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Minimal Padé Model Reduction for Multivariable Systems

The approximation of high order linear multivariable systems in the Padé sense is considered. A unified treatment is presented by which various models of minimal order are found which match given sequences of time moments and Markov matrices. The uniqueness of these models is investigated and in cases where there exist more than one minimal model for a given sequence the set of all the distinct models is characterized by a minimal set of independent parameters which can be assigned arbitrary values. The possible instability of Padé reduced models for stable systems is considered and a method is suggested which yields stable models that approximate the high order system, or at least its magnitude, in the Padé sense.

I Introduction

The methods of approximating linear invariant large scale systems by lower order models in the Padé sense find models that match the first terms of expansion of the system transfer function matrix about zero and/or infinity, cf. [1] and references therein. Several techniques have been developed for single-input single-output (SISO) systems [1]-[5] and attempts to extend these techniques to the multi-input multi-output (MIMO) case using matrix Padé equations, matrix continued fractions, moment matching and partial realizations have been made [1], [6]-[11]. None of these extensions provide a general method for MIMO systems that obtains models of minimal order to match a given sequence of matrix terms of predetermined length of the transfer function matrix expansions. Application of Padé equation, continued fraction and related equivalent techniques that were used in the SISO case, for MIMO systems encounters several problems. An important shortcoming is that they may lead to models whose order is higher than intended, and sometimes even higher than the order of the approximated systems. This difficulty can be traced in the numerical examples used in, say, [9], [14]-[16], [24]. Another difficulty which is well recognized and that may occur even in the SISO case is that Padé approximation of stable system may yield unstable models [12]-[16]. The present paper suggests a new approach for the Padé approximation problem that overcomes many previous difficulties and limitations. A short conference version of this paper was presented earlier in [34].

II The Minimal Padé Approximation (MPA) Problem

We consider a high order linear time invariant multivariable system of m inputs and l outputs which is described by a transfer function matrix $H(s)$. Let the series expansion of $H(s)$ about zero (assuming analytically there) and infinity be

$$H(s) = -\sum_{i=1}^{\infty} T_i s^{i-1} \quad \text{and} \quad H(s) = \sum_{i=1}^{\infty} M_i s^{-i} \qquad (2.1a,b)$$

respectively. The matrices M_i, T_i, $i=1,2,\ldots$ are the Markov and the (modified) time moment matrices of the system, respectively.

Given a sequence of $r=p+q$ time moment and Markov matrices of the high order system $\sigma(p,q) = \{T_p, T_{p-1}, \ldots, T_1, M_1, M_2, \ldots, M_q\}$, the definition of the (p,q) mixed mineral Padé approximation (MPA) reduction problem is the following.

Find a model whose representation (A,B,C), $A\epsilon R^{n\times n}$, $B\epsilon R^{nxm}$, $C\epsilon R^{lxn}$, satisfies for a minimal value of n.

$$CA^{-i}B = T_i \quad i=1,\ldots,p \qquad (2.2a)$$

$$CA^{i-1}B = M_i \quad i=1,\ldots,q \qquad (2.2b)$$

Denoting the class of all representations equivalent to (A,B,C) under the regular state-space coordinate transformation by $E_n(A,B,C)$, a solution to a (p,q) mixed MPA reduction problem is called unique if all the minimal representations that satisfy (2.2a,b) belong to the same $E_n(A,B,C)$.

In the special case of $p=0$, the above defined problem coincides with the minimal partial realization problem [19], [20] which has recently been investigated using an invariant description approach [17]. We shall apply the approach in [17] to solve the above model reduction problem also for $p>0$.

We redenote the mixed sequence $\sigma(p,q)$ by $\{G_1, G_2, \ldots, G_r\}_q^p$ and construct the following incomplete Hankel matrix

$$K(p,q) = \begin{bmatrix} G_1, \ldots, G_r, G_{r+1}^* \ldots \\ \cdot \\ \cdot \\ \cdot \\ G_r \\ G_{r+1}^* \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad (2.3)$$

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where asterisked entries G^*_{r+1}, G^*_{r+2} represent some unspecified extension matrices which, following the "asterisked convention" [17], may participate in $K(p,q)$ and its submatrices without changing the minimal rank of these matrices. Denoting by n the rank of the above matrix, $\rho K(p,q)$, we have the following.

Lemma 1 A solution to the (p,q) mixed MPA reduction problem is a model of order n. Such a solution always exists but it is not necessarily unique.

The proof follows from the well-known minimal partial realization theory [19], [20], noting that a triple (A,B,C) matches $\{G_1 \ldots G_r\}^p_q$ iff the triple (A,A^pB,C) matches a finite Markov sequence $M_i = G_i$, $i=1, \ldots, r$ of some different system.

