them from the triple of matrices $(\hat{A}, \hat{B}, \hat{C})$ associated with $\mathcal{B}$.

THEOREM 3.2. *Given the Markov basis $\mathcal{B} = (I_n, J_n; \mathcal{G})$ for the equivalence class $E_n(A, B, C)$ of minimal realization of G_i, $i = 1, 2, \cdots$, two possible canonical realizations and two corresponding canonical bases of invariants for $E_n(A, B, C)$ are the following*:

1a) *The realization $(A_1, B_1, C_1) \in E_n(A, B, C)$ where*

$$A_1 = Q^{-1}\hat{A}, \quad B_1 = Q^{-1}\hat{B}, \quad C_1 = \hat{C}. \tag{3.8}$$

1b) *The columns J_n are the first independent columns of the controllable pair $[B_1, A_1]$ and they form the $n \times n$ identity matrix. The entries in the remaining m columns of $[B_1, A_1]$, denoted by S_1, form part of the corresponding canonical basis $\mathcal{B}_1$ defined below.*

1c) *The canonical basis of invariants for (A_1, B_1, C_1) is $\mathcal{B}_1 = (J_n; \mathcal{G}_1)$ where*

$$\mathcal{B}_1 = \mathrm{Se}\,\{C_1\} \cup S_1 \tag{3.9}$$

Se $\{C_1\}$ *denotes the set of entries in C_1 and S_1 is defined in statement* (1b).

2a) *The realization $(A_2, B_2, C_2) \in E_n(A, B, C)$ where*

$$A_2 = \hat{A}Q^{-1}, \quad B_2 = \hat{B}, \quad C_2 = \hat{C}Q^{-1}. \tag{3.10}$$

2b) *The rows I_n are the first independent rows of the observable pair $\left[\begin{smallmatrix} C_2 \\ A_2 \end{smallmatrix}\right]$ and they form the $n \times n$ identity matrix. The set of entries in the remaining rows of $\left[\begin{smallmatrix} C_2 \\ A_2 \end{smallmatrix}\right]$, denoted by S_2, form part of the corresponding canonical basis $\mathcal{B}_2$ defined below.*

2c) *The canonical basis of invariants for (A_2, B_2, C_2) is $\mathcal{B}_2 = (I_n; \mathcal{G}_2)$ where*

$$\mathcal{G}_2 = \mathrm{Se}\,\{B_2\} \cup S_2. \tag{3.11}$$

Se $\{B_2\}$ *denotes the set of entries of B_2, and S_2 is defined in statement* (2b).

Proof. See Appendix 1.

The two canonical forms and their corresponding canonical bases of invariants may be derived without explicit calculations involving the matrix Q. To achieve this purpose we define the following restricted elimination procedure.

DEFINITION 3.2. A *row* (*column*) *reserving elimination operation* represented by the matrix T_1^r of size $p \times p$ (T_2^r of size $q \times q$), is defined as a restricted Gaussian elimination procedure that acts only on the rows (columns) of some matrix M of rank n and size $p \times q$ $p, q \geqq n$. The action of T_1^r (T_2^r) is to change the first n independent rows (columns) of $T_1^r M$ (MT_2^r) to unity column (row) vectors without interchanging row (column) positions.

Note that T_1^r brings the first n columns of $T_1^r M$ to n unity vectors that may form a nonordered arbitrary selection of n columns of the $p \times p$ identity matrix. The procedure that changes the first n independent columns of M to n ordered unity vectors will be called a *complete row elimination* and is denoted by T_1. T_1 combines the action of T_1^r followed by a proper row interchange procedure. Similarly for the dual case we shall denote by T_2 the complete column elimination procedure that changes the first n independent rows of M to the ordered sequence of unity row vectors of the $q \times q$ identity matrix.

Note that Q^{-1} in (3.8) and (3.10) stands for the operations of T_1 and T_2, respectively, so that the canonical forms can be derived from $(\hat{A}, \hat{B}, \hat{C})$ by a complete elimination procedure without finding Q^{-1} explicitly. Definition 3.2 suggests the following even more simple algorithm.

ALGORITHM 3.1

1. To obtain (A_1, B_1, C_1)

(i) $C_1 = \hat{C}$;

(ii) $T_1'[\hat{B}, \hat{A}] = [\tilde{B}_1, \tilde{A}_1]$ where $\tilde{B}_1 \in R^{n\times m}$, $\tilde{A}_1 \in R^{n\times n}$ are intermediate matrices resulting from the implicit action of the row reserving operation T_1' of Definition 3.2;

(iii) $[B_1, A_1] = P^t[\tilde{B}_1, \tilde{A}_1]$ where P is a permutation of the $n \times n$ identity matrix formed by columns J_n of $[\tilde{B}_1, \tilde{A}_1]$. Columns J_n are identified at stage (ii) as the pivotal columns of the action of T_1'.

2. To obtain (A_2, B_2, C_2)

(i) $B_2 = \hat{B}$

(ii)
$$\begin{bmatrix} \hat{C} \\ \hat{A} \end{bmatrix} T_2' = \begin{bmatrix} \tilde{C}_2 \\ \tilde{A}_2 \end{bmatrix},$$

where T_2' is the column reserving elimination of Definition 3.2 and $\tilde{C}_2 \in R^{l\times n}$, $\tilde{A}_2 \in R^{n\times n}$ are the intermediate results of its action;

(iii)
$$\begin{bmatrix} C_2 \\ A_2 \end{bmatrix} = \begin{bmatrix} \tilde{C}_2 \\ \tilde{A}_2 \end{bmatrix} P^t,$$

where the permutation P is the matrix formed by rows I_n of $\left[\begin{smallmatrix} \tilde{C}_2 \\ \tilde{A}_2 \end{smallmatrix}\right]$ which are the pivotal rows revealed at stage (ii).

