Zero Location with Respect to the Unit Circle of Discrete-Time Linear System Polynomials

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The location of the zeros of discrete systems characteristic polynomials with respect to the unit circle is investigated. A new sequence of symmetric polynomials of descending degrees is defined for the characteristic polynomial and the number of zeros inside and outside the unit circle is shown to be related to a certain sign variation pattern of the polynomials in the sequence. A stability table based on this sequence is presented to obtain the sought distribution of zeros. The new table is close in appearance, size, and number of arithmetic operations to the Routh table used for continuous-time system polynomials. By comparison with the table of Jury, based on the theory of Marden, Cohn, and Schur, the new table involves about half the number of entries and a corresponding significant saving in computations. The study includes a detailed consideration of all possible cases of singular conditions so that the complete information on the number of zeros inside, outside, and on the unit circle is always obtained (including some additional information on possible reciprocal pairs of zeros). Necessary and sufficient conditions for a polynomial to have all its zeros inside the unit circle are obtained as a special outcome. Other additional necessary conditions, that are useful to shorten procedures when the table is used only for testing the stability of the system, are also given.

I. Introduction

An important topic in the analysis and design of discrete time systems is the determination of the location of the zeros of its characteristic polynomial with respect to the unit circle. The characteristic polynomial may be found from the difference equations, the characteristic equation of the state-space matrix presentation, or from the least common denominator of all the minors of the transfer function matrix of a reachable and observable linear time-invariant system. The mathematical formulation of the problem is the following. Given a real polynomial

$$D(z) = d_0 + d_1 z + \cdots + d_n z^n = d_n \prod_{i=1}^{n} (z - z_i),$$
$$d_n > 0, \qquad D(1) \neq 0 \quad (1)$$

with known d_i, $i = 0, \cdots, n$, find the location of the (unknown) zeros z_i, $i = 1, \cdots, n$ with respect to the z-plane unit circle

$$C = \{ z | z = e^{j\psi}, \quad \psi \in [-\pi, \pi] \}. \quad (2)$$

Two assumptions are made in (1), not one of them is practically restrictive. If $d_n < 0$ then $-D(z)$ can be considered. If $D(1) = 0$ then zeros at $z = 1$ can first be removed and leave a lower degree polynomial for consideration. The convenience in the assumption $D(1) \neq 0$ will be clarified later (remark 4.1 in Section IV).

This paper presents a new method to determine the number of inside the unit circle (IUC), on the unit circle (UC), and outside the unit circle (OUC) zeros of $D(z)$. The polynomial $D(z)$ will be called stable if it is a possible characteristic polynomial of a stable discrete time system, namely, if all its zeros are IUC. In general $D(z)$ may have $\alpha_n \geq 0$ IUC zeros, $\gamma_n \geq 0$ OUC zeros, and, if $\alpha_n + \gamma_n < n$, additional $\beta_n = n - (\alpha_n + \gamma_n)$ UC zeros.

Well-known stability tests and methods to locate the zeros of polynomials with respect to the unit circle were derived by Schur [1], Cohn [2], Marden [3], Jury [4], and Astrom [5]. These methods are related to a certain sequence of polynomials of descending degrees that are defined for $D(z)$ from which the location of the zeros can be obtained. The implementation of the procedure is best carried out in a table form. Several variations for a table form have been proposed, including [3, p. 151], [4, p. 98], [5, p. 124], [8, the appendix], [9].

The present paper introduces a different approach that, based on the definition of a sequence of symmetric polynomials for $D(z)$, finds new conditions from which the numbers α_n and γ_n can be determined. The method culminates on a new stability table for discrete system polynomials. This table has many features in common with the Routh table, the more important among these being a comparable number of involved entries and arithmetic operations. The number of IUC or OUC zeros can be obtained from this table in all cases including possible singular cases that are all carefully treated. The new table exhibits additional information on possible pairs of reciprocal zeros and zeros on the unit circle.

The subject that is studied in the present paper is useful for the design of discrete systems with complicated control rules or with feedback or feedforward delays where it is required to adjust system parameters within stable ranges. For these purposes, the importance of the topic has not been diminished since its introduction to control theory decades ago, in spite of the presently available fast digital means to calculate the zeros explicitely. When using numerical methods to determine the zeros of a polynomial, the stability bounds on the possible range of parameters are not available and their derivation becomes a tedious root

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locus problem of difficulty that increases rapidly with the degree of the polynomial. It seems that the new stability table for discrete-time systems and the other results of the study in this paper and in [6] reduce, for the first time, the long-standing difference that has existed, both in size and computational effort, between the two acknowledged standard textbook methods to determine the number of "stable" and "unstable" zeros of discrete and continuous system polynomials, the (z-plane) Marden–Jury and the (s-plane) Routh tables, respectively.

The paper is constructed as follows. Section II defines for $D(z)$ the sequence of symmetric polynomials and establishes relations of the number of IUC and OUC zeros to this sequence for normal cases. The computational aspects of the method are considered in Section III where the method is implemented by a new stability table whose rows are formed by the coefficients of the symmetric polynomials. Section IV extends the method to all the possible cases of singularities. The singularities are classified into two types and a modification is provided for each type such that the table can always be completed and the number of IUC, UC, and OUC zeros found. In the last section, Section V, a comparison of the new table with the tables of Routh and Marden–Jury is presented. The comparison reveals a close similarity of the new table to the Routh table and consequently the advantages of the new table over the table of Marden–Jury. The presentation also contains subsections for the special case of stable polynomials (Section II-C) and the application of the new table for merely testing stability (Section III-B). Three numerical examples are included and illustrate the use of the table in the normal and in the first and second type singular cases.

II. The Number of Zeros Inside the Unit Circle

A. A Sequence of Symmetric Polynomials

The determination of the number of IUC, UC, and OUC zeros of $D(z)$ will be based on the properties of a special sequence of polynomials that will be associated with $D(z)$ (we shall, in fact, present and conveniently use two closely related sequences). We first define the new sequences formally and then establish their properties. Note that the other studies on the distribution of zeros mentioned above are also based on some associated sequences of polynomials. The relative simplicity of the method in this paper follows from a new choice of the underlying sequence of polynomials.

Let us denote by $D^*(z)$ the reciprocated polynomial of $D(z)$, namely,

$$D^*(z) = z^n D(z^{-1}) = d_n + d_{n-1}z + \cdots + d_0 z^n. \quad (3)$$

The zeros of $D^*(z)$ are the inverses z_i^{-1} of the zeros of $D(z)$. Thus (α, γ) IUC and OUC zeros of $D(z)$ are replaced by (γ, α) IUC and OUC zeros for $D^*(z)$. Also note that $D^*(z)$ is obtained from $D(z)$ simply by reversing the order of its coefficients.

A polynomial $S_k(z)$ of degree k is called a symmetric (or mirror) polynomial if

$$S_k^*(z) = S_k(z) \quad (4)$$

while $A_k(z)$ is called an antisymmetric (or antimirror) polynomial if

$$A_k^*(z) = -A_k(z). \quad (5)$$

Therefore, $S_k(z)$ is symmetric if and only if

$$S_k(z) = \sum_{i=0}^{k} s_i z^i, \qquad s_i = s_{k-i}, i = 0, 1, \cdots, k \quad (6)$$

and $A_k(z)$ is antisymmetric if and only if

$$A_k(z) = \sum_{i=0}^{k} a_i z^i, \qquad a_i = -a_{k-i}, i = 0, 1, \cdots, k. \quad (7)$$

Lemma 2.1: A real polynomial $D_k(z)$ is symmetric or antisymmetric if and only if it has only UC or reciprocal pairs of IUC and OUC zeros.

Proof: If z_i is a zero of a symmetric or an antisymmetric polynomial then z_i^{-1} is also its zero. Therefore, either z_i and z_i^{-1} are both UC zeros or they form a reciprocal pair of IUC and OUC zeros. To prove the converse, assume that a real polynomial $D_k(z)$ has only UC or reciprocal pairs of zeros. $D_k(z)$ may have i) a quadruple of zeros formed by two complex conjugate pairs of reciprocal zeros; ii) a real pair of reciprocal zeros; iii) a complex conjugate pair of UC zeros; iv) real zeros at $z = -1$; v) real zeros at $z = 1$. Each of cases i)–iv) contributes a symmetric factors while zeros at $z = 1$ of even or odd multiplicity contribute, respectively, a symmetric or an antisymmetric factor. It follows that $D_k(z)$ is either symmetric or antisymmetric.

An arbitrary real polynomial of degree k can always be written as the sum of a symmetric and an antisymmetric polynomial

$$D_k(z) = \frac{1}{2} S_k(z) + \frac{1}{2} A_k(z) \quad (8)$$

where

$$S_k(z) = D_k(z) + D_k^*(z) \quad (9)$$

and

$$A_k(z) = D_k(z) - D_k^*(z). \quad (10)$$

The polynomial $D_k(z)$ can, therefore, be further written in the form

$$D_k(z) = \frac{1}{2} S_k(z) + \frac{1}{2}(z-1) S_{k-1}(z) \quad (11)$$

since an antisymmetric polynomial $A_k(z)$ must have a zero at $z = 1$ (substitute $z = 1$ into (5)) and if this zero is divided out, the resulting polynomial, of degree $k - 1$, has to be symmetric (see Lemma 2.1).