Let the first n independent rows and columns in $K(p,q)$ be denoted by the set of indices $I_n = \{i_1, \ldots, i_n\}$, and $J_n = \{j_1, \ldots, j_n\}$, respectively. Following the "asterisked convention" of G^*_{r+i}, $I_n\,(J_n)$ represents the first n rows which, regarding only its numerically specified column (row) positions, do not depend on preceding rows (columns). Therefore, I_n and J_n are determined only by $\{G_1, \ldots, G_r\}$ where we further prove the following.

Lemma 2 The sets I_n and J_n are independent of p, $0 \le p \le r$.
Proof: Let (A_H,B_H,C_H) be a minimal state space representation of the given high order system, $A_H \epsilon R^{N\times N}$, $B_H \epsilon R^{N\times m}$, $C_H \epsilon R^{l\times n}$. The proof follows from the following possible decomposition of $K(p,q)$, where $C_H A_H^{i-1} B_H$ for $i>q$ are the terms with asterisks.

$$K(p,q) = [C^t_H, A_H{}^t C^t_H \ldots]^t A_H^{-p} [B_H, A_H B_H, \ldots] \quad (2.4)$$

The sets I_n and J_n are arithmetic invariants of $E_n(A,B,C)$. They are bijectively related to the observability and the controllability indices $I_n = \{\nu_1, \ldots, \nu_l\}$ and $J_n = \{\mu_1, \ldots, \mu_m\}$ of the models by the following; $\nu_k\,(\mu_k)$ is the number of elements in common to the set $\{i_k, i_k+l, i_k+2l, \ldots\}$ (the set $\{j_k, j_k+m, j_k+2m, \ldots\}$) and the set I_n (the set J_n). Defining

$$\alpha = \max_j \mu_j \qquad \text{and} \qquad \beta = \max_j \nu_i$$
$$j=1 \ldots m \qquad\qquad i=1 \ldots l \quad (2.5)$$

The next theorem follows from [19], [20] and the last two lemmas.

Theorem 1 The solution of the (p,q) mixed MPA model reduction problem is unique iff $p+q \ge \alpha+\beta$. Hence, the condition for uniqueness of the models is independent of the emphasis that is put on the approximation at zero and infinity.

Theorem 2 The set of entries of the matrices (p,q)

$$\mathcal{G}_{p,q} = \{g_{ijk} \mid (k=1,2 \ldots, \nu_i+\mu_j);$$
$$i=1, \ldots, l, j=1, \ldots, m\} \quad (2.6)$$

with $g_{ijk} = (G_k)_{ij}$, is a sufficient set of parameters, in addition to the sets I_n and J_n for the complete determination of all the possible solutions of the (p,q) mixed MPA model reduction problem.

Theorem 2 follows directly from [17], [21] and the two lemmas, where a more precise statement of the theorem could have been that $\mathcal{B} = \{I_n, J_n, \mathcal{G}_{p,q}\}$ is a (nested) basis of invariants for the set of all solutions to the (p,q) mixed MPA problem. Such a statement also implies that the sufficiency above can generically be replaced by minimality [17].

If $r<\alpha+\beta$ then by Theorem 1 the problem has more than one solution. In this case the set $\mathcal{G}_{p,q}$ contains the following subset of numerically unspecified parameters which is formed by entries of $\{G_{r+1} \ldots G_{\alpha+\beta}\}$

$$\mathcal{P} = \{g_{ijk} \mid (k=r+1, \ldots, \nu_i+\mu_j);$$
$$i=1 \ldots l, j=1 \ldots m; \nu_i+\mu_j > r\} \quad (2.7)$$

For this set we have by [17] and the two lemmas, the following

Theorem 3 The set $\mathcal{P}$ is the maximal set of degrees of freedom for the (p,q) mixed MPA models. For any arbitrarily assigned values of the elements of $\mathcal{P}$ there corresponds a unique model and all the solutions correspond to some numerical specification of $\mathcal{P}$.

A simple procedure for the derivation of a solution (A,B,C) for the (p,q) mixed MPA that uses I_n, J_n and the set of entries $\mathcal{G}_{p,q}$ is given as follows [17] [18].

An algorithm. Given $K(p,q)$, I_n and J_n,

(*i*) Construct the following matrices

$\hat{C}$ The $l\times n$ matrix formed from $K(p,q)$ by the intersection of the rows $\{1, \ldots, l\}$ with the columns J_n
$\hat{B}$ The $n\times m$ matrix formed from $K(p,q)$ by the intersection of the rows I_n with the columns $\{1, \ldots, m\}$
$\hat{A}$ The $n\times n$ matrix formed from $K(p,q)$ by the intersection of the rows I_n with the columns $\{j_1+m, j_2+m, \ldots, j_n+m\}$, where $J_n = \{j_1, \ldots, j_n\}$

(*ii*) Perform on $[\hat{B},\hat{A}]$ a row elimination procedure that brings the columns J_n to form the $n\times n$ identity matrix and results in, say, $[\bar{B},\bar{A}]$.