Any canonical representation can be derived from its Markov basis of invariants. The two canonical forms of Theorem 3.2 have been chosen as suitable forms for system invariant descriptions in having a structure that reflects the output or the input structural properties of the system and as forms that are easily derived from the basis. The derivation of the canonical bases of invariants $\mathcal{B}_1$ and $\mathcal{B}_2$ shows the connection between the Markov sets of invariants and previous descriptions of canonical invariants. The significance of canonical forms that reflect some of the invariant properties has been recognized in [10] and more recently in [11]. In fact the second canonical form and canonical basis of Theorem 3.2, derived here from the Markov basis, are identical to the results of Rissanen which are derived in [10] directly from the Hankel matrix. The first canonical form also coincides with a realization obtained by the Silverman procedure [14]. Silverman has not considered the invariant structure of the pair $[B_1, A_1]$ or any related invariant aspect of his realization. The main reason for stating Theorem 3.2 and subsequent algorithms is to provide for the next section alternative equivalent descriptions for the solution of the minimal partial realization problem other than the nested bases. However, these results are also significant for the previous invariant descriptions and derivation of complete minimal realizations. They show that the realization obtained by Rissanen [10] is a dual form of the earlier realization derived by Silverman [14]. These results also supply a simplified elimination procedure for the derivation of the invariants of Rissanen and provide a system invariant description framework for the realization of Silverman.

Example 3.2. We illustrate Theorem 3.2 and the subsequent algorithm by continuation of Example 3.1. To derive from $\hat{A}$, $\hat{B}$, $\hat{C}$, the first canonical form we follow Algorithm 3.1 to obtain:

(i)
$$C_1 = \hat{C} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

(ii) Perform reserving row elimination on $[\hat{B}, \hat{A}]$

$$[\hat{B}, \hat{A}] = \left[\begin{array}{cc|ccc} \boxed{1} & 1 & 1 & 2 & 5 \\ 1 & 1 & 1 & 4 & 9 \\ 1 & 1 & 3 & 5 & 6 \end{array}\right] \to \begin{bmatrix} 1 & 1 & 1 & 2 & 5 \\ 0 & 0 & 0 & \boxed{2} & 4 \\ 0 & 1 & 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & \boxed{1} & 2 & 0 & -5 \end{bmatrix}$$

$$\to \left[\begin{array}{cc|ccc} 1 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & -5 \end{array}\right] = [\tilde{B}_1, \tilde{A}_1],$$

where the squared entries indicate the pivotal element at each step.

(iii) Rearranging the rows of $[\tilde{B}_1, \tilde{A}_1]$ to obtain the identity matrix at columns $J_3 = \{1, 2, 4\}$ or equivalently extracting P from these columns and performing the row changes by premultiplication by P^t results in

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to P^t[\tilde{B}_1, \tilde{A}_1] = \left[\begin{array}{cc|ccc} 1 & 0 & -1 & 0 & 6 \\ 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] = [B_1, A_1],$$

note that columns $J_3 = \{1, 2, 4\}$ of $[B_1, A_1]$ form the identity matrix. The first canonical form is therefore

$$A_1 = \begin{bmatrix} -1 & 0 & 6 \\ 2 & 0 & -5 \\ 0 & 1 & 2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

The basis is $\mathcal{B}_1 = (J_3; \mathcal{G}_1)$ where $\mathcal{G}_1$ is formed by the set of entries of C_1 and of columns 1 and 3 of A_1.

Using the second part of Algorithm 3.1, the dual canonical form (A_2, B_2, C_2) is

$$A_2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 3 \\ -2 & -3 & 2 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The corresponding canonical basis is $\mathcal{B}_2 = (I_3; \mathcal{G}_2)$ where $\mathcal{G}_2$ is formed by the entries of B_2 and of rows 2 and 3 of A_2.

4. Nested bases of invariants and minimal partial realizations. Given a finite sequence of r Markov matrices $\{G_1, \cdots, G_r\}$. We construct the Hankel matrix

(4.1)
$$H^r = \begin{bmatrix} G_1 & G_2 & \cdots & G_r & G_{r+1}^* & \cdots \\ G_2 & & & & & \\ \vdots & & & & & \\ G_r & & & & & \\ G_{r+1}^* & & & & & \end{bmatrix}$$

where $\{G^*_{r+1}, G^*_{r+2}, \cdots\}$ represents some unknown complementary sequence. This matrix is closely related to the incomplete Hankel matrix used by Tether [2] with the slight difference that we explicitly write entries $(G^*_k)_{ij} = g^*_{ijk}$ for $k > r$ instead of the common asterisks put, in [2] and [7], [8], in all the locations of the unknown data. This modification proves to be powerful if the following "asterisk convention" is adopted: (i) Asterisked entries g^*_{ijk} and their combinations are carried along in any submatrix of H^r and any operation on such submatrices. (ii) Asterisked entries of matrices are assumed not to influence the internal dependencies between the rows and columns that are determined by the numerically specified entries. Consequently the indices of the first independent rows and columns and the rank of the matrix are not changed by any specific choice of values for the asterisked entries. The rank of a matrix that contains asterisked entries is by this convention the minimal rank that is admissible by its numerically specified parts.

Following the above convention, let n_r be the rank of H^r and denote the indices of the first n_r independent rows and columns of H^r by $I^r_{n_r}$ and $J^r_{n_r}$, respectively. Thus, $I^r_{n_r}$ ($J^r_{n_r}$) are the first n_r rows (columns) in H^r which, considering for each row (column) only columns (rows) that correspond to its numerically specified positions, do not depend linearly on preceding rows (columns).