Given the polynomial $D(z)$ of (1), we assign to it a sequence of $n + 1$ polynomial $T_n(z), T_{n-1}(z), \cdots, T_0(z)$ according to the following formal definition:

i)

$$T_n(z) = D(z) + D^*(z) \quad (12a)$$

$$T_{n-1}(z) = [D(z) - D^*(z)]/(z-1). \quad (12b)$$

Obviously, $T_n(z)$ and $T_{n-1}(z)$ form the two symmetric polynomials in a decomposition (11) of $D(z)$.

ii) The other polynomials are constructed from $T_n(z)$ and $T_{n-1}(z)$ by the recursion

$$T_{k-2}(z) = z^{-1}[\delta_k(z+1) T_{k-1}(z) - T_k(z)], \quad k = n, n-1, \cdots, 2 \quad (13a)$$

with

$$\delta_k = T_k(0)/T_{k-1}(0). \quad (13b)$$

The recursion requires the normal conditions

$$T_{n-i}(0) \neq 0, \qquad i = 1,2,\cdots,n. \tag{14}$$

The construction is interrupted when a $T_k(0) = 0$ occurs. Such singular cases will be discussed in detail later (Section IV). In the following we assume that the normal conditions of (14) are satisfied.

Next, we study some of the properties of the sequence that has just been defined. To start with, we denote for $k = n$ and $k = n - 1$

$$T_k(z) = \sum_{i=0}^{k} t_i^{(k)} z^i \tag{15}$$

and

$$Q_k(z) = \delta_k(z+1)T_{k-1}(z) - T_k(z) = \sum_{i=0}^{k} q_i^{(k)} z^i. \tag{16}$$

The polynomial $Q_k(z)$ is symmetric, for $k = n$, since $T_k(z)$ and $T_{k-1}(z)$ are symmetric for $k = n$. Substitution of $z = 0$ in (16) leads to

$$q_0^{(k)} = \left(t_0^{(k)}/t_0^{(k-1)}\right)t_0^{(k-1)} - t_0^{(k)} = 0 \tag{17}$$

which shows that, for $k = n$, $q_k^{(k)} = q_0^{(k)} = 0$. Therefore, $T_{k-2}(z)$ in (13a) is a symmetric polynomial of degree $k - 2$ (for $k = n$). Repeating the above argument for $k = n - 1$, $n - 2,\cdots,$ we obtain the following:

Lemma 2.2: The polynomials $T_n(z), T_{n-1}(z),\cdots,T_0(z)$ defined for $D(z)$ in (12)–(14) form a sequence of symmetric polynomials of descending degrees, where $T_k(z)$ is a polynomial of degree k, $k = 0,1,\cdots,n$. ($T_n(z)$ is of degree $n - 1$ iff $d_0 = -d_n$).

Remark 2.1: We do not require $T_n(0) \neq 0$ in the normal conditions of (14). It is possible to have $T_n(0) = 0$ and, consequently, $T_n(z)$ of actual degree $n - 1$ and still continue the construction of $T_k(z)$ for $k = n - 1, n - 2,\cdots,0$. This special case occurs when $D(z)$ has $d_0 = -d_n$. This case for $T_n(z)$ is also the only exception (for normal conditions) for a $T_k(z)$ not to be of full degree k. It is seen from (13) that a $T_n(z)$ of degree less than $n - 1$, having more than one zero at $z = 0$ (caused by $D(z)$ having also $d_1 = -d_{n-1}$ and so forth), implies the singular condition $T_{n-2}(0) = 0$ and the construction of the sequence breaks down. The last condition of (14), $T_0 = 0$ is not required for completing the sequence. However, $T_0 = 0$ implies a zero at $z = -1$ which we shall consider similarly to other cases of UC zeros as a singular case (of type I, see Theorem 4.2).

Once the sequence $\{T_k(z)\}_{k=0}^{n}$ is formally established we can associate with $D(z)$ the following second sequence $\{D_k(z)\}_{k=0}^{n}$ defined by $D_0(z) = \frac{1}{2}T_0(z)$ and

$$D_k(z) = \frac{1}{2}T_k(z) + \frac{1}{2}(z-1)T_{k-1}(z), \qquad k = 1,\cdots,n. \tag{18}$$

It is obvious from the definition of $\{T_k(z)\}$ and Lemma 2.2 that $D(z) = D_n(z)$ in (18) and that $T_k(z)$ and $T_{k-1}(z)$ play the roles of the two symmetric polynomials in the decomposition (11) of $D_k(z)$ for all $k < n$.

B. The Number of Zeros in Normal Conditions

So far we have associated with $D(z)$ two sequences, a sequence of symmetric polynomials $\{T_k(z)\}$, uniquely defined by (12), (13) if (14) holds, and a second sequence of polynomials $\{D_k(z)\}$ that is defined via $\{T_k(z)\}$ by (18). This subsection will show how the required information on the number of IUC and OUC zeros of $D(z)$ is directly obtained from these sequences. The following theorem is a basis for a subsequent principal theorem in this paper.

Theorem 2.1: Given the sequence $\{D_k(z)\}_{k=0}^{n}$, defined for $D(z)$ by (12), (13), and (18), if $D_k(z)$, $k < n$, has (α_k, γ_k) IUC and OUC zeros, $\alpha_k + \gamma_k = k$, then the zeros distribution of $D_{k+1}(z)$ with respect to the unit circle is given by

i)
$$(\alpha_k + 1, \gamma_k) \qquad \text{if } \operatorname{sgn} D_{k+1}(1) = \operatorname{sgn} D_k(1)$$

ii)
$$(\alpha_k, \gamma_k + 1) \qquad \text{if } \operatorname{sgn} D_{k+1}(1) = -\operatorname{sgn} D_k(1).$$

The proof for Theorem 2.1 is given in the Appendix. Let us denote by $\operatorname{Var}\{a_0,\cdots,a_n\}$ the number of sign changes of the sequence of real numbers $\{a_0,\cdots,a_n\}$. The next theorem is our main result.

Theorem 2.2: Given $D(z)$ and the sequence $\{D_k(z)\}_{k=0}^{n}$ or $\{T_k(z)\}_{k=0}^{n}$ the number of zeros of $D(z)$ inside the unit circle is given by $\alpha_n = n - \nu_n$ where

$$\nu_n = \operatorname{Var}\{D_n(1),\cdots,D_0(1)\} \tag{19}$$

or equivalently

$$\nu_n = \operatorname{Var}\{T_n(1),\cdots,T_0(1)\}. \tag{20}$$

The validity of (19) is easily deduced from Theorem 2.1, and setting $z = 1$ into (18) to obtain $D_k(1) = \frac{1}{2}T_k(1)$, $k = 0,\cdots,n$, shows the equivalence of (20) and (19). If $D(z)$ has no UC zeros then the number of OUC zeros is ν_n. Zeros on the unit circle will be shown in Section IV (Theorem 4.2) to imply singular conditions, therefore we have

Corollary 2.1: $D(z)$ has ν_n OUC zeros, no UC zeros, and $n - \nu_n$ IUC zeros whenever the construction of $\{D_k(z)\}$ and $\{T_k(z)\}$ obey the normal conditions of (14).

Remark 2.2: We shall extend in Section IV the method to also cover all kinds of possible singular cases. These extensions will present modifications for the sequences $\{D_k(z)\}$ and $\{T_k(z)\}$ for cases where a term $T_k(0) = 0$ is encountered. The modified sequence will always retain $n - \nu_n$, with ν_n given by (19) and (20), as the number of IUC zeros. Corollary 2.1 will also be shown to hold for a larger class that includes some singular cases as well, but, evidently, it cannot be true at the presence of UC zeros.

We consider next the interrelations between the two sequences $\{D_k(z)\}$ and $\{T_k(z)\}$. The sequence $\{D_k(z)\}$ plays an essential role in the proof of Theorem 2.1 on which Theorem 2.2 is based. It has also the common property with previous sequences that were used to study the distribution of zeros in [1]–[5] that the highest degree polynomial in the auxiliary sequence is the examined polynomial. The sequence $\{D_k(z)\}$ will be also used to prove the modifications required in singular cases. On the other hand, $\{D_k(z)\}$ is defined via $\{T_k(z)\}$ and Theorem 2.2 indicates that the number of IUC zeros of $D(z)$ can be equally determined from $\{T_k(z)\}$. A similar state is revealed also in the forthcoming discussion of singular cases; once a suggested modification has been justified by using the sequence $\{D_k(z)\}$, the same information on the position of the zeros can be obtained equally from $\{T_k(z)\}$. In fact, a question that arises is why to take in practice the additional effort to calculate also $\{D_k(z)\}$? We shall use $\{D_k(z)\}$ and $\{T_k(z)\}$ interchangeably to set up the results but we shall convert all the results into properties of the sequence $\{T_k(z)\}$. We tend to consider $\{T_k(z)\}$, rather than $\{D_k(z)\}$, as the main sequence in association with $D(z)$ and show, in the next section, an algorithm based on $\{T_k(z)\}_{k=0}^n$ that reduces the computational effort involved in the determination of the location of zeros with respect to the unit circle to a minimum never attained before.