(*iii*) The solution (A,B,C) is given by

$$A=\bar{A}, \quad B=\bar{B}, \quad C=\hat{C}\bar{A}^p \quad (2.8)$$

The above algorithm reduces for the special case of $p=0$ and sufficiently large r to the derivation of minimal (complete) realizations from Markov sequences. For this case the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ form the submatrices of the Hankel matrix used by Silverman [18] for the derivation of minimal realizations.

The triple of matrices $(\hat{A}, \hat{B}, \hat{C})$ involves exactly all the entries of $G_{p,q}$. In the case where $r<\alpha+\beta$ the triple $(\hat{A}, \hat{B}, \hat{C})$ contains combinations of the unspecified parameters of and it therefore represents, in terms of a minimal set of independent parameter, the set of all distinct models that solve the (p,q) mixed MPA problem. The above algorithm represents a derivation of the solution in the first of the two canonical forms (the controllable form) of [17]. An equivalent dual form (an observable form) can be obtained by column elimination of $[\hat{C}^t \hat{A}^t]^t$. A simple example clarifies the method.

Example 2.1 The matrices T_1, M_1 and M_2 of a high order system of $m=2$ inputs and $l=3$ outputs are given by

$$T_1 = \begin{bmatrix} \textcircled{1} & \textcircled{1} \\ \textcircled{1} & \textcircled{2} \\ \textcircled{2} & \textcircled{1} \end{bmatrix} \quad M_1 = \begin{bmatrix} \textcircled{3} & \textcircled{5} \\ \textcircled{2} & \textcircled{1} \\ \textcircled{7} & 14 \end{bmatrix} \quad M_2 = \begin{bmatrix} \textcircled{7} & 7 \\ \textcircled{6} & \textcircled{7} \\ 15 & 14 \end{bmatrix} \quad (2.9)$$

A minimal reduced order model is required to match these matrices.

The incomplete $K(1,2)$ Hankel matrix of (2.3) for $G_1 = T_1$, $G_2 = M_1$, $G_3 = M_2$, is

$$K(1,2)=\left[\begin{array}{cc:cc:cc} \boxed{1} & 1 & 3 & 5 & 7 & 7 \\ 1 & \boxed{2} & 2 & 1 & 6 & 7 \\ 2 & 1 & 7 & 14 & 15 & 14 \\ \hdashline 3 & 5 & 7 & 7 & & \\ 2 & 1 & \boxed{6} & 7 & & \\ 7 & 14 & 15 & 14 & & \\ \hdashline 7 & 7 & & & & \\ 6 & 7 & & & & \\ 15 & 14 & & & & \end{array}\right] \qquad (2.10)$$

The rank of $K(1,2)$ and the first independent rows and columns can be determined by applying elementary row operations on $K(1,2)$ without changing row positions. The pivotal entries of such elimination procedure were denoted in (2.10) by squares. Note that, for example, the incomplete row 4 is eliminated in such a row operation because its specified part, columns 1-4, can be expressed as linear combination of corresponding column positions of rows 1 and 2. We have that $\rho K(1,2)=3$ and that the first independent rows and columns are $I_3=\{1,2,5\}$ and $J_3=\{1,2,3\}$, respectively. Thus, $\nu=\{1,2,0\}$ and $\mu=\{2,1\}$. The solution is not unique because $\beta=\max\nu_i=2$, $\alpha=\max\mu_j=2$ and $r=3<\alpha+\beta=4$. The set of entries $\mathcal{G}_{1,2}$ of (2.6) that is involved in determining the solution consists of the encircled entries of (2.9) and one free parameter $\mathcal{P}=\{g_{214}\}$ which may assign arbitrary values. It follows also from Lemma 2 that MPA for any (p,q) matrices, $p+q=3$, of the same system are nonunique models of order $n=3$ and that the set of all the solutions in each case can be constructed by entries of the same positions in G_1, G_2, G_3 and g_{214}. Applying step (*i*) of the algorithm we obtain the following

$$\hat{C}=\begin{bmatrix}1 & 1 & 3\\ 1 & 2 & 2\\ 2 & 1 & 7\end{bmatrix}\quad \hat{B}=\begin{bmatrix}1 & 1\\ 1 & 2\\ 2 & 1\end{bmatrix}\quad \hat{A}=\begin{bmatrix}3 & 5 & 7\\ 2 & 1 & 6\\ 6 & 7 & g_{214}\end{bmatrix}$$