Let β_r denote the smallest integer for which every row of the block row $\beta_r + 1$ of H^r (i.e., $[G_{\beta r+1}, G_{\beta r+2}, \cdots]$) depends on the previous rows and similarly let α_r denote the smallest integer for which every column of the block column $\alpha_r + 1$ depends on the preceding columns. We have the following important result on the existence of m.p.r.'s [2], [3].

THEOREM 4.1. *Given the finite sequence* $\{G_1, \cdots, G_r\}$: (1) *There exists an extension sequence* $\{G_{r+1}, G_{r+2}, \cdots\}$ *for which* n_r *is the dimension of the minimal realization of the infinite sequence* G_i, $i = 1, 2 \cdots$. *This realization is not, in general unique.* (2) *Every extension fixed up to* $r_0 = \alpha_r + \beta_r$ *is uniquely determined thereafter.*

The invariants description approach developed and discussed in this section will provide an alternative verification of this well known theorem. The theorem indicates that values for the g^*_{ijk} exist for which the structure of the Hankel matrix as well as the row and column dependencies are retained. Later we shall be able to specify the required g^*_{ijk} values and construct the minimal extension sequences. Let G_i, $i = 1, 2, \cdots$ be the infinite Markov sequence associated with some equivalence class in Σ_n and let $\mathcal{B} = (I_n, J_n; \mathcal{G})$ be its Markov basis described in Theorem 3.1. The next theorem establishes $\mathcal{B}$ as a nested basis of invariants (Definition 2.5).

THEOREM 4.2. *Let* $\{G_1, G_2, \cdots, G_r\}$ *be an rth order subsequence of the infinite sequence* G_i, $i = 1, 2, \cdots$ *whose Markov basis is* $\mathcal{B} = (I_n, J_n; \mathcal{G})$. *Let also* $n_r = \rho H^r$ *where* H^r *is the incomplete Hankel matrix associated with the finite subsequence. There exist subsets of* $I^r_{n_r} \subset I_n$, $J^r_{n_r} \subset J_n$ *and a subset* $\mathcal{G}_r$ *of* $n_r(m + l)$ *elements of* $\mathcal{G}_r \subset \mathcal{G}$ *such that* $\mathcal{B}_r = (I^r_n, J^r_n; \mathcal{G}_r)$ *forms a Markov basis for a m.p.r. of* $\{G_1, \cdots, G_r\}$ *of dimension* n_r.

Proof. Let $I^r_{n_r}$ and $J^r_{n_r}$ be the indices of the first independent rows and columns of H^r of (4.1). Let $\hat{A}_r \in R^{n_r \times n_r}$, $B_r \in R^{n_r \times m}$ and $C_r \in R^{l \times n_r}$ be the submatrices of H of (3.1) derived in association with I_n and J_n in accordance with Definition 3.1. Note that the matrices $\hat{A}_r$, $\hat{B}_r$, $\hat{C}_r$ are derived from H of (3.1), not from H^r of (4.1), and thus all their entries are specified and completely determined by $I^r_{n_r}$, $J^r_{n_r}$ and the infinite sequence G_i, $i = 1, 2, \cdots$. Clearly $n_r \leqq n$ and as the process of successive replacement of asterisked entries in H^r by numerically specified entries G_{r+i}, $i = 1, 2, \cdots$ may add new independent rows and columns but cannot cancel former independencies, we have $I^r_{n_r} \subset I_n$ and $J^r_{n_r} \subset J_n$. It therefore follows that the following algebraic set of invariants defined for $I^r_{n_r}$ and $J^r_{n_r}$

$$\mathcal{G}_r = \{g_{ijk} \mid k = 1, \cdots, \bar{\nu}_i + \bar{\mu}_j, i \in \mathbf{l}, j \in \mathbf{m}\}, \tag{4.2}$$

where

$$\bar{\nu}_i = \#I^r_{n_r}/i, \qquad \bar{\mu}_j = \#J^r_{n_r}/j \tag{4.3}$$

is a subset of $\mathcal{G}$. Construct from $(\hat{A}_r, \hat{B}_r, \hat{C}_r)$ a representation $(\bar{A}, \bar{B}, \bar{C}) \in \Sigma_{n_r}$ (using Algorithm 3.1 say) clearly $\mathcal{B}_r = (I^r_{n_r}, J^r_{n_r}; \mathcal{G}_r)$ is a Markov basis for $E_{n_r}(\bar{A}, \bar{B}, \bar{C})$. The Markov entries $\bar{G}_i = \bar{C}_i\bar{A}^{(i-1)}\bar{B}$ satisfy $\bar{G}_i = G_i$ for (at least) $i = 1, \cdots, r$ therefore $\mathcal{B}_r$ is a Markov basis for a m.p.r. of $\{G_1, \cdots, G_r\}$ where we have also shown that $I^r_{n_r} \subset I^r_n$, $J^r_{n_r} \subset J_n$ and $\mathcal{G}_r \subset \mathcal{G}$. □

It follows from Theorem 4.2 and Definition 2.5 that all the bases $\mathcal{B}_r$ are nested bases. The set of invariants $\mathcal{G}_r$ is either completely composed of entries that are selected from $\{G_1, \cdots, G_r\}$, in which case $\mathcal{B}_r$ represents a basis for the unique m.p.r. of the rth order sequence, or it contains also entries from $\{G_{r+1}, G_{r+2}, \cdots\}$. In the latter case $\mathcal{B}_r$ represents a basis of an equivalence class in $S^r_{n_r}$, the one which is induced by the higher order basis $\mathcal{B}$. In this case it is understood that other infinite Markov sequences of minimal dimensions n^*, $n^* \geqq n_r$ that have $\{G_1, \cdots, G_r\}$ for their first r matrices may induce other sub-bases for equivalence classes in $S^r_{n_r}$.

The last observation leads to the following condition for the uniqueness of a m.p.r.