C. Stable Polynomials

Stable polynomials, which have all their zeros inside the unit circle, form a special but very important group of polynomials $D(z)$. Some conclusions for a stable polynomial can already be deduced from the last two theorems. We shall return to this subject also in Section III after presenting a tabular algorithm.

Lemma 2.3 [4]: The following are three necessary conditions for a polynomial $D(z)$ given by (1) to be stable:

i) $d_n > |d_0|$
ii) $D(1) > 0$
iii) $(-1)^n D(-1) > 0$.

Proof: The proofs of conditions i)–iii) follow easily by substitution of $z = 0$, $z = 1$, and $z = -1$, respectively, into (1) and using the stability condition $|z_i| < 1$, $i = 1, \cdots, n$. If $d_n > 0$ is not assumed, the three conditions should be replaced by i) $|d_n| > |d_0|$, ii) $d_n D(1) > 0$, and iii) $d_n(-1)^n D(-1) > 0$.

Theorem 2.3: If $D(z)$ is stable then $D_i(z)$, $i = 1, \cdots, n-1$, are all stable.

Proof: Assume $D(z)$ is a stable polynomial given by (1), but that there exists an unstable $D_k(z)$, $k < n$. Let $D_k(z)$ have α_k IUC zeros where $\alpha_k < k$. The building-up pattern in Theorem 2.1 for the number of IUC zeros allows $D_n(z)$ to have at most $n - k$ additional IUC zeros. Namely, $\alpha_n = n - k + \alpha_k < n$ IUC zeros. This contradicts the assumption that $D(z)$ is stable.

Theorem 2.4: $D(z)$ is stable if and only if the normal conditions of (14) hold for the construction of $\{T_k(z)\}_{k=0}^n$ and ν_n of (20) is zero.

Proof: The only significant complementary statement here is that stable polynomials fall into the category of normal conditions. We have to show that $T_k(0) \neq 0$ for all k if $D(z)$ is stable. Assume that $D(z)$ is stable but a $T_{k-1}(z)$ with $T_{k-1}(0) = 0$ has occurred. We have from (18)

$$D_k(z) = \frac{1}{2}T_k(z) + \frac{1}{2}(z-1)T_{k-1}(z)$$
$$= d_0^{(k)} + d_1^{(k)}z + \cdots + d_k^{(k)}z^k$$

and using for $T_k(z)$ the notation of (15), we have that $2d_0^{(k)} = t_0^{(k)}$ and $2d_k^{(k)} = t_k^{(k)}$ because $T_{k-1}(0) = 0$ means $t_0^{(k-1)} = t_{k-1}^{(k-1)} = 0$. Therefore, $d_0^{(k)} = d_k^{(k)}$ and the necessary condition for stability of $D_k(z)$, $|d_k^{(k)}| > |d_0^{(k)}|$ is not satisfied. If $D_k(z)$ is not stable then by Theorem 2.3 $D(z)$ cannot be stable. Therefore, stable $D(z)$ implies normal conditions.

Remark 2.3: In many cases it is only required to determine whether a given polynomial is stable or not. The last theorem suggests an instructive piece of information for this purpose by stating that singular conditions already indicate an unstable polynomial. Additional useful necessary conditions for such cases will be brought in Section III-B.

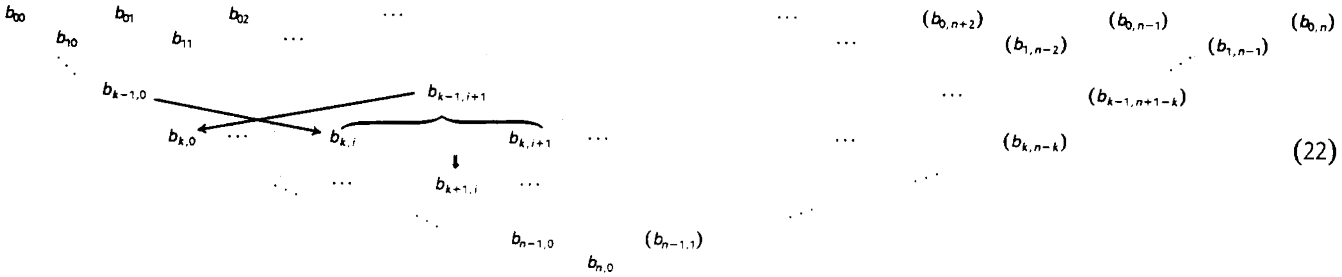
III. Computational Procedures

A. A Table Form

A demonstrative way to use Theorem 2.2 is to present the polynomials $\{T_k(z)\}_{k=0}^n$ in a tabular form. Let $\{T_k(z)\}$ be written in the following explicit form:

$$T_{n-k}(z) = \sum_{i=0}^{n-k} b_{k,i}z^i, \qquad k = 0, 1, \cdots, n \tag{21}$$

and construct an array of $n + 1$ rows where row k presents the coefficients of $T_{n-k}(z)$ in ascending (or descending, since $T_{n-k}(z)$ are symmetric) powers of z^i.



(22)

The first two rows correspond to $T_n(z)$ and $T_{n-1}(z)$

$$T_n(z) = b_{00} + b_{01}z + \cdots + b_{0,n}z^n \tag{23a}$$
$$T_{n-1}(z) = b_{10} + b_{11}z + \cdots + b_{1,n-1}z^{n-1} \tag{23b}$$

which were defined for a given $D(z)$ in (12). The subsequent rows present $T_{n-2}(z), T_{n-3}(z), \cdots, T_0(z) = b_{n,0}$. These polynomials were defined by the recursion (13), which will now be replaced by a procedure that is more adequate for the table form.

First we redenote the δ_k's of (15b) in reversed order

$$\tilde{\delta}_k = \delta_{n-k+1}, \qquad k = 1, \cdots, n \tag{24}$$

namely, $(\tilde{\delta}_1, \cdots, \tilde{\delta}_n) = (\delta_n, \cdots, \delta_1)$ and we have from (13b) that

$$\tilde{\delta}_k = b_{k-1,0}/b_{k,0}, \qquad k = 1, \cdots, n. \tag{25}$$

Next, we substitute (21) into (13a)

$$\sum_{i=0}^{n-k-1} b_{k+1,i}z^i$$
$$= z^{-1}\left[\tilde{\delta}_k(z+1)\sum_{i=0}^{n-k} b_{k,i}z^i - \sum_{i=0}^{n-k+1} b_{k-1,i}z^i\right]$$

and compare coefficients of similar power of z to obtain for $k = 1, \cdots, n-1$ and $i = 0, \cdots, n-k-1$

$$b_{k+1,i} = \tilde{\delta}_k(b_{k,i} + b_{k,i+1}) - b_{k-1,i+1}. \tag{26}$$

The last two equations, (25) and (26), form the sought procedure to construct the table (22) from its first two rows (23a) and (23b). These two equations can also be combined into a determinantal rule

$$b_{k+1,i} = \frac{1}{b_{k,0}}\begin{vmatrix} b_{k-1,0} & b_{k-1,i+1} \\ b_{k,0} & b_{k,i} + b_{k,i+1} \end{vmatrix}. \tag{27}$$

This form is schematically indicated by arrows in (22). It is noted, however, that the use of (25) and (26) involves less arithmetic operations than (27). The number of IUC and OUC zeros can be determined directly from the table by the number of sign changes

$$\nu_n = \mathrm{Var}\{\sigma_0,\cdots,\sigma_n\} \tag{28a}$$

where $\sigma_k = T_{n-k}(1)$ is simply the sum of entries in row k, i.e.,

$$\sigma_k = \sum_{i=0}^{n-k} b_{k,i}, \qquad k = 0,\cdots,n. \tag{28b}$$

The number of IUC zeros is $n - \nu_n$ and the number of OUC zeros is ν_n (Corollary 2.1).

Remark 3.1: The symmetry of $T_k(z)$ for all k imposes a similar symmetry on the table. The right half of each row is a mirror reflection of its left half. It is sufficient, therefore, to calculate only the left half of the table and to complete its right half by reflection. In fact, once enough familiarity with the method has been gained, the right half of the table can completely be dropped out. We are not adopting this approach in this exposition, we put however parenthesis on the entries of the right half of the table in (22) and in the coming numerical illustrations to remind of their redundancy.