Applying row elimination on $[\hat{B},\hat{A}]$ we find, letting $g=g_{214}$, that

$$[\bar{B},\bar{A}]=\begin{bmatrix}1 & 0 & : & 0 & -19 & 4g-52\\ 0 & 1 & : & 0 & 3 & 14-g\\ 0 & 0 & : & 1 & 7 & 15-g\end{bmatrix}$$

and thus the triple of matrices (A,B,C)

$$B=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix},\quad A=\begin{bmatrix}0 & -19 & 4g-52\\ 0 & 3 & 14-g\\ 1 & 7 & 15-g\end{bmatrix},\quad C=\hat{C}\bar{A}=\begin{bmatrix}3 & 5 & 7\\ 2 & 1 & 6\\ 7 & 14 & 15\end{bmatrix}$$

represents the set of all minimal models that match T_1, M_1, and M_2.

III Stability

It is well known that the Padé approximation to stable systems may yield unstable models both in the SISO and the MIMO cases. Several methods have been suggested for SISO systems [12], [13], [23], [24]. Extension to multivariable (m, $l>1$) systems, suggested in [14]-[16], may again encounter a dimensional problem similar to the one mentioned in the introduction and that will be explained in Section V.

The theory of Section II may be used to provide stable models in two ways. In the first, if the (p,q)th problem has more than one solution, values for the free parameters in $\mathcal{P}$ may be searched that yield stable models. The second method is to try different weights on the approximations at $s=0$ and $s=\infty$ and to choose those models (if any) that are stable.

A completely different approach that always provides stable low order models to a given stable system is given below. It stems from the importance of the squared magnitudes of transfer function matrices in system theory. The squared magnitude of the transfer function $H(s)$ plays a major role, for instance, in the linear filtering and the optimal control problems. It is shown in [27]-[29] that the optimal filter in the stationary filtering problem is determined by $H(s)H'(-s)$ rather than by $H(s)$. Similarly, it is shown in there, that the optimal control for the linear steady state regulator problem does not depend on $H(s)$ explicitly but only on the term $H'(-s)H(s)$. The importance of the square magnitude of transfer function matrices lies however far beyond its use in optimal control and filterings. It is shown in [31] [32] that the control properties of multivariable systems are largely dictated by their singular values. These values that play a "generalized Bode plots role" in system designs depend only on the squared magnitude of the transfer function matrices.

Consider the case where the high order system is asymptotically stable and where no stable solution to the (p,q) mixed MPA reduction problem, for a specific p, $q>0$, can be found. Let (A,B,C) be an unstable solution to the problem, it is required to obtain new stable models (A_c, B_c, C_c) and (A_0, B_0, C_0) of similar order such that their transfer function matrices $G_C(s)=C_c(sI-A_c)^{-1}B_c$ and $G_0=C_0(sI-A_0)^{-1}B_0$ yield *a* (p,q) mixed Padé approximation of $H'(-s)H(s)$ and $H(s)H'(-s)$, respectively.

The two theorems below show how to find models that satisfy, respectively

$$G_c'(-s)G_c(s)=G'(-s)G(s),\quad G_c(0)=G(0) \qquad (3.1a,b)$$

$$G_0(s)G_0'(-s)=G(s)G'(-s),\quad G_0(0)=G(0) \qquad (3.2a,b)$$

where $G(s)=C(sI-A)^{-1}B$. Clearly such models form (p,q) mixed Padé approximations to $H'(-s)H(s)$ and $H(s)H'(-s)$, respectively.

Denoting the spectral decomposition of the matrix A by

$$A=[U_1,U_2]\begin{bmatrix}J_1 & 0\\ 0 & J_2\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix},\quad [U_1,U_2]\begin{bmatrix}V_1\\ V_2\end{bmatrix}=I \qquad (3.3a,b)$$

where J_1 and J_2 are the $(\xi\times\xi)$ and $(n-\xi)\times(n-\xi)$ Jordan blocks that correspond to the eigenvalues $(\lambda_1,\ldots,\lambda_\xi)$ and $(\lambda_{\xi+1},\ldots,\lambda_n)$ in the right and the left half planes, respectively, we have the following

Theorem 4 The matrix $A_C=A-P_cC'C$ (3.4)

is a stability matrix with eigenvalues $(-s_1,\ldots,-\lambda_\xi,\lambda_{\xi+1}\ldots\lambda_n)$ where

$$P_c=U_1Q_c^{-1}U_1^+ \qquad (3.5a)$$

Q_c is the solution of

$$Q_cJ_1+J_1^+Q_c=U_1^+C'CU_1,\quad Q_c>0 \qquad (3.5b)$$

and $(\ )^+$ denotes Hermitian transpose. The triples (A_c, B, C) and $(A_c, B, CA^{-1}A_c)$ satisfy (3.1*a*) and both (3.1*a*) and (3.1*b*), respectively.