PROPOSITION 4.3. *The sequence $\{G_1, \cdots, G_r\}$ yields a unique m.p.r. if and only if it acquires a Markov basis $\mathcal{B}_r = (I^r_{n_r}, J^r_{n_r}; \mathcal{G}_r)$ for which the set $\mathcal{G}_r$, defined in* (4.2), *is completely formed by entries of the sequence $\{G_1, \cdots, G_r\}$.*

Define for $\bar{\nu}_i$ and $\bar{\mu}_j$ of (4.3)

$$\beta_r = \max_{i \in \mathbf{l}} \bar{\nu}_i, \quad \alpha_r = \max_{j \in \mathbf{m}} \bar{\mu}_j, \quad \nu_0 = \alpha_r + \beta_r. \tag{4.4}$$

It follows from (4.1) that the condition expressed in Proposition 4.3 is satisfied if and only if $r_0 = \alpha_r + \beta_r \leqq r$. It is easy to verify that α_r and β_r of (4.4) are identical to the integers in Theorem 4.1. This proposition therefore assures the uniqueness conditions stated in Theorem 4.1.

We now proceed to investigate the case where $\{G_1, \cdots, G_r\}$ has more than one m.p.r. Assume that $r < r_0$ and thus that the set $S^r_{n_r}$ of all m.p.r.'s of $\{G_1, \cdots, G_r\}$ consists of distinct equivalence classes to each of which there corresponds a different extension sequence. Denote a general form of an infinite Markov sequence whose first r Markov matrices are $\{G_1, \cdots, G_r\}$ by

$$\{G_1, G_2, \cdots, G_r, G^*_{r+1}, G^*_{r+2}, \cdots\}, \tag{4.5}$$

where G^*_{r+1}, G^*_{r+2} are some unknown matrices. The sequence (4.5) may have realizations of any minimal dimension $n^* \geqq n_r$.

Applying the derivation of the rth order Markov basis as in the proof of Theorem 4.2, to the sequence (4.5) and following the discussion that preceded this theorem the Markov bases $\mathcal{B}^*_r = (I^r_{n_r}, J^r_{n_r}; \mathcal{G}^*_r)$ are obtained where $\mathcal{G}^*_r$ may be divided into two disjoint sets $\mathcal{G}^*_r = \tilde{\mathcal{G}}_r \cup \mathcal{P}_r$. The first set

$$\tilde{\mathcal{G}}_r = \{g_{ijk} \mid k = 1, \cdots, \min(\bar{\nu}_i + \bar{\mu}_j, r), i \in \mathbf{l}, j \in \mathbf{m}\} \tag{4.6}$$

with $\bar{\nu}_i$ and $\bar{\mu}_j$ as defined in (4.3) represents the specified invariants that form a selection of entries of $\{G_1, \cdots, G_r\}$ while the second set

$$\mathcal{P}_r = \{g_{ijk} \mid k = r+1, \cdots, \bar{\nu}_i + \bar{\mu}_j > r, i \in \mathbf{l}, j \in \mathbf{m}\} \tag{4.7}$$

represents a complementary set of unspecified invariants that form a selection of entries of the extension segment $\{G^*_{r+1}, \cdots, G^*_{r_0}\}$ of positions specified by $I^r_{n_r}$ and $J^r_{n_r}$. It follows from the nested property of Markov bases (Theorem 4.2), that the above set of bases $\mathcal{B}^*_r$ represents the set of rth order sub-bases of any general sequence (4.5). Therefore, any admissible extension sequence of dimension n_r is represented by $\mathcal{B}^*_r$ for some suitable choice of values for $\mathcal{P}_r$. The set $\mathcal{P}_r$ is a complete set of parameters $\{g^*_{ijk}\}$ for $S^r_{n_r}$, labelled by the locations of the required unspecified entries in the extension sequence. Two m.p.r.'s of $\{G_1, \cdots, G_r\}$ that assign values to $\mathcal{P}_r$ and are different even in one labelled parameter value represent different equivalence classes in $S^r_{n_r}$. The question now arises whether by arbitrarily assigning numerical values to the set of parameters $\mathcal{P}_r$, the resultant set of invariants $\{I^r_{n_r}, J^r_{n_r}; \tilde{\mathcal{G}}_r \cup \mathcal{P}_r\}$ is a basis of some m.p.r. of $\{G_1, \cdots, G_r\}$, or in other words, whether the relation between $S^r_{n_r}/E_{n_r}$ and $\mathcal{P}_r$ is also surjective (onto). Since the set of parameters $\mathcal{P}_r$ is taken from locations in H^r whose specification cannot affect the rank condition $n_r = \rho H^r$, we get the following result:

PROPOSITION 4.4. *There exists a one-to-one and onto (a bijective) relationship between $S^r_{n_r}/E_{n_r}$, the set of equivalent classes in $S^r_{n_r}$, and the set of parameters $\mathcal{P}_r$.*

It has been noted, in the paragraph following Theorem 3.1, that other bases which correspond to nice selections of arithmetic invariants other than the choice of the first set of independent rows and columns may be found. Choice of such bases for the partial realization would lead to an algebraic set of invariants which would contain both specified and unspecified invariants. It can be shown that though the unspecified invariants form alternative candidates for the parametrization of the set $S^r_{n_r}$ and satisfy the one-to-one relationship of the last proposition, they do not satisfy the onto relationship. The set $\mathcal{P}_r$ is the largest set of unspecified Markov entries to which we may assign values independently and the smallest set of parameters for $S^r_{n_r}$ that covers all m.p.r.'s of order r. We restate and prove this claim as Proposition 4.5.

PROPOSITION 4.5. *The set $\mathcal{P}_r$ is* (i) *an independent set, and equivalently,* (ii) *a minimal set of parameters for the parametrization of the set of all minimal partial realizations of $\{G_1, \cdots, G_r\}$.*

Proof. See Appendix 2.