Example 1: Consider the polynomial

$$D(z) = 1.5z^5 - 13.5z^4 + 28.5z^3 + 3.5z^2 - 4.5z + 0.5$$

we have from (12)

$$T_5(z) = D_5(z) + D_5^*(z) = 2z^5 - 18z^4 + 32z^3 + 32z^2 - 18z + 2$$

$$T_4(z) = [D_5(z) - D_5^*(z)]/(z-1) = (z^5 - 9z^4 + 25z^3 - 25z^2 + 9z - 1)/(z-1)$$

$$= z^4 - 8z^3 + 17z^2 - 8z + 1$$

and using (23) and (27) the following table is constructed:

2	−18	32	(32)	(−18)	(2)
1	−8	17	(−8)	(1)	
4	−14	(−14)	(4)		
5.5	−24	(5.5)			
6/11	(6/11)				
35					

Calculating the sums of the entries in each row, by (28),

$$\nu_5 = \mathrm{Var}\{32, 3, -20, -13, 12/11, 35\} = 2.$$

Therefore, $D(z)$ has 3 IUC zeros and (as the table is normal) 2 OUC zeros.

B. Stable Polynomials

Let us return to the special case of stable polynomials in order to obtain additional characterizations of this case for the table form. We first recall that the normal conditions are necessary for stability (Theorem 2.4). Necessary and sufficient conditions for stability are found by (20) and (28) to be the following equivalent conditions:

a) $\mathrm{Var}\{T_n(1),\cdots,T_0(1)\} = 0$ (29)

b) $\mathrm{Var}\{\sigma_0,\cdots,\sigma_n\} = 0$ (30)

c) $\sigma_i > 0, \quad i = 0,\cdots,n$ (assuming $D(z)$ with $d_n > 0$). (31)

The third form follows from (30) by noting that, for a polynomial $D(z)$ as in (1), $D(1) > 0$ is a necessary condition for stability and $\sigma_0 = T_n(1) = \frac{1}{2}D(1) > 0$.

It is instructive to have simple necessary conditions for stability which become obvious during the construction of the table and allow its premature interruption in cases when the table is used only to determine stability. Theorem 2.3 has already presented one such condition; namely, the normality condition of (14). Note that the normal conditions, $T_i(0) \neq 0$, $i = 0,\cdots,n$, are equivalent to the necessary conditions $b_{i,0} \neq 0$, $i = 0,\cdots,n$, in the table form. A still stronger necessary condition can however be shown.

Theorem 3.1: A necessary conditions for stability is that

$$\delta_i > 0, \qquad \text{for all } i = 1,\cdots,n \tag{32}$$

or equivalently (assuming $d_n > 0$), that all the first entries of the rows of the table are positive

$$b_{i,0} > 0, \qquad i = 0,\cdots,n. \tag{33}$$

Proof: Substituting $z = 1$ into (13a) to obtain

$$\delta_k = [T_k(1) + T_{k-2}(1)]/2T_{k-1}(1) > 0 \tag{34}$$

because all $T_k(1)$, $k = 0,\cdots,n$, must have the same sign if $D(z)$ is stable. If in addition $D(z)$ is given by (1), i.e., if $d_n > 0$ then $d_n > |d_0|$ being a necessary condition for a stable polynomial implies also that $b_{00} = d_n + d_0 > 0$. Therefore, by (25), the conditions $\delta_i > 0$ and $b_{i,0} > 0$, $i = 1,\cdots,n$, are the same.

Remark 3.2: The necessary conditions of Theorem 3.1 are not sufficient for stability. The polynomial in Example 1 that satisfies these conditions but is not stable is an appropriate counterexample.

Remark 3.3: An obvious sufficient condition for stability is provided by the case where all the entries in the table have the same (positive) sign. However, interior negative entries in a table do not exclude stability.

Remark 3.4: The "visual" necessary condition (33) (assuming $d_n > 0$) and $\sigma_{2i} > 0$, $i = 0,1,\cdots,[n/2]$ (half of the conditions (31)) are also sufficient for stability. This is so because $\delta_i > 0$ and $\sigma_{2i} > 0$ implies by (34) also $\sigma_{2i+1} > 0$, $i = 0,1,\cdots,[n/2]$. Therefore if (33) is found to be true it is sufficient for stability to evaluate and verify the signs of half of the σ_i's. (A result similar to this has been obtained for the table form in [6] from different considerations.)

Remark 3.5: If $D(z)$ is stable and $d_n > 0$ then all polynomials $D_k(z)$, $k < n$, are also stable (Theorem 2.3) and have a leading positive coefficient $d_k^{(k)} > 0$ because, from

(18), $d_k^{(k)} = \frac{1}{2}b_{k,k} + \frac{1}{2}b_{k-1,k-1} = \frac{1}{2}b_{k,0} + \frac{1}{2}b_{k-1,0} > 0$ using the property of symmetry and (33).

IV. Zeros on the Unit Circle and Other Singularities

A. Classification of Singularities

So far we assumed normal conditions for the stability table. In this section the method is extended to deal also with possible singular cases when a $T_k(0) = 0$ does appear. We already know from Theorem 2.4 that singular cases always indicate unstable polynomials. However, unstable polynomials, as the numerical Example 1 shows, do not necessarily lead to singular conditions in the construction of the table. In a complementary manner to the class of stable polynomials, that implies normal conditions, we shall soon identify some special patterns of zeros distribution that imply a singular condition (and of a special form).

We shall distinguish between two types of singularities and refer to them as the "first type" or "type I" and the "second type" or "type II." The first type of singularity considers the case where $T_k(0) = 0$ due to $T_k(z)$, that is, identically zero. In the table form, type I singularity is featured by the appearance of a row whose entries are all zeros. The singularity of the second type considers the case where $T_k(0) = 0$ but $T_k(z) \neq 0$ identically. In the table form, this second case is characterized by a row that has some first zero entries (and, consequently, some last zero entries as well) but is not completely vanishing. We shall consider separately these two types of singularity and show for each type how the table can be completed to yield the sought numbers of IUC, UC, and OUC zeros.

B. Type I Singularity

Let us first reveal a close relation between the first type singularity and zeros of $D(z)$ which are on the unit circle or appear in reciprocal pairs. Denote by $z_c \neq 1$ a possible UC zero and let z_r and z_r^{-1} denote a possible pair of reciprocal zeros. Let $k > 0$ denote the smallest integers for which $T_n(z),\cdots,T_k(z)$ are all normal, namely, $T_{k+i}(0) \neq 0$, $i = 0,\cdots,n-k$ but $T_{k-1}(0) = 0$. Let $D_n(z),\cdots,D_k(z)$ be the corresponding polynomials of (18). A first characterization of the case is given in the next theorem.

Theorem 4.1: i) If z_c or z_r and z_r^{-1} are zeros of $D(z) = D_n(z)$ then they are also zeros of $D_{n-i}(z)$, $i = 1,\cdots,n-k$. ii) If z_c or z_r and z_r^{-1} are zeros of $D_m(z)$, $k \leqslant m < n$ then they are also zeros of $D_{m+i}(z)$, $i = 1,\cdots,n-m$.

Proof: Let $D_m(z)$ be some intermediate polynomial in the nonsingular partial sequence, $k < m < n$. If z_c or z_r and z_z^{-1} are zeros of $D_m(z)$ then they are also zeros of $D_m^*(z)$. Therefore, they are zeros of $T_m(z)$ and $T_{m-1}(z)$, because from (18),

$$T_m(z) = D_m(z) + D_m^*(z) \tag{35}$$

and

$$T_{m-1}(z) = \left[D_m(z) - D_m^*(z)\right]/(z-1). \tag{36}$$

The recursion (13a) and its reversed form

$$T_m(z) = \delta_m(z+1)T_{m-1}(z) - zT_{m-2}(z) \tag{37}$$

transfer these zeros downward to $T_{k-1}(z)$ and upward to $T_n(z)$, respectively ($T_{k-1}(z)$ will soon be shown to be identically zero). Consequently, by (18), all the corresponding polynomials $D_n(z),\cdots,D_k(z)$ have zeros at z_c or z_r and z_r^{-1}.

Remark 4.1: The assumption of $z_c \neq 1$ is convenient but not necessary for the development of the theory. It is not restrictive because zeros at $z_c = 1$ are identified immediately by $T_n(1) = 0$ and $D(1) = 0$ and can be eliminated prior to the usage of the table. The cost of the involved arithmetic (additive arithmetic only) is compensated by the correspondingly shortened table. If $D(z)$ has no zeros at $z = 1$, or after these zeros are divided out, all the elements in the sign rules (19) or (20) are nonzero, not only for the normal conditions discussed thus far, but also for the extensions of the method to the singular cases of this section. (This remark clarifies the assumption in (1) that $D(1) \neq 0$.)

The next theorem establishes an "if and only if" relation between UC or reciprocal pairs of zeros and the type I singularity.