Theorem 5 The matrix $A_0=A-BB'P_0$ (3.6)

is a stability matrix with eigenvalues $(-\lambda_1,\ldots,-\lambda_\xi,\lambda_{\xi+1},\ldots,\lambda_n)$ where

$$P_0=V_1^+Q_0^{-1}V_1 \qquad (37a)$$

and

$$J_1Q_0+Q_0J_1=V_1BB'V_1^+,\ Q_0>0 \qquad (3.7b)$$

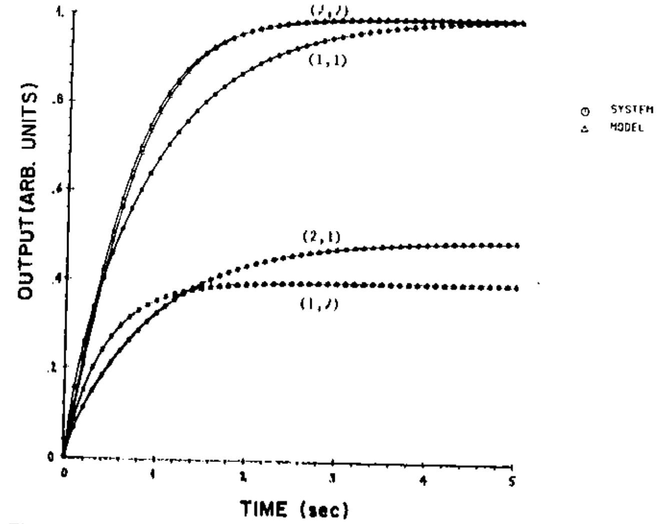

Fig. 1 Unit step responses for system and model (A_1, B_1, C_1) (i,j) represent the responses of outputs i to a unit step at inputs j

The triples (A_0,B,C) and $(A_0,A_0A^{-1}B,C)$ satisfy (3.2a) and both (3.2a) and (3.2b), respectively.

The proof of theorem 5 is dual to the proof of theorem 4. The latter is given in the Appendix.

IV A Numerical Illustration

A system is given whose transfer function matrix is

$$H(s)=\begin{bmatrix} \dfrac{2(s+5)}{(s+1)(s+10)} & \dfrac{s+4}{(s+2)(s+5)} \\[2ex] \dfrac{s+10}{(s+1)(s+20)} & \dfrac{s+6}{(s+2)(s+3)} \end{bmatrix}$$

This system can be easily verified to be of order $N=6$. Its eigenvalues are -20, -10, -5, -3, -2, -1. The simple way to obtain the first time moment or Markov matrices of $H(s)$ is to expand each of its four entries about $s=0$ or $s=\infty$, (2.1a,b), respectively. The first time moment matrices are

$$T_1=\begin{bmatrix} -1 & -0.4 \\ -0.5 & -1 \end{bmatrix}\; T_2=\begin{bmatrix} 0.9 & 0.18 \\ 0.475 & 0.667 \end{bmatrix}$$

$$T_3=\begin{bmatrix} -0.89 & -0.086 \\ -0.474 & -0.389 \end{bmatrix}\; T_4=\begin{bmatrix} 0.889 & 0.042 \\ 0.474 & 0.213 \end{bmatrix}$$

This $H(s)$ was used before to illustrate the multivariable Padé method of [9]. The two models that were derived in [9] to match $\{T_1,T_2,T_3,T_4\}$ and $\{M_2,M_1,T_1,T_2\}$ are both of order $n=8$. To see what is the minimal order of models that match 4 matrices of the sequence $\sigma(p,q)$, for any $p+q=4$, it is sufficient to investigate the incomplete Hankel matrix for $\sigma(4,0)$. The rank of $K(4,0)$ is $\rho K(4,0)=4$. Its first independent rows and columns are $I_n=J_n=\{1,2,3,4\}$. Therefore, the MPA approach can match any sequence of four matrices $\sigma(p,4-p)$, $0\le p\le 4$, by models of order $n=4$. The structural indices $I_n=J_n=\{1,2,3,4\}$ are common for all $0\le p\le 4$ and it yields $\nu=\mu=\{2,2\}$ and $\alpha+\beta=4$. Since $\alpha+\beta=p+q=4$ the MPA model is unique for any p.