It is considered important in some fields of system theory, such as certain problems of adaptive modelling identification to have a description of the set of all m.p.r.'s with the least possible set of parameters. It follows from the last proposition that only $\mathcal{P}_r$ results in such a description. This useful complete invariant description of the set $S^r_{n_r}$ of all minimal partial realization is summarized by the following:

THEOREM 4.6. *The set of all minimal partial realizations $S^r_{n_r}$ of a finite rth order sequence is completely determined by the set of nested Markov bases $\mathcal{B}^*_r = (I^r_{n_r}, J^r_{n_r}; \tilde{\mathcal{G}}_r \cup \mathcal{P}_r)$ where $\tilde{\mathcal{G}}_r$ and $\mathcal{P}_r$ are defined in* (4.6) *and* (4.7) *respectively. $\mathcal{P}_r$ is a minimal set of independent parameters for $S^r_{n_r}$ and the relation between the set of equivalence classes in $S^r_{n_r}$ and the set of parameters $\mathcal{P}_r$, $S^r_{n_r}/E_{n_r} \to \mathcal{P}_r$ is one-to-one and onto.*

Remark 4.1. The number of parameters in $\mathcal{P}_r$ is determined by the arithmetic invariants (implicitly via (4.7)).

Remark 4.2. The equivalence classes in $S^r_{n_r}$ have the following list of system invariants in common: The arithmetic invariants $I^r_{n_r}, J^r_{n_r}$ (and as a consequence the controllability and observability indices), the subset of the algebraic invariants $\tilde{\mathcal{G}}_r$ of (4.6) and $\#\mathcal{P}_r$, the minimal number of the above-mentioned parameters. These equivalence classes in $S^r_{n_r}$ differ only in the numerical values acquired by the set $\mathcal{P}_r$.

Remark 4.3. In the special case where the m.p.r. of $\{G_1, G_2, \cdots, G_r\}$ is unique the theorem implies the following: $S^r_{n_r}$ reduces to a single equivalence class for which $\mathscr{B}_r = (I^r_{n_r}, J^r_{n_r}; \mathscr{G}_r)$ is the corresponding Markov basis and the minimal set of parameters, $\mathscr{P}_r$, is empty.

The description of the minimal partial realizations need not be confined to nested Markov bases of invariants. It has been mentioned in the preceding section that any canonical representation can be derived from an ordinary Markov basis. In a similar manner any canonical representation can be derived from $\mathscr{B}^*_r$ for equivalent descriptions of $S^r_{n_r}$. Let $(\hat{A}_r, \hat{B}_r, \hat{C}_r)$ be the triple of matrices of Definition 3.1, derived from H^r in association with $\mathscr{B}^*_r = (I^r_{n_r}, J^r_{n_r}; \mathscr{G}_r \cup \mathscr{P}_r)$. The first and the second canonical forms of Theorem 3.2, (A_1, B_1, C_1), $(A_2, B_2, C_2) \in S^r_{n_r}$, can be derived from $(\hat{A}_r, \hat{B}_r, \hat{C}_r)$ by using a method analogous to the method of § 3,

$$[B_1, A_1] = T_1[\hat{B}_r; \hat{A}_r] \quad \text{and} \quad C_1 = C_r \tag{4.8}$$

and

$$B_2 = B_r \quad \text{and} \quad \begin{bmatrix} C_2 \\ A_2 \end{bmatrix} = \begin{bmatrix} \hat{C}_r \\ \hat{A}_r \end{bmatrix} T_2, \tag{4.9}$$

where T_1 and T_2 represent, respectively, the row and the column elimination operations, of Algorithm 3.1. Bases of canonical invariants $\mathscr{B}^*_1 = (I^r_{n_r}; \mathscr{G}^*_1)$ and $\mathscr{B}^*_2 = (J^r_{n_r}; \mathscr{G}^*_2)$ can also be derived for these canonical representations in accordance with Theorem 3.2. The difference between the m.p.r. canonical descriptions in the present case and the minimal (complete) realization description by system invariants of § 3 becomes significant in the case of $r < r_0$. In this case, which corresponds to the existence of more than one solution to the m.p.r. problem, the canonical representations as well as their corresponding canonical bases of invariants contain undetermined entries which are expressed by combinations of the minimal set of parameters $\mathscr{P}_r$.

Some other points of significance about the set $\mathscr{P}_r$ that make it further useful in certain problems of system identification are as follows. The set $\mathscr{P}_r$ is formed by assembling parameters in a form that can directly use further data that may be available under excessive measurements. Furthermore, as the parameters $\{g^*_{ijk}\}$ in the set $\mathscr{P}_r$ are labelled by their position in the extending data set, $\mathscr{P}_r$ contains information that indicates precisely which output-input pairs of relations (i, j) require further exploration and in what way can the model be completely specified.

Example 4.1. We shall illustrate the invariant description concepts presented above for m.p.r.'s by deriving nested bases of invariants and canonical realizations for sequences of order $r = 2, 3, 4$ for the numerical example of Tether [2].

$$G_1, G_2, G_3, G_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 10 & 7 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 22 & 15 \\ 3 & 3 \end{bmatrix}. \tag{4.10}$$

Fourth order m.p.r.'s for this example were also derived in [5]–[8]. Nested bases of invariants are suggestive of recursive algorithms of realizations of sequences of successive higher orders. Since an efficient algorithm of this kind requires details which were not discussed in the present context, we shall derive invariant descriptions of m.p.r.'s separately for each order. For the sake of brevity we shall derive realizations only in the second canonical forms.