Theorem 4.2: If $\Phi(z)$ is a factor of $D(z)$ of degree k that contains all its UC zeros and reciprocal pairs of zeros then (up to a possible constant) $T_k(z) = \Phi(z)$ and $T_{k-1}(z) = 0$ (assuming, for the moment, no earlier type II singularity, see Remark 4.2 below). Conversely, if $T_k(z) = \Phi(z)$ and $T_{k-1}(z) = 0$ then $\Phi(z)$ is a factor of $D(z)$ that contains all its UC and reciprocal pairs of zeros.

Remark 4.2: Theorem 4.2 will be extended also for the case where an earlier type II singularity does occur. At this stage, however, if a $T_r(z) \neq 0$ with $T_r(0) = 0$ occur and $r > k$ then $T_{r-1}(z),\cdots,T_k(z)$ are not yet defined.

Proof of Theorem 4.2: If $\Phi(z)$ is the factor that contains all the UC and the reciprocal pairs of zeros of $D(z)$ then it is a symmetric polynomial (assuming no zero at $z = 1$). Applying Theorem 4.1, $\Phi(z)$ is also a factor of all $D_{n-i}(z)$ and $T_{n-i}(z)$, $i = 1,2,\cdots$, for which $T_{n-i}(0) \neq 0$. Therefore, it must be that $T_k(z) = K_1\Phi(z)$, for some constant K_1, and $T_{k-1}(0) = 0$ or else, one has from Theorem 4.1 the contradiction that the polynomial of degree k, $\Phi(z)$, is also a factor of $D_{k-1}(z)$. Since $\Phi(z)$ has to be a factor of $D_k(z)$ these two polynomials may differ only by a constant K_2, namely $D_k(z) = K_2\Phi(z)$. Then, by (21),

$$D_k(z) = K_2\Phi(z) = \frac{1}{2}T_k(z) + \frac{1}{2}(z-1)T_{k-1}(z)$$
$$= \frac{1}{2}K_1\Phi(z) + \frac{1}{2}(z-1)T_{k-1}(z).$$

Setting $z = 0$ yields $K_2 = \frac{1}{2}K_1$ because $T_{k-1}(0) = 0$ and $\Phi(0) \neq 0$. Substituting back $\frac{1}{2}K_1 = K_2$ implies $T_{k-1}(z) = 0$. Therefore, the singularity is of type I. To prove the second part, $T_k(z) = \Phi(z)$ and $T_{k-1}(z) = 0$ imply $D_k(z) = \frac{1}{2}\Phi(z)$. As $\Phi(z)$ is symmetric it can have only UC or reciprocal pairs of zeros. Then by the second part of Theorem 4.1 such zeros of $D_k(z)$ are also zeros of $D(z) = D_n(z)$. Therefore, $\Phi(z)$ is a factor of $D(z)$. $D(z)$ cannot have zeros z_c or z_r and z_r^{-1} that are not included in $\Phi(z)$ since this would cause an earlier ocurrance of the type I singularity (by the first part of this theorem).

The Procedure for a Type I Singularity: If $T_n(z),\cdots,T_k(z)$ is a partial normal sequence and the next polynomial is $T_{k-1}(z) = 0$, the partial sequence should be continued with $\hat{T}_{k-1}(z),\cdots,\hat{T}_0(z)$ where $\{\hat{T}_i(z)\}_{i=0}^{k-1}$ is the sequence that corresponds to the polynomial

$$D_{k-1}(z) = -P_{k-1}^*(z) \tag{38}$$

by the usual assignment (12), and where $P_{k-1}(z)$ is the derivative of $T_k(z)$ with respect to z, i.e.,

$$P_{k-1}(z) = T_k'(z). \tag{39}$$

In other words, set the next two rows of the table to be the coefficients of

$$\hat{T}_{k-1}(z) = -P_{k-1}^*(z) - P_{k-1}(z) \tag{40a}$$

and

$$\hat{T}_{k-2}(z) = \left[-P_{k-1}^*(z) + P_{k-1}(z)\right]/(z-1) \tag{40b}$$

respectively, and resume the recursion.

$D(z)$ is given by the sum

$$2\nu_k - k + \mathrm{Var}\{T_n(1), \cdots, T_k(1)\} + k - \nu_k$$

or

$$\nu_n = \mathrm{Var}\{T_n(1), \cdots, T_k(1)\} + \nu_k$$

which is equal to (41).

Example 2: Consider the polynomial

$$D(z) = z^5 + 1.8z^4 - 0.35z^3 + 0.8z^2 + 1.65z + 0.5.$$

The first two rows are by (12) and (23)

1.5		3.45		0.45		(0.45)		(3.45)		(1.5)
	0.5		0.65		−0.5		(0.65)		(0.5)	
		0		0		(0)		(0)		

Theorem 4.3: The number of IUC zeros of $D(z)$ with the above type I singularity, occurring right after $T_k(z)$ is $\alpha_n = n - \nu_n$ where

$$\nu_n = \mathrm{Var}\{T_n(1), \cdots, T_k(1), \hat{T}_{k-1}(1), \cdots, \hat{T}_0(1)\}. \tag{41}$$

The number of UC zeros is $\beta_n = 2\nu_k - k$ where

$$\nu_k = \mathrm{Var}\{T_k(1), \hat{T}_{k-1}(1), \cdots, \hat{T}_0(1)\} \tag{42}$$

and the number of reciprocal pairs is $k - \nu_k$.

Proof: Assume that $T_k(z) = 2D_k(z)$ has α_k IUC zeros then it can be deduced from a theorem by Cohn [2] ([3, p. 159], [4, p. 134], [7], [9]) that $P_{k-1}^*(z)$ also has α_k IUC zeros.

They lead, using (27), to a first type singularity in the third row. To apply the modification for type I singularity, we note that in this case $n = 5$, $k = 4$ and

$$T_4(z) = 0.5z^4 + 0.65z^3 - 0.5z^2 + 0.65z + 0.5$$

$$P_3(z) = T_4'(z) = 2z^3 + 1.95z^2 - z + 0.65.$$

The table has to be continued by the rows of the table that corresponds to

$$D_3(z) = -P_3^*(z) = -0.65z^3 + z^2 - 1.95z - 2.$$

The complete table is therefore

1.5		3.45		0.45		(0.45)		(3.45)		(−1.5)
	0.5		0.65		−0.5		(0.65)		(0.5)	
		−2.65		−0.95		(−0.95)		(−2.65)		
			1.35		4.3		(1.35)			
				−10.14		(−10.14)				
					−4.6					

Since the number of OUC zeros of $P_{k-1}^*(z)$ is

$$\nu_{k-1} = \mathrm{Var}\{\hat{T}_{k-1}(1), \cdots, \hat{T}_0(1)\}$$

the polynomial $T_k(z)$ must have $\nu_{k-1} + 1$ OUC zeros. This number will be given by ν_k of (42) if we show that $\mathrm{sgn}\, T_k(1) = -\mathrm{sgn}\, \hat{T}_{k-1}(1)$. We take the derivatives of the two sides of $T_k(z) = z^k T_k(z^{-1})$ and substitute in the resulting equality $z = 1$, to obtain

$$2T_k'(1) = kT_k(1).$$

Then the application of (40a)

$$\hat{T}_{k-1}(1) = -2T_k'(1) = -kT_k(1)$$

shows that, $\hat{T}_{k-1}(1)$ and $T_k(1)$ have oposite signs. Since the number of IUC zeros of $P_{k-1}^*(z)$ is $\alpha_k = k - \nu_k$, where $\nu_k = \nu_{k-1} + 1$ (and is also given by (42)), α_k is also the number of IUC zeros of $T_k(z)$. Since $T_k(z)$ has either a UC or a reciprocal pair of zeros, it has α_k reciprocal pairs of IUC and OUC zeros and $k - 2\alpha_k = 2\nu_k - k$ UC zeros. Since, by Theorem 4.2, $T_k(z)$ is the factor of all such zeros of $D(z)$, $D(z)$ has $2\nu_k - k$ UC zeros and $k - \nu_k$ reciprocal pairs of zeros. Finally, the total number of UC and OUC zeros of

Using Theorem 4.3 we conclude that the polynomial has $\alpha_5 = 5 - \nu_5$ IUC zeros where

$$(n = 5) \quad \nu_5 = \mathrm{Var}\{10.8, 1.8, -7.2, 7, -20.28, -4.6\} = 3$$

i.e., $\alpha_5 = 2$ IUC zeros. The number of UC zeros $\beta_5 = 2\nu_4 - 4$, is found from

$$(k = 4) \quad \nu_4 = \mathrm{Var}\{1.8, -7.2, 7, -20.28, -4.6\} = 3$$

to be $\beta_5 = 2$. The number of reciprocal pairs of zeros is $4 - \nu_4 = 1$. The zeros of $D(z)$ are at

$$0.5, \quad 0.5, \quad 0.6 \pm j0.8, \quad \text{and } 2$$

which verify our findings. The example also illustrates Theorem 4.2. The factor formed by the collection of UC and reciprocal pair of zeros is

$$\Phi(z) = (z - 0.5)(z - 2)\left[(z - 0.6)^2 + 0.64\right]$$
$$= z^4 + 1.3z^3 - z^2 + 1.3z + 1.$$

It has degree 4 and it causes a first type singularity right after the row for $T_4(z)$. The above factor is also equal to $2T_4(z)$, again in agreement with Theorem 4.2.