The MPA model that matches $\sigma(4,0)=\{T_4,T_3,T_2,T_1\}$ is found to be

$$A_1=\begin{bmatrix} 0 & 0 & -10.43 & -1.415 \\ 0 & 0 & .4926 & -6.039 \\ 1 & 0 & -11.43 & -1.417 \\ 0 & 1 & .2711 & -5.018 \end{bmatrix}\quad B_1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_1=\begin{bmatrix} 2.026 & 1.064 & -12.64 & -4.380 \\ 8.193 & 1.029 & -4.365 & -.4209 \end{bmatrix}$$

The eigenvalues of A_1 are -10.381, -3.078, -1.994, -1.000. Mixed MPA models to yield better approximation of the transient response can equally be derived [33]. The model for $\sigma(3,1)=\{T_3, T_2, T_1, M_1\}$ is found to be stable. It has eigenvalues at -20.905, -3.261, -1.962, -0.995. The model that matches $\sigma(2,2)=\{T_2,T_1,M_1,M_2\}$ is another stable model with eigenvalues given by -18.680, -12.076, -3.641, -0.919.

The model that matches $\sigma(1,3)=\{T_1,M_1,M_2,M_3\}$ is as follows

$$A_2=\begin{bmatrix} 0 & 0 & 3.256 & -.0742 \\ 0 & 0 & 24.42 & -6.558 \\ 1 & 0 & -14.02 & -.0123 \\ 0 & 1 & 29.07 & -5.493 \end{bmatrix}\quad B_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_2=\begin{bmatrix} 2 & 1 & -12 & -3 \\ 1 & 1 & -11 & 1 \end{bmatrix}$$

The eigenvalues of A_2 are -13.96, -4.056, -1.703 and $\lambda=.2026$. Applying the method of theorem 4, a stable model $(\hat{A}_c, B_2, C_2)$ that approximates $H^t(-s)H(s)$ is obtained through the following steps. Eigenvector u that satisfies $A_2u=\lambda u$ is $u^t=(-.4947, .2261, -.0338, -.1329)$. The solution of the scalar ($\xi=1$) equation (3.6) for $J_1=\lambda$, and $U_1=u$ is $q=.00635$. Therefore by (3.5),

$$P_c=\begin{bmatrix} 38.54 & -17.61 & 2.635 & 10.36 \\ -17.61 & 8.050 & -1.204 & -4.733 \\ 2.635 & -1.204 & .1801 & .7079 \\ 10.36 & -4.733 & .7079 & 2.782 \end{bmatrix}$$

and by (3.4)

$$A_c=\begin{bmatrix} 4.131 & .9161 & -10.04 & -12.02 \\ -1.888 & -.4187 & 30.49 & -1.100 \\ 1.282 & .0626 & -14.93 & -.9188 \\ 1.110 & 1.246 & 25.50 & -8.702 \end{bmatrix}$$

A model that satisfies both (3.1a) and (3.1b) is (A_c, B_2, C_c) where

$$C_c=CA_2^{-1}A_c=CA_c^{-1}A_2=\begin{bmatrix} .2995 & -7.064 & -136.7 & 387.5 \\ 2.216 & 6.766 & 78.20 & -278.3 \end{bmatrix}$$

Figure 1 compares the unit step responses of all the input-output combinations for the model (A_1, B_1, C_1) and the system $H(s)$.

V Other Padé Approximations

Tether and Kalman were the first to note that minimal partial realization may be useful also for reduced order modelling [19], [20]. However, the literature on Padé methods for model reduction has not realized for many subsequent years the advantage of minimal partial realizations over the extensively treated Padé equations, continued fractions and some related equivalent techniques.

The SISO complete Padé approximation (PA) problem asks for a model of order n that matches $2n$ time moments and Markov parameters. The problem can be solved by a variety of equivalent techniques, that were unified in [5] into a single set of equations that is applicable for any p and q, where $p+q=2n$. The comparison of the formulation in [5] with the implication of section II for SISO systems yields the following relations between the two methods: The SISO complete (p,q) PA problem (of [5] or equivalents) has a solution if and only if the (p,q) MPA problem, $p+q=2n$, is characterized by $I_n = J_n = \{1,2, \ldots ,n\}$. For this case the unique model of the MPA solutions and the conventional complete (p,q) PA model coincide for any given p and q, $p+q=2n$. In any other case the complete PA techniques encounter singularities (the continued fraction expansion, for example, requires a division by a zero) and a solution is said not to exist. Therefore, the singularities in the conventional Padé techniques indicate the existence of models of order less than n that match the sequence $\sigma(p,q)$. In these complementary cases the MPA approach yields the (unique or the set of different) models of minimal order $(<n)$ that match the sequence $\sigma(p,q)$, $p+q=2n$.