(a) *Fourth order sequence.* Construct for $r=4$, H^4, the fourth order incomplete Hankel matrix of (4.1)

$$
(4.11) \qquad H^r = \left[\begin{array}{cccc|cc|cc}
\boxed{1} & 1 & 4 & 3 & 10 & 7 & 22 & 15 \\
0 & 0 & 0 & 0 & \boxed{1} & 1 & 3 & 3 \\
4 & \boxed{3} & 10 & 7 & 22 & 15 & g_{115} & g_{125} \\
0 & 0 & \boxed{1} & 1 & 3 & 3 & g_{215} & g_{225} \\
10 & 7 & 22 & 15 & g_{115} & g_{125} & g_{116} & g_{126} \\
1 & 1 & 3 & \boxed{3} & g_{215} & g_{225} & g_{216} & g_{226} \\
22 & 15 & g_{115} & g_{125} & g_{116} & g_{126} & g_{117} & g_{127} \\
3 & 3 & g_{215} & g_{225} & g_{216} & g_{226} & g_{217} & g_{227}
\end{array}\right].
$$

The squared entries in (4.11) represent the pivotal elements determined by the numerically specified entries. (In a numerical example we may drop the asterisks used to mark unspecified entries. These can be found, for example, by the row reserving elimination operation (Definition 3.2). We observe that a m.p.r. of order $r=4$ is of dimension $\rho H^4=5$. The first independent rows are $I_5^4=\{1,2,3,4,6\}$ thus $\bar{\nu}_i=\#\{1,3\}=2$, $\bar{\nu}_2=\#\{2,4,6\}=3$ by which $\nu^4=\{2,3\}$. Similarly, the first independent columns are $J_5^4=\{1,2,3,4,5\}$ hence $\mu^4=\{3,2\}$ and a fourth order m.p.r. is determined by entries of the first $r_0=3+3=6$ Markov matrices. The set of Markov bases therefore consists of I_5^4, J_5^4 as the arithmetic invariants and $\tilde{\mathscr{G}}_4\cup\mathscr{P}_4$ as the algebraic invariants, where $\tilde{\mathscr{G}}_4$ and $\mathscr{P}_4$ are determined by (4.6) and (4.7), respectively, to be $\tilde{\mathscr{G}}_4=\{(g_{11k}, g_{12k}, g_{21k}, g_{22k}), k\in 4\}$ and $\mathscr{P}_4=\{g_{115}, g_{215}, g_{225}, g_{216}\}$. These invariants are summarized in the upper part of Table 4.1 where the algebraic invariants appear as encircled entries in the Markov matrices. Associated with the set of bases $\mathscr{B}_4^*=\{I_5^4, J_5^4; \tilde{\mathscr{G}}_4\cup\mathscr{P}_4\}$ are the triple of matrices $(\hat{A}, \hat{B}, \hat{C})$ of Definition 3.1,

$$
\hat{A}=\begin{bmatrix}
4 & 3 & 10 & 7 & 22 \\
0 & 0 & 1 & 1 & 3 \\
10 & 7 & 22 & 15 & g_{115} \\
1 & 1 & 3 & 3 & g_{215} \\
3 & 3 & g_{215} & g_{225} & g_{216}
\end{bmatrix}, \qquad
B=\begin{bmatrix}
1 & 1 \\
0 & 0 \\
4 & 3 \\
0 & 0 \\
1 & 1
\end{bmatrix},
$$

$$
\hat{C}=\begin{bmatrix}
1 & 1 & 4 & 3 & 10 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

from which the second canonical form (A_2, B_2, C_2) can be obtained by Algorithm 3.1 resulting in $B_2=\hat{B}$ and

$$
\begin{bmatrix} C_2 \\ A_2 \end{bmatrix}=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-2 & a & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
b & d & 0 & c & -b+3
\end{array}\right],
$$

where $a=g_{115}-46$, $b=g_{215}-g_{225}$, $c=g_{225}-9$, $d=g_{216}-3g_{215}+6-(g_{215}-7)\times(g_{125}g_{215}+3)$. The corresponding canonical invariant basis is $\mathscr{B}_2^*=(I_5^4;\mathscr{G}_2^*)$ where $\mathscr{G}_2^*$ is formed by the entries of B_2 and of rows 3, 5 of A_2.

(b) *Third order sequence*: $\{G_1, G_2, G_3\}$. The three upper-block diagonals of (4.8) reveal that H^3, the incomplete Hankel matrix required for $r=3$, is characterized by $n_3=4$ and that the first four independent rows and columns of H^3 are, respectively, $I_4^3=\{1,2,3,4\}\rightarrow\nu^3=\{2,2\}$ and $J_4^3=\{1,2,3,5\}\rightarrow\mu^3=\{3,1\}$ from which $r_0=5$. The set of Markov bases are $\mathscr{B}_3^*=\{I_4^3, J_4^3; \tilde{\mathscr{G}}_3\cup\mathscr{P}_3\}$ where $\tilde{\mathscr{G}}_3$ and $\mathscr{P}_3$ are formed by the encircled entries in the middle part of Table 4.1. The associated triple of matrices $(\hat{A},\hat{B},\hat{C})$ for these invariants are,

$$\hat{A}=\begin{bmatrix}4 & 3 & 10 & g_{114}\\ 0 & 0 & 1 & g_{214}\\ 10 & 7 & g_{114} & g_{115}\\ 1 & 1 & g_{214} & g_{215}\end{bmatrix},\qquad \hat{B}=\begin{bmatrix}1 & 1\\ 0 & 0\\ 4 & 3\\ 0 & 0\end{bmatrix},\qquad \hat{C}=\begin{bmatrix}1 & 1 & 4 & 10\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

The second canonical form (A_2, B_2, C_2) is readily obtained from these matrices by Algorithm 3.1; $B_2=\hat{B}$ and

$$\begin{bmatrix}C_2\\ A_2\end{bmatrix}=\left[\begin{array}{cccc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -2 & c & 3 & a\\ 1 & d & 0 & b\end{array}\right],$$

where $a=g_{114}-22$, $b=g_{214}-4$, $c=g_{115}-3g_{114}-g_{214}(g_{114}-20)+20$, $d=g_{215}-g_{214}(g_{214}-4)-10$. The corresponding canonical invariant bases are $\mathscr{B}_2^*=(I_4^3;\mathscr{G}_2^*)$ where $\mathscr{G}_2^*$ is composed of the entries of B_2 and of rows 3, 4 of A_2.