Remark 4.3: Type I singularities will occur more than once in a table if (and only if) $D(z)$ has z_c or z_r, z_r^{-1} pairs of zeros of multiplicity higher than one. The number of occurrances is equal to the highest multiplicity of a z_c or a z_r, z_r^{-1} pair among the UC and reciprocal pairs of zeros of $D(z)$. The above follows from the fact that UC or reciprocal pairs of zeros of multiplicity higher than one are retained as such zeros also by $D_{k-1}(z) = [-T_k'(z)]^*$ with multiplicity lowered by one. Obviously, Theorem 4.3 still applies both for the table as a whole as well as for each subtable defined at each interruption. Furthermore, one may superimpose conclusions from Theorems 4.1–4.3 for each subtable and obtain additional information on the multiplicities of the UC and reciprocal pairs of zeros of $D(z)$ (or, occasionally even identify them).

C. *Type II Singularities*

In difference from the first type singularity, the appearance of a second type singularity is not specific to a special pattern of zeros position, except that it implies an unstable polynomial. A second type singularity was defined as a case when $T_r(0) \neq 0$, $T_{r-1}(0) = 0$, but $T_{r-1}(z) \neq 0$ identically. In table form the situation is as follows, letting $k = n - r$:

$$\begin{array}{lll} T_r(z): & b_{k,0} b_{k,1} \cdots & \cdots b_{k,1}, b_{k,0} \\ T_{r-1}(z): & \underbrace{0 \cdots 0}_{q}, b_{k+1,q} \cdots & \cdots b_{k+1,q}, \underbrace{0 \cdots 0}_{q} \end{array} \tag{43a}$$

where $b_{k,0} \neq 0$, and q is the number of first (and, consequently, last) vanishing entries in the singular row

$$\begin{aligned} b_{k+1,i} &= 0, \qquad i = 0, \cdots, q-1 \\ b_{k+1,q} &\neq 0, \qquad 2q < r. \end{aligned} \tag{43b}$$

The Procedure for a Type II Singularity: Replace the rows that correspond to $T_r(z)$ and $T_{r-1}(z)$ by rows that correspond to the following modified polynomials, respectively:

$$\hat{T}_r(z) = T_r(z) + (z-1)T_{r-1}(z)[z^q - z^{-q}] \tag{44a}$$

$$\hat{T}_{r-1}(z) = T_{r-1}(z)[K + z^q + z^{-q}], \qquad K > 2 \tag{44b}$$

where K is an arbitrary (> 2) real constant. Observe that in the table context z^q and z^{-q} represent simply shift of entries q positions to the right and to the left, respectively. Note also that the modified polynomials $\hat{T}_r(z)$ and $\hat{T}_{r-1}(z)$ are symmetric and of the claimed degree and that the second substituted row is regular because

$$\hat{T}_{r-1}(0) = b_{k+1,q} \neq 0. \tag{45}$$

The table can, therefore, be continued with the two modified rows. If it is completed by polynomials $\hat{T}_r(z), \hat{T}_{r-1}(z), \cdots, \hat{T}_0(z)$ then according to the next theorem the number of IUC zeros of $D(z)$ is given by $n - \nu_n$ where

$$\nu_n = \mathrm{Var}\{T_n(1), \cdots, \hat{T}_r(1), \hat{T}_{r-1}(1), \cdots, \hat{T}_0(1)\} \tag{46}$$

because the modified first two rows retain the signs at $z = 1$ of the two rows they replace

$$\hat{T}_r(1) = T_r(1) \quad \text{and} \quad \operatorname{sgn}\hat{T}_{r-1}(1) = \operatorname{sgn} T_{r-1}(1). \tag{47}$$

Theorem 4.4: i) The modified polynomial

$$\hat{D}_r(z) = \tfrac{1}{2}\hat{T}_r(z) + \tfrac{1}{2}(z-1)\hat{T}_{r-1}(z) \tag{48}$$

has as many IUC zeros as

$$D_r(z) = \tfrac{1}{2}T_r(z) + \tfrac{1}{2}(z-1)T_{r-1}(z). \tag{49}$$

ii) $D_r(z)$ and $\hat{D}_r(z)$ have the same UC zeros and the same pairs of reciprocal zeros (if any).

Proof: Let $K > 2$ in (44b) be written as $K = 2 + 2\epsilon$, $\epsilon > 0$. We substitute into (48) the expressions (44) for $\hat{T}_r(z)$ and $\hat{T}_{r-1}(z)$ and obtain

$$\begin{aligned} \hat{D}_r(z) &= D_r(z) + (z-1)T_{r-1}(z)[0.5 + \epsilon + z^q] \\ &= D_r(z)[1.5 + \epsilon + z^q] - D_r^*(z)[0.5 + \epsilon + z^q]. \end{aligned}$$

In the second equality we used $D_r(z) - D_r^*(z) = (z-1) \cdot T_{r-1}(z)$. Since $\epsilon > 0$, the polynomial $P_{r+q}(z) = D_r(z)[1.5 + \epsilon + z^q]$ has as many IUC zeros as $D_r(z)$. Consider the quotient

$$\frac{\hat{D}_r(z)}{P_{r+q}(z)} = 1 - \frac{1 + 2\epsilon + 2z^q}{3 + 2\epsilon + 2z^q} \cdot \frac{D_r^*(z)}{D_r(z)}.$$

Since for any z on the unit circle

$$\left|\frac{D_r^*(z)}{D_r(z)}\right| = 1 \quad \text{and} \quad \left|\frac{1 + 2\epsilon + 2z^q}{3 + 2\epsilon + 2z^q}\right| < 1 \qquad (\text{for } \epsilon > 0)$$

we find that

$$\frac{\hat{D}_r(z)}{P_{r+q}(z)} > 0, \qquad \forall z \in C.$$

It thus follows from the argument theorem that $\hat{D}_r(z)$ has as many IUC zero as $P_{r+q}(z)$, which has as many IUC zeros as $D_r(z)$. If $D_r(z)$ has any UC or reciprocal pairs of zero these must be zeros of both $T_r(z)$ and $T_{r-1}(z)$ and therefore they are zeros of $\hat{D}_r(z)$ as well. This proves part ii) and also shows that the number of IUC zeros is $n - \nu_n$ where ν_n is given by (46).

Remark 4.4: The merit of the preservation property of the UC and reciprocal pairs of zeros by the suggested type II modification as expressed in part ii) of Theorem 4.4, is most significant in an additional way. It means that a type II modification does not interfere with any features of a later or earlier occurrence of a type I singularity. Assume that $D(z)$ has UC or reciprocal pairs of zeros in a total number of k. If a second type of singularity occurs in the first $n - k$ rows (once or several times), and is treated (each time) by the modification of (44) then the validity of all the features and conclusions of the previous subsections with regard to the first type of singularity are guaranteed by part ii) of Theorem 4.4. The UC and reciprocal pairs of zeros still produce a row of zeros right after the polynomial $\hat{T}_k(z)$, now the result of one or more type II modifications, and the type I modification may be applied to continue the table. This polynomial $\hat{T}_k(z)$ is still, as stated in Theorem 4.2, the factor of all UC and reciprocal pairs of zeros of $D(z)$ and the number of these zeros is given by Theorem 4.3. Note that this remark removes the temporary restriction imposed on Theorem 4.2 (see Remark 4.2 there).

Example 3: Consider the polynomial

$$D(z) = 6z^4 + 5z^3 + 8z^2 + 7z + 2.$$

The first three rows of the table are

R_1	8		12		16		(12)		(8)	
R_2		4		2		(2)		(4)		
R_3			0		-8		(0)			

The third row is singular and $q = 1$.

Applying the type II modification of (44) with say $K = 2.5$ (we use this time a self-explanatory notation for direct operations on rows instead of using the $T_k(z)$ polynomials).

$$R_2 \to R_2 + z^1(0,8,-8,0) - z^{-1}(0,8,-8,0)$$
$$=(4,2,2,4)+(0,0,8,-8)-(8,-8,0,0)$$
$$=(-4,10,10,-4)$$
$$R_3 \to 2.5(0,-8,0)+(0,0,-8)+(-8,0,0)$$
$$=(-8,-20,-8).$$

The modified resulting table is

$$\begin{array}{ccccccccc} 8 & & 12 & & 16 & & (12) & & (8) \\ & -4 & & 10 & & (10) & & (-4) & \\ & & -8 & & -20 & & (-8) & & \\ & & & 4 & & (4) & & & \\ & & & & 4 & & & & \end{array}$$

$$\nu_4 = \mathrm{Var}\{56, 12, -36, 8, 4\} = 2.$$

Therefore, $D(z)$ has 2 IUC zeros and (as no first type singularity is apparent) 2 OUC zeros.