The matrix continued fraction approach of [6]–[8] and the Matrix Padé equation method of [9] were suggested to extend corresponding SISO techniques to multivariable systems. They are essentially restricted to systems of an equal number of inputs and outputs ($m=l$, in spite of the effort in [8] to remove this limitation). Like the SISO versions, and even more often so, they may fail whenever a singular matrix has to be inverted. These methods can produce only models of orders n', with n' a multiple of $m=l$. A more severe difficulty is the aforementioned dimensional difficulty. A model intended to be of order n' will be in general of a higher order $n=n'm(=n'l)$. The reason for this phenomenon is that the degree of a transfer function matrix $G(s)$ that is composed of a minimal common denominator of degree n' and an $m \times l$ polynomial matrix may range between $n' \leq \delta G(s) \leq n' \cdot \min(m,l)$. Since $G(s)$ is the unconstrained result of an approximation, $\delta G(s) = n' \min(m,l)$ in the generical case. (A similar dimensional difficulty arises, for similar reasons, in the matrical versions for the aggregation-Padé and the Routh-Pade methods in [14] and [15], [16]). Therefore, even for $m=l$ systems and when the matrix continued fraction and Padé equations have a solution, this may not achieve significant reduction in order and sometimes may produce models of order that, unnoticeably, exceed the order N of the original system. These observations were made before in [33] and motivated there the need for a proper multivariable Pade method that incorporates minimality of order and removes the other limitations. It is noted that when $m,l << N$ the matricial Padé methods may produce reasonable low order models. However, the MPA method will always yield models of a lower order of the same accuracy or models of a similar order (if acceptable) that match a longer sequence of matrices.

Partial realization to achieve reduced order Padé models has been first applied in [10] and [11]. As presented, the methods are applicable only for cases characterized (using our terminology) by simple structures of the sets I_n and J_n and they do not deal with the problem of finding models of minimal order for a given sequence of a length r. It is interesting that although a minimal partial realization technique always circumvents the dimensional pitfall the solutions in [10] and [11] were suggested merely as an alternative technique to derive the former Padé models.

The paper of Hickin and Sinha [25] is, to the authors' best knowledge, the first published paper that appreciates and applies the advantages of partial realization to overcome limitations of former Padé methods for multivariable systems. In [25] and in a subsequent paper [26], stable multivariable Padé models of prescribed order have been proposed by first chosing a predetermined stability matrix A and then completely the state space presentation by using partial realization. In [25] the matrix A is chosen to retain eigenvalues of A_H whereas in [26] approximated Routh polynomials are fitted into a canonical structure of A. Thus, the methods in [25] and in [26] may be viewed as dimensionally proper multivariable extensions for the aggregation-Padé method of [14] and the Routh-Padé method of [15] and [16], respectively. The methods in [25] and [26] match about n/l or n/m matrices for a system with m inputs, l outputs and a model of order n. By comparison, the MPA approach proposed presently relates r, the maximal number of matched matrices to the subset of controllability and observability indices, that is common to all the mixed (p,q) $p+q=r$, reduced order models, and it can be shown from (2.7) that $r \geq n(m+l)/m \cdot l$. This is a significant improvement which, say for $m=l$, yields an improvement factor of accuracy of at least 2. The stable models in [25] and [26] require the exact knowledge of a canonical structure of the system matrix A_H (in [25] the eigenvalues of A_H are also needed). This is in general an acceptable requirement because often the 'original' system is merely another known model of dimension that is too high. However, there are other practical cases when the first few time moment and Markov matrices, as obtained from input-output measurements, are the only available data about a stable high order system. The MPA method can be applied equally when all that is known about a system is its stability and some first terms of time moment and Markov matrices. It always yields the set of all models of lowest order that fully use the available data.

It is possible to combine the proposed method with the nonminimal partial realization concept of [25] and [26] to obtain a generalization that provides some extra flexibility to the design of Padé models. Let the solution to the MPA problem of r matrices be models of minimal order n. Assume they are characterized by structural indices I_n and J_n and sets $\mathcal{G}$ and $\mathcal{P}$ of (2.7) and (2.8). Consider a case that $\mathcal{P}$ does not provide enough free parameters for some definite design specifications. It is possible then to add right-most specified entries $g_{ijk} \in \mathcal{G}$ to the set $\mathcal{P}$ and treat them as unspecified free and independent parameters of the model, [17]. Obviously, whenever the remaining number of matched parameters r' $(r' < r)$ admits a MPA whose order is $n' < n$, the resulting model is not minimal any more. Such models deserve the name nonminimal Padé approximations to distinguish them from MPA's. (They should also be distinguished from the aforementioned Padé techniques that may encounter the dimensional difficulty.) Nonminimal PA may be used as one more possibility to obtain stable models (e.g., when $\mathcal{P} = \phi$, and the unique MPA for a specific p and q is not stable). Applied as a stability guarantor, nonminimal PA generalize the stability approach in [25], and [26]. Stable models that match a sequence of matrices of length r' that is larger than that attained in there, may in general be expected. Nonminimal PA may also be used to meet other design specifications while minimizing the necessary decrease in the Padé sense of accuracy.