(c) *Second order sequence* $\{G_1, G_2\}$. Repetition of the above procedure for $r=2$ yields $n_2=2$ $I_2^2=\{1,3\}\rightarrow\nu^2=\{2,0\}$, $J_2^2=\{1,2\}\rightarrow\mu^2=\{1,1\}$, by which $r_0=3$. The set of Markov bases are $\mathscr{B}_2^*=\{I_2^2, J_2^2; \tilde{\mathscr{G}}_2\cup\mathscr{P}_2\}$ where $\tilde{\mathscr{G}}_2$ and $\mathscr{P}_2$ respectively are formed by the specified and the unspecified encircled entries in the $r_0=3$ Markov matrices in the lower part of Table 4.1. From the associated triple of matrices $(\hat{A},\hat{B},\hat{C})$,

$$\hat{A}=\begin{bmatrix}4 & 3\\ g_{113} & g_{123}\end{bmatrix},\qquad \hat{B}=\begin{bmatrix}1 & 1\\ 4 & 3\end{bmatrix},\qquad \hat{C}=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix},$$

the following realization in the second form is found

$$A_2=\begin{bmatrix}0 & 1\\ a & b\end{bmatrix},\qquad B_2=\begin{bmatrix}1 & 1\\ 4 & 3\end{bmatrix},\qquad C_2=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix},$$

where $a=4g_{113}-3g_{123}$, $b=g_{113}-g_{123}$ and the corresponding canonical set of bases are $(I_2^2;\mathscr{G}_2^*)$ with $\mathscr{G}_2^*$ containing the entries of B_2 and the second column of A_2.

Table 4.1 summarizes the invariants of the realizations of orders $r=4,3,2$ and exhibits their nested property. Our results can be compared for the $r=4$ case, with the previous realizations in [2], [5]–[8]. Tether [2] suggests a minimal extension segment $\{G_{r+1},\cdots,G_{r_0}\}=\{G_5, G_6\}$ that contains only two free parameters which in comparison with our results corresponds to two unnecessary constraints on $\mathscr{P}_4$, namely $g_{115}=46$, $g_{215}=g_{225}$. The realization in [5] identifies only three free parameters for $S_{n_4}^4$ and has other weaknesses discussed in [6]. The authors in [7] and [8] correctly

TABLE 4.1
Nested bases of invariants for $r = 2, 3, 4$ *realization of* (4.10).

$r=4, \quad n_4=5, \quad I_5^4=\{1,2,3,4,6\}, \quad J_5^4=\{1,2,3,4,5\}, \quad r_0=6$

$$G_1, G_2, G_3, G_4, G_5^*, G_6^* = \begin{bmatrix} \textcircled{1} & \textcircled{1} \\ \textcircled{0} & \textcircled{0} \end{bmatrix}, \begin{bmatrix} \textcircled{4} & \textcircled{3} \\ \textcircled{0} & \textcircled{0} \end{bmatrix}, \begin{bmatrix} \textcircled{10} & \textcircled{7} \\ \textcircled{1} & \textcircled{1} \end{bmatrix}, \begin{bmatrix} \textcircled{22} & \textcircled{15} \\ \textcircled{3} & \textcircled{3} \end{bmatrix}, \begin{bmatrix} \textcircled{g_{115}} & g_{125} \\ \textcircled{g_{215}} & \textcircled{g_{225}} \end{bmatrix}, \begin{bmatrix} g_{116} & g_{126} \\ \textcircled{g_{216}} & g_{226} \end{bmatrix}$$

$\mathscr{P}_4 = \{g_{115}, g_{215}, g_{225}, g_{216}\}$

$r=3, \quad n_3=4, \quad I_4^3=\{1,2,3,4\}\subset I_5^4, \quad J_4^3=\{1,2,3,5\}\subset J_5^4, \quad r_0=5$

$$G_1, G_2, G_3, G_4^*, G_5^* = \begin{bmatrix} \textcircled{1} & \textcircled{1} \\ \textcircled{0} & \textcircled{0} \end{bmatrix}, \begin{bmatrix} \textcircled{4} & \textcircled{3} \\ \textcircled{0} & \textcircled{0} \end{bmatrix}, \begin{bmatrix} \textcircled{10} & \textcircled{7} \\ \textcircled{1} & \textcircled{1} \end{bmatrix}, \begin{bmatrix} \textcircled{g_{114}} & g_{124} \\ \textcircled{g_{214}} & g_{224} \end{bmatrix}, \begin{bmatrix} \textcircled{g_{115}} & g_{125} \\ \textcircled{g_{215}} & g_{225} \end{bmatrix}$$

$\mathscr{P}_3 = \{g_{114}, g_{214}, g_{115}, g_{215}\}$

$r=2, \quad n_2=2, \quad I_2^2=\{1,3\}\subset I_4^3, \quad J_2^2=\{1,2\}\subset J_4^3, \quad r_0=3$

$$G_1, G_2, G_3^* = \begin{bmatrix} \textcircled{1} & \textcircled{1} \\ \textcircled{0} & \textcircled{0} \end{bmatrix}, \begin{bmatrix} \textcircled{4} & \textcircled{3} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \textcircled{g_{113}} & \textcircled{g_{123}} \\ g_{213} & g_{223} \end{bmatrix}$$

$\mathscr{P}_2 = \{g_{113}, g_{123}\}$

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identify four independent parameters. Their descriptions use Popov type system invariants [9] and the representation is admitted in [7] to be nonunique. The realization there is into arbitrary Luenberger forms [13] by which the unity vectors in $\begin{bmatrix} C \\ A \end{bmatrix}$ or $[B, A]$ appear in arbitrary order and in positions that are not related to the system output or input structure. The computation in [7] requires solutions of sets of linear equations and the elimination procedure in [8] requires an auxiliary matrix. By comparison with these former invariant description approaches to m.p.r. our method is also advantageous computationally.