D. A Unifying Summary of the Approach for Normal and Singular Cases

We end the presentation of the method of this paper by unifying and combining our main findings. Allow $T_n(z), T_{n-1}(z), \cdots, T_0(z)$ to present a complete sequence of both normal and modified polynomials. That is, a $T_k(z)$ may be either a normal polynomial or the result or successor of second or first type modifications

We have shown the following:

1) The number of IUC zeros is always given by $\alpha_n = n - \nu_n$ where

$$\nu_n = \mathrm{Var}\{T_n(1), \cdots, T_0(1)\}. \tag{50}$$

In other words, Theorem 2.2 holds for normal, type I, and type II singular conditions.

2) If a type I singularity is not apparent then there are no UC zeros and consequently the number of OUC zeros is ν_n (or Corollary 2.1 holds for normal and type II singular conditions).

3) If a type I singularity occurs because of $T_{k-1}(z) = 0$, prior to substitution, then the number of UC zeros is $\beta_n = 2\nu_k - k \geqslant 0$, where

$$\nu_k = \mathrm{Var}\{T_k(1), \cdots, T_0(1)\} \tag{51}$$

and, consequently, the number of OUC zeros for this case is $\gamma_n = \nu_n - \beta_n$ (Theorem 4.3).

4) Normal conditions and $\nu_n = 0$ are necessary and sufficient for stable polynomials (Theorem 2.4).

The possible combinations are conveniently summarized also in Table 1.

Table 1 A summary for Normal and Singular Cases (ν_n and ν_k are given by (50) and (51))

No. of Zeros	Normal	Normal and Type II	Normal or Type II and Type I
IUC	$n - \nu_n \le n$	$n - \nu_n < n$	$n - \nu_n < n$
UC	none	none	$\beta_n = 2\nu_k - k \ge 0$
OUC	$\nu_n \ge 0$	$\nu_n > 0$	$\nu_n - \beta_n \ge 0$

V. Comparison with Other Stability Tables

This paper presents a new method to determine the distribution with respect to the unit circle of the zeros of a discrete system polynomial $D(z)$. The distribution is efficiently obtained by applying the new stability table. A worthwhile concluding discussion would be a comparison of the new table with the former table of Marden and Jury and with the table of Routh that is designed to determine the distribution of zeros of a continuous-time system polynomial $H(s)$ in the left and the right halves of the s-plane. The comparison with the table of Routh is done first and it exhibits a remarkable formal similarity as well as analogous interpretations. The indicated similarity with the Routh table also makes the new table easy to remember. The subsequent comparison with the Marden–Jury table emphasizes, among other advantages, the significant computational saving provided by the new table.

A. Comparison with the Routh Table

We assume the familiarity with the celebrated Routh table and present the following itemized comparison.

i) *Structure:* The new table has for $D(z)$ the size and form of the Routh table for a polynomial $H(s)$ of a same degree n. The two tables have $n + 1$ rows and the same number of entries in respective rows (counting for the new symmetric table only its left half entries). This results in the same total number of entries to be calculated. The first two rows on the new table are formed by the coefficients, in ascending powers of z, of $\frac{1}{2}[D(z) + D^*(z)]$ and $\frac{1}{2}[D(z) - D^*(z)]/(z - 1)$ (where the $\frac{1}{2}$ factor is added to emphasize the similarity). The first two rows in the Routh table are formed by the coefficients, in ascending powers of $x = s^2$, of $\frac{1}{2}[H(s) + H(-s)]$ and $\frac{1}{2}[H(s) - H(-s)]/s$, a self-evident analogy is apparent. The next rows in the new table are obtained by the determinant rule (27) which is slightly different from the determinant rule for the Routh table in having $b_{k,i} + b_{k,i+1}$ instead of an only $b_{k,i}$ term.

ii) *Arithmetics:* The construction of the two tables requires the calculation of the same number of entries. The involved number of elementary multiplicative operations is exactly equal. The number of elementary additive operation is higher by one operation per each entry in the new table (due to the above mentioned difference and the setup of the first two rows).

iii) *Analysis:* The number of "unstable" zeros is determined by the number of sign changes of a sequence of $n + 1$ numbers; given by (28) in the new table and by the entries in the first column (nonsingular situations in both tables is tacitly assumed although the comparison can be extended to appropriate modifications, etc.).

iv) *Singularities:* The meaning and the treatment of the two types of singularities in the new table can be shown to be in close parallelism with the Routh table. This is so both for the interpretation of the first singularity and its treatment by differentiation and for the removal of the second type singularity by row shift operations.

v) *Application as Stability Test Tables:* For both tables no sign variation indicates stability and a singularity indicates instability. Some similar "shortcuts" to indicate instability or stability may also be indicated. A negative first entry implies instability in both tables while positivity of all entries is sufficient for stability. However, a warning against a wrong impression of a complete similarity should be given. After all, the two tables do not function identically. (For example, a negative entry anywhere in the Routh table implies instability whereas an inner negative entry in the new table does not exclude stability.)

B. Comparison with the Marden–Jury Table

The Marden–Jury table is different in structure and interpretation from the present table. It is suggested in several variations by Marden [3], Jury [4], Astrom [5], modifications by Jury [8] and Raible [9]. They all have $n+1$ rows (not counting a reverse-order image by which each row is conveniently followed in [3]–[5] and [8]). The first row has $n+1$ entries formed by the coefficients of $D(z)$ and each succeeding row has one entry less than its predecessor (unlike the new table that has, say for $n=2m$, rows with $m+1, m+1, m, m, \cdots$, etc., entries). To get an estimate of the total number of additive and multiplicative operations, denoted by A and M, respectively, we shall proceed as follows. The total number of entries in the old table is $\mathcal{O}(0.5n^2)$, not counting reversed older rows, whereas the number of entries in the new table is $\mathcal{O}(0.25n^2)$, not counting its right half reflection. (The notation $f=\mathcal{O}(\alpha n^2)$ is used here to mean that f/n^2 tends to α for large n.) The number of additive operations is one per entry in any of the versions [3]–[5], [8], [9] by which $A_0=\mathcal{O}(0.5n^2)$. The number of multiplicative operations is two per entry in [3] and [4], giving $M_0=\mathcal{O}(n^2)$, three per entry in [8] with $M_0=\mathcal{O}(1.5n^2)$, and only one per entry in [5] and [9], by which $M_0=\mathcal{O}(0.5n^2)$. The new table requires two additive operations per entry but has half the number of entries, therefore $A_n=\mathcal{O}(0.5n^2)$. The number of multiplicative operations in the new table is one per entry and therefore $M_n=\mathcal{O}(0.25n^2)$. The number of multiplications is one fourth of that in [3], [4], one sixth of that in [8], and one half of that in [5] and [9]. Since the exact expressions for A and M are in both tables quadratic in n, the computational advantage of the new table is already apparent for polynomials of large degrees. The new table exhibits significant computational saving also for low degree polynomials. However, it is difficult to set unobjectionable exact figures for the number of additive and multiplicative operations for the two different tables and hold a precise comparison. The difficulties stem from the following three reasons: i) Each table applies differently for only testing stability and for the more general zeros location problem. ii) Each table admits certain shortcuts (not discussed here for the new table), advantage of which can be taken by an experienced user. iii) There exists in both tables a tradeoff, to some extent, between the M and A values. Fortunately, the variations implied by i)–iii) seem to influence only the linear parts of the expressions for the A's and M's, leaving the leading coefficients unaffected and the above obtained estimates valid.

The procedure for counting the number of IUC, UC, and OUC zeros is considered to be by far more effective in the new table then the procedure of [3] or [4]. The count of the OUC zeros, at least in normal cases, by a simple sign variation rule is possible in the tables of [8] and [9]. However, these tables still involve twice the number of entries and in [8] the amount of computation is even higher than in [4]. The singularities in the old table are, like in the new one, of two types; a type I that is associated with UC or reciprocal pairs of zeros and a complementary type II. The removal of the type I singularity by differentiation is due to Cohn [2] and is used in the new table essentially in the same way as in [3], [4], [7], and [9], where in [7] the possibility of UC and reciprocal pair of zeros of multiplicity higher than one is also mentioned (our Remark 4.3). Since, also in the old table, a type I singularity follows a symmetric row (an antisymmetric, if $z_c=1$ is a zero of odd multiplicity of $D(z)$), the always symmetric rows of the new table retain the "factor two" advantage for type I singularities, in the sense that in the new table the two next rows, compared to one, are gained through the differentiation. Possibly, more significant is the contribution in the treatment for the second type singularity. The modification for the type II singularity in the new table is simple (shifting and summing up rows), it always removes the singularity and is computationally less consuming than the respective treatments in [3], [4], [9]–[12]. The removal of the singularities by perturbation of the polynomial, of the unit circle, or by the ϵ-method also suggested in [4], [10]–[12] is a great deal less satisfactory and incomplete. Not less important than the simplicity of the second modification in the new table is its integrity with the first type modification. Unlike an ϵ-method or the radial perturbation of the z variable, the new type II modification does not interfere with a type I singularity, nor does it complicate, by changing zeros positions, a subsequent identification of the UC or reciprocal pairs of zeros. The second modification in the new table is close in this sense to (but is more direct than) a corresponding modification due to Cohn [2] (also cited in [3, p. 158], [4, p. 131], and [9]) that similarly has this property in combination with a type I singularity treated by differentiation.