Conclusions

The present paper solves the approximation of linear multivariable systems in the Padé sense by minimal low order models. A unified approach is presented that for any given sequence $\sigma(p,q)$ of p time moment and q Markov matrices $p+q=r$, provides the set of all minimal models that match these sequences. A systematic procedure that resolves the stability problem is given. It provides a modification by which an unstable minimal model is replaced by a stable model of the same order that approximates the magnitude effect of the system. The importance of this approach is that in all cases it provides a stable model that approximates at least the singular values of the system transfer function matrix. Other possibilities to achieve stable models follow from the available choice among models that differently approximate the low and high frequencies and the freedom of assigning arbitrary values to the parameters of .

As a pure Padé model approximation, the method of this paper requires the knowledge of the input-output matrices only and it can therefore be used in identification problems. The derivation of the models is based on a well specified generically minimal, set of elements in the matrix sequence $\sigma(p,q)$ which forms a basis of invariants for the problem.

The common structural properties of the mixed (p,q) models can be used for significant computational saving in the derivation of various models of a same minimal order with different emphasis on the approximation of the low and the high frequencies. A recursive scheme for models of increasing order that match sequences of successive length may be developed using their nesting property [17]. The minimal order of the derived models guarantees that these models converge and reproduce the original system from any sequence of a large enough length r and any $0 \leq p \leq r$.

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APPENDIX

Proof of Theorem 4

It follows from (3.5) that

$$J_1 Q_c^{-1} + Q_c{}^{-1} J_1^{+} = Q_c{}^{-1} U_1^{+} C' C U_1 Q_c^{-1} \tag{A.1}$$

and hence that

$$J_1 - Q_c{}^{-1} U_1{}^{+} C' C U_1 = - Q_c{}^{-1} J_1{}^{+} Q_c \tag{A.2}$$

We find from (A.1) that

$$\mathrm{diag}(J_1, J_2)\bar{P} + \bar{P}\mathrm{diag}(J_1, J_2)^{+} = \bar{P} U_1^{+} C' C U_1 \bar{P}, \tag{A.3a}$$

where

$$\bar{P} = \mathrm{dig}(Q_c^{-1}, 0) \tag{A.3b}$$

and thus multiplying both sides of the last equation from the left and from the right by $[U_1, U_2]$ and $[U_1, U_2]^{+}$, respectively, we obtain

$$A P_c + P_c A' = P_c C' C P_c \tag{A.4a}$$

where

$$P_c = [U_1, U_2]\bar{P}[U_1, U_2]^{+} = U_1 Q_c{}^{-1} U_1^{+} \tag{A.4b}$$

It follows from (A.4a) that

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} [A - P_c C' C][U_1, U_2] =$$

$$\begin{bmatrix} J_1 - Q_C^{-1} U_1^{+} C' C U_1 & -Q_c^{-1} U_1^{+} C' C U_2 \\ 0 & J_2 \end{bmatrix} \tag{A.5}$$

and thus by (A.2) it is found that all the eigenvalues of A_c lie in the left half plane.

Applying the method of [30] it also follows from (A.4a), adding $-sP_c+sP_c$ to the left-hand side of the equation and multiplying the resulting equation by $C(sI-A)^{-1}$ and $(-sI-A^t)^{-1}C^t$ from the left and from the right, respectively, and rearranging the resulting equation, that

$$[I+C(sI-A)^{-1}P_cC^t][I+CP_c(-sI-A^t)^{-1}C^t]=I \quad (A.6)$$

Considering the stable transfer function matrix

$$\begin{aligned} G_c(s) &= C(sI-A_c)^{-1}B \\ &= C[I+(sI-A)^{-1}P_cC^tC]^{-1}(sI-A)^{-1}B \\ &= [I+C(sI-A)^{-1}P_cC^t]^{-1}G(s) \end{aligned} \quad (A.7)$$

In order to satisfy both (3.1a) and (3.1b) we observe from (A.6) that

$$[I-CA^{-1}P_cC^t][I-CP_cA^{-t}C^t]=I$$

and thus that $C^tC=C^t[I-CP_cA^{-t}C^t][I-CA^{-1}P_cC^t]C$

Equations (3.1a) and (3.1b) follows, respectively, from

$$CA^{-1}A_c=[I-CA^{-1}P_cC^t]C \quad (A.8a)$$

and

$$CA^{-1}A_c(sI-A_c)^{-1}B\Big|_{s=0}=CA^{-1}B=G(0) \quad (A.8b)$$