5. Conclusions. This paper studies the minimal partial realization (m.p.r.) problem using system invariant descriptions. The concept of nested bases of invariants is introduced and these bases are derived from entries in specified positions of the Markov sequences. These bases form invariant descriptions for m.p.r.'s which, in contrast to previous approaches, do not depend on any particular choice of a canonical representation. The existence of a unique solution to the m.p.r. problem can be tested on these invariants and when more than one solution exists the set of all m.p.r.'s for the given finite sequence can be expressed as a set of bases that contains a subset of undetermined invariants. The nesting property of these bases is used to prove that this set of undetermined invariants forms a minimal set of independent parameters that covers all possible m.p.r.'s of the sequence and that for any arbitrarily assigned values of these parameters there corresponds some admissible solution.

Two canonical state space representations have been suggested and an efficient algorithm for their derivation from the nested bases is provided. These canonical forms reflect the structural properties of the underlying system and also compare favorably in their numerical aspects with previous approaches to m.p.r.'s.

Any other canonical representation can alternatively be derived from these bases, and the solution to the m.p.r. problem can be expressed by combinations of the minimal set of parameters obtained. The complete freedom in assigning values to these parameters may be used to search for further properties of the constructed models. (e.g., to ask for stable models). These parameters form entries of specific positions in the unknown extension sequence of the Markov matrices which may be of importance in certain identification problems. The formulation may also be advantageous in building adaptive real time identification models from input-output data. In this latter case an estimated state space model can be continuously updated by measuring only specific locations in the input-output map prescribed by the basis of invariants (where the model may be taken to be valid so long as the arithmetic invariants remain unchanged).

An obvious property of the suggested nested bases of invariants which has not been put to use in the present context is that these bases are ideal for recursive m.p.r. algorithms for sequences of Markov matrices of successive orders. This stems from the projective property of nested bases, i.e., they present sub-bases not only for all possible minimal extension sequences but also for arbitrary extension sequences of higher dimensions. Such a sequential algorithm, whose detailed numerical aspects have yet to be developed, will have the following features. The dimension of a realization of a sequence of a given length need not be known in advance. Subsequent order realizations require the calculation of only a few new invariants which add to the former set of invariants to form the new basis. The final important feature is that at each stage either the unique m.p.r. or in the nonunique case, the set of all possible m.p.r.'s are obtained and in the latter case these m.p.r.'s are described in terms of a minimal set of parameters.

Appendix 1. Proof of Theorem 3.2. We shall prove only the first part of the theorem, as the second part follows by an obvious dual reasoning. Statements (1a) and (1b) follow from Remarks 3.2 and 3.3. We have to show that $\mathcal{B}_1=(I_n; \mathcal{G}_1)$ is a basis of invariants. I_n represents the arithmetic invariants associated with the observability indices (3.4) of the underlying system. The elements of the set $\mathcal{G}_1$ are entries in a canonical representation, thus they are canonical invariants. The set $(I_n; \mathcal{G}_1)$ is complete because it completely determines (A_1, B_1, C_1) via statements (1a) and (1b). The pair (B_1, A_1) is controllable by statement (1b) by which an arbitrary choice of $(I_n; \mathcal{G}_1)$ fails to give rise to a representation $(A_1, B_1, C_1)\in\Sigma_n$ if and only if it yields an unobservable pair (A_1, C_1). This condition is equivalent to $\rho[{}^{C}_{A}]<n$ and it can be expressed by suitable sets V_i of (2.8). Consequently the map $\mathcal{G}_1:\Sigma_n\to R^{n(m+l)}$ is surjective except possibly on some hypersurfaces of "measure zero" in its codomain, thus $\mathcal{G}_1$ is also an independent set of invariants in the sense of Definition 2.3.

Appendix 2. Proof of Proposition 4.5. Assume that $\mathcal{P}_r$ is not independent and let $\mathcal{P}_r=\mathcal{P}_1\cup\mathcal{P}_2$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are subsets of independent and dependent parameters, respectively. Once $\mathcal{P}_1$ has been arbitrarily assigned values the set $\mathcal{P}_2$ is uniquely determined in contrast to the surjective relationship between $S^r_{n_r}/E_{n_r}$ and $\mathcal{P}_r$ stated in Proposition 4.4. Therefore all the parameters in $\mathcal{P}_r$ can be assigned values independently. Now we show the equivalence of (i) and (ii). For (ii)$\to$(i), a minimal set of parameters has to be independent or else a smaller set can be extracted for the parametrization of $S^r_{n_r}$. For the converse, (i)$\to$(ii), assume there exists another basis $\hat{\mathcal{B}}^*_r$ for $S^r_{n_r}$ whose set of algebraic invariants $\hat{\mathcal{G}}_r\cup\hat{\mathcal{P}}_r$ is composed of a smaller set of unspecified values $\#\hat{\mathcal{P}}_r<\#\mathcal{P}_r$. Then $\mathcal{G}_r\cup\mathcal{P}_r$ could be expressed as a function of $\hat{\mathcal{G}}_r\cup\hat{\mathcal{P}}_r$ which implies the contradiction that not all the parameters in $\mathcal{P}_r$ can be assigned values independently.

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