VI. Conclusions

The paper has presented a new method to determine the location of the zeros of a discrete system polynomial with respect to the z-plane unit circle. The method is based on a sequence of symmetric polynomials and the analysis is best carried out in a tabular formulation. The new stability table is shown to be very similar to the Routh table. Its both formal and meaningful similarity to the Routh table makes the new table easy to remember. A thorough analysis and treatment of all possible singular cases is included such that the sought information on the number of zeros inside, on, and outside the unit circle can always be obtained (as well as additional information on the number, location, and multiplicity of UC and reciprocal pairs of zeros). The new table has half the size and involves less computation than the Marden–Jury table. (In certain possible digital implementations the main reduction in arithmetics being in the number of multiplications becomes even more significant because the time needed to a multiply is often greater than that needed for an addition). The systematic consideration and removal of all possible singularities is also considered superior over corresponding treatments for the former table. This can best be demonstrated by devising a highly singular polynomial (e.g., with multiple UC or reciprocal pairs of zeros plus type II singularities) or by resolving the numerical illustrations in [4], [10]–[12], by the method of the paper.

The table of this paper may be viewed as an improved version of a table, similar in size and computations, that has been presented in [6]. The table in [6] has been derived based on a new continued fraction expansion in the z-domain and presents (only) necessary and sufficient conditions for stability. A modified version of the continued fraction form of [6] can also be associated with the table in this paper. The possible association of a continued fraction expansion to the new table is one more link of similarity between the new table and the Routh table (which is well known to be equivalent to certain analogous s-plane CF expansions, [6]) and a difference from the Marden–Jury table. It may "explain" the closeness of the new table to the Routh table and as a result its advantages over former z-plane stability tables.

Appendix
Proof of Theorem 2.1

The proof is based on the arugment theorem. To prove parts i) and ii) we show, respectively, that a) If $\operatorname{sgn} D_{k+1}(1) = \operatorname{sgn} D_k(1)$ then $\alpha_{k+1} = \alpha_k + 1$ and b) If $\operatorname{sgn} D_{k+1}(1) = -\operatorname{sgn} D_k(1)$ then $\alpha_{k+1} = \alpha_k$. Note that case b) means that $D_{k+1}(z)$ has $\beta_{k+1} = \beta_k + 1$ OUC zeros because $D_{k+1}(z)$ has no UC zeros if $D_k(z)$ has no UC zeros, $\alpha_k + \gamma_k = k$ (see Theorem 4.1).

The symmetric real polynomials $T_k(z)$ for values of $z \in C$, say $z = e^{j\psi}$, take the forms

$$T_{2m+1}(e^{j\psi}) = (e^{j\psi} + 1)e^{mj\psi}R_{2m}(\cos\psi) \tag{A1}$$

and

$$T_{2m}(e^{j\psi}) = e^{mj\psi}\hat{R}_{2m}(\cos\psi) \tag{A2}$$

for odd and even k's, respectively, where $R_{2m}(x)$ and $\hat{R}_{2m}(x)$ are real polynomials in the real variable $x = \cos\psi$. This can be shown from the fact that if $z_k \neq \pm 1$ is a zero of $T_k(z)$ then z_k^{-1} is also a zero of $T_k(z)$. Note also that from the symmetry of $T_k(z)$ and the recursion (13) we also have that at $z = -1$ for all $m = 0, 1, \cdots$

$$T_{2m+1}(-1) = 0 \tag{A3a}$$

and

$$T_{2m}(-1) = T_0 \text{ (a common constant).} \tag{A3b}$$

Consider for the proof of part a) the quotient

$$f(z) = \frac{zD_k(z)}{D_{k+1}(z)} = \frac{z[T_k(z) + (z-1)T_{k-1}(z)]}{T_{k+1}(z) + (z-1)T_k(z)}. \tag{A4}$$

We proceed to show that as z traverses the unit circle C, $f(z)$ does not encircle the origin. The proof is by verifying that for any $\psi_0 \in [0, 2\pi]$ for which $f(e^{j\psi_0})$ is real it has a same (positive) sign. We have to consider separately the cases of even and odd k's. So, assume first that $k = 2m - 1$, we have that at $z = 1$ ($\psi = 0, 2\pi$)

$$f(1) = \frac{T_{2m-1}(1)}{T_{2m}(1)} > 0 \tag{A5}$$

where $f(1) > 0$ because of the sign assumption. Similarly at $z = -1$, using (A3) we find

$$f(-1) = \frac{2T_{2m-2}(-1)}{T_{2m}(-1)} = 2 > 0. \tag{A6}$$

To check for values of $z = e^{j\psi}$ other than $z = \pm 1$, we substitute (A1) and (A2) into (A4). After dividing out a factor $e^{jm\psi}$, common to the numerator and denominator, the resulting expression is

$$\begin{aligned} f(e^{j\psi}) = \{&R_{2m-2}(\cdot)[\cos\psi + j\sin\psi + 1] \\ &+ \hat{R}_{2m-2}(\cdot)[\cos\psi + j\sin\psi - 1]\} \\ &/\{\hat{R}_{2m}(\cdot) + 2j\sin\psi R_{2m-2}(\cdot)\}. \end{aligned} \tag{A7}$$

$f(e^{j\psi})$ is real for values of ψ_0 for which either the two imaginary or the two real parts become zero simultaneously in the numerator and the denominator. The two imaginary parts may become zero when

$$\sin\psi_0[R_{2m-2}(\cos\psi_0) + \hat{R}_{2m-2}(\cos\psi_0)] = 0 \tag{A8a}$$

$$\sin\psi_0 R_{2m-2}(\cos\psi_0) = 0. \tag{A8b}$$

Solutions to (A8) other then $\sin\psi_0 = 0$ ($z = \pm 1$) require $R_{2m-2}(\cos\psi_0) = \hat{R}_{2m-2}(\cos\psi_0) = 0$ and imply that $D_k(z)$ has UC zeros, $D_{2m-2}(e^{j\psi_0}) = 0$, against the assumption. Next, the two real parts may become zero for values of ψ_0 that solve

$$\begin{aligned}(\cos\psi_0 + 1)&R_{2m-2}(\cos\psi_0) \\ &+ (\cos\psi_0 - 1)\hat{R}_{2m-2}(\cos\psi_0) = 0\end{aligned} \tag{A9a}$$

$$\hat{R}_{2m}(\cos\psi_0) = 0. \tag{A9b}$$

If $z \neq \pm 1$ then $|\cos\psi_0| < 1$, the next quantity is positive

$$\epsilon = (1 + \cos\psi_0)/(1 - \cos\psi_0) > 0 \tag{A10}$$

and (A9a) can be rewritten as

$$\hat{R}_{2m-2}(\cos\psi_0) = \epsilon R_{2m-2}(\cos\psi_0). \tag{A11}$$

To see what real value takes $f(e^{j\psi})$ for ψ_0 that solve (A9) we substitute (A11) and (A9b) into (A7) to find

$$f(e^{j\psi_0}) = \frac{1 + \epsilon}{2} > 0. \tag{A12}$$

So, we have shown that there is no $z \in C$ for which $f(z)$ takes a negative real value. Applying the argument theorem $D_{k+1}(z)$ has as many IUC zeros as $zD_k(z)$, namely, $\alpha_{k+1} = \alpha_k + 1$. By placing in (A4) the expression (A1), (A2) of $T_{k-1}(z)$, $T_k(z)$, $T_{k+1}(z)$ for $k = 2m$ it can similarly be shown that $f(e^{j\psi})$ never takes a negative real values and the same conclusion $\alpha_{k+1} = \alpha_k + 1$ is reached.

The proof of part b) uses instead of (A4) the quotient

$$g(z) = \frac{D_k(z)}{D_{k+1}(z)}. \tag{A13}$$

The proof then proceeds in a similar manner. First it is shown that by the sign assumption in b), $g(z)$ is negative at $z = \pm 1$. Next, for $z \neq \pm 1$ the two imaginary parts cannot be simultaneously zeros for that would again imply UC zeros for $D_k(z)$. The two real parts may become simultaneously zeros only at values of ψ_0 for which

$$g(e^{j\psi_0}) = -\frac{1+\epsilon}{2} < 0 \tag{A14}$$

where $\epsilon > 0$ is again given by (A10). Since $g(z)$ takes for $z \in C$ either complex or negative real values, it cannot encircle the origin as z traverses C and therefore $D_{k+1}(z)$ has as many IUC zeros as $D_k(z)$, namely, $\alpha_{k+1} = \alpha_k$.

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