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Direct bilinear Routh stability criteria for discrete systems

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The bilinear transformation is applied to Routh conditions for Hurwitz polynomials to obtain a variety of equivalent direct z-plane continued fraction (CF) expansions and stability conditions for discrete system polynomials. Efficient computational algorithms are provided. They can be used to test the stability of a discrete system and to determine the number of zeros of unstable system polynomials outside the unit circle. The many new CF forms may be also applicable to the design of bilinear transformed sampled data ladder filters.

Keywords: Discrete stability criterion, z-plane continued fractions, Bilinear transformation, Ladder networks.

1. Introduction

A possible approach to test the stability of a discrete system with a characteristic polynomial

$$D_n(z) = d_0 + d_1 z + \cdots + d_n z^n \tag{1}$$

is to transform this polynomial to the z-plane using the bilinear transformation

$$z = \frac{1+s}{1-s}, \qquad s = \frac{z-1}{z+1}, \tag{2}$$

to obtain say

$$H_n(s) = h_0 + h_1 s + \cdots + h_n s^n \tag{3}$$

and then to apply the well known Routh criterion to test $H_n(s)$. Evidently $D_n(z)$ is stable, namely has all its zeros inside the unit circle, i.e. $|z_i| < 1$, $i = 1,\ldots,n$, where

$$D_n(z) = d_n \prod_{i=1}^{n} (z - z_i),$$

if and only if $H_n(s)$ is Hurwitz, i.e. all its zeros satisfy Re $s_i < 0$, where

$$H_n(s) = h_n \prod_{i=1}^{n} (s - s_i).$$

Explicit formulas to express h_i in terms of d_i for this approach are known [12]. It is more attractive however to take the counter approach; instead of transforming $D_n(z)$ into $H_n(s)$, to convert the Routh stability conditions into z-plane stability theorems. This approach has been firstly taken in [1] to derive a direct Routh–Padé model reduction method. The purpose of this paper is to develop a full extent of direct discrete stability conditions that can be obtained from respective Hurwitz conditions and to provide computational algorithms for them. The resulting algorithms can be used, along with other available methods [6]–[10], to test stability of discrete systems as well as to find the number of zeros of $D_n(z)$ outside the unit circle.

2. Discrete bilinear Routh theorems

Define for $D_n(z)$ the two polynomials

$$M_n(z) = \tfrac{1}{2}\left[D_n(z) + D_n^*(z)\right], \tag{4}$$

$$A_n(z) = \tfrac{1}{2}\left[D_n(z) - D_n^*(z)\right], \tag{5}$$

where $D_n^*(z)$ is the reciprocated polynomial

$$D_n^*(z) = \sum_{i=0}^{n} d_{n-i} z^i = z^n D_n(z^{-1}). \tag{6}$$

It is noted that the polynomial $M_n(z)$ is a mirror (or a symmetric) polynomial with the property

$$M_n(z) = \sum_{i=0}^{n} m_i z^i = M_n^*(z)$$

or (7)

$$m_i = m_{n-i}, \quad i = 0,\ldots,n,$$

whereas the polynomial $A_n(z)$ is an anti-mirror (or

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an anti-symmetric) polynomial with

$$A_n(z) = \sum_{i=0}^{n} a_i z^i = -A_n^*(z)$$

or (8)

$$a_i = -a_{n-i}, \quad i = 0, \dots, n,$$

The mirror and anti-mirror properties of the above polynomials present features that will be useful for the derivation of the direct Routh theorems and their computational algorithms.

Let $H_n(s)$ be the polynomial (3) whose zeros are the bilinear mapping of the zeros of $D_n(z)$. Define for $H_n(s)$ the tangent function

$$\rho_n(s) = \frac{H_n(s) - H_n(-s)}{H_n(s) + H_n(-s)}. \tag{9}$$

The Routh stability table is well known to be a row by row inscription of the coefficients of the polynomials involved in the continued fraction (CF) expansion about $s = 0$ of $\rho_n(s)$,

$$\rho_n(s) = \frac{1}{\gamma_n/s} \; \frac{1}{+\;\gamma_{n-1}/s} \; + \cdots + \; \frac{1}{\gamma_1/s}. \tag{10}$$

A necessary and sufficient condition for $H_n(s)$ to be Hurwitz is that $\gamma_i > 0$ for all $i = 1, \dots, n$. It is known that $\rho_n(s)$ and the discrete tangent function

$$\rho_n(z) = \frac{D_n(z) - D_n^*(z)}{D_n(z) + D_n^*(z)} \tag{11}$$

are mapped one into the other by (2), see [1], [2]. Therefore, since the bilinear transformation maps the unit circle $|z| = 1$, its interior and its exterior, one to one and onto the $s = jw$ axis, the left half and the right half s-planes, respectively, we readily obtain:

Theorem 1. *A real polynomial $D_n(z)$ is stable (has all its zeros inside the unit circle $|z| = 1$) if and only if the following CF exists for $\rho_n(z)$:*

$$\rho_n(z) = \frac{1}{\gamma_n\left(\frac{z+1}{z-1}\right)} \; \frac{1}{+\;\gamma_{n-1}\left(\frac{z+1}{z-1}\right)} \; + \cdots + \; \frac{1}{\gamma_1\left(\frac{z+1}{z-1}\right)} \tag{12}$$

and $\gamma_i > 0$ for all $i = 1, \dots, n$.

The Routh necessary and sufficient conditions for $H_n(s)$ can alternatively be presented as conditions on CF expansions of $\rho_n(s)$ about $s = \infty$. The expansion of $\rho_n(s)$ about $s = \infty$ is given by

$$\rho_{2m+1}(s) = \delta_{2m+1}s + \frac{1}{\delta_{2m}s} \; + \cdots + \; \frac{1}{\delta_1 s} \tag{13a}$$

and

$$\rho_{2m}(s) = \frac{1}{\delta_{2m}s} \; \frac{1}{+\;\delta_{2m-1}s} \; + \cdots + \; \frac{1}{\delta_1 s} \tag{13b}$$

for $n = 2m + 1$ and $n = 2m$, respectively. The necessary and sufficient conditions for $H_n(s)$ to be Hurwitz are $\delta_i > 0$ for all $i = 1, \dots, n$. The expansion (13) and the $\delta_i > 0$ conditions can be deduced from (10) and $\gamma_i > 0$, and vice versa, by noting that $H_n(s)$ is Hurwitz if and only if $H_n^*(s) = s^n H_n(1/s)$ is Hurwitz. The CF expansions (13) of $\rho_n(s)$ about $s = \infty$ yield the next theorem in a similar way as Theorem 1 follows from the expansion (10) of $\rho_n(s)$ about $s = 0$.

Theorem 2. *A real polynomial $D_n(z)$ is stable if and only if the following CF exists for $\rho_n(z)$:*

$$\rho_{2m+1}(z) = \delta_{2m+1}\left(\frac{z-1}{z+1}\right) + \frac{1}{\delta_{2m}\left(\frac{z-1}{z+1}\right)} \; + \cdots + \; \frac{1}{\delta_1\left(\frac{z-1}{z+1}\right)}, \tag{14a}$$

$$\rho_{2m}(z) = \frac{1}{\delta_{2m}\left(\frac{z-1}{z+1}\right)} \; \frac{1}{+\;\delta_{2m-1}\left(\frac{z-1}{z+1}\right)} \; + \cdots + \; \frac{1}{\delta_1\left(\frac{z-1}{z+1}\right)}, \tag{14b}$$

for $n = 2m + 1$ and $n = 2m$, respectively, and $\delta_i > 0$ for all $i = 1, \dots, n$.

The above two theorems were firstly obtained in [5] based on theory developed in [4] from only z-plane considerations. The bilinear transformation provides both simple proofs as well as demonstrations of their relations via the bilinear transformation to the Routh s-plane conditions. The following two corrolaries form the basis for a set of generalized s-plane stability conditions that involve CF expansions in terms of both $(z-1)/(z+1)$ and $(z+1)/(z-1)$.

Corollary 1. *$D_n(z)$ is a table if and only if $\rho_n(z)$ can be written as*

$$\rho_n(z) = \cfrac{1}{\gamma_n\left(\cfrac{z+1}{z-1}\right) + \rho_{n-1}(z)} \tag{15}$$

where $\gamma_n > 0$ and $\rho_{n-1}(z)$ is the discrete tangent function of a stable polynomial.

Corrolary 2. *$D_n(z)$ is stable if and only if $\rho_n(z)$ can be written as*

$$\rho_{2m+1}(z) = \delta_{2m+1}\left(\frac{z-1}{z+1}\right) + \rho_{2m}(z) \tag{16a}$$

or

$$\rho_{2m}(z) = \cfrac{1}{\delta_{2m}\left(\cfrac{z-1}{z+1}\right) + \rho_{2m-1}^{-1}(z)} \tag{16b}$$

for $n = 2m+1$ or $n = 2m$, respectively, where $\delta_n > 0$ and $\rho_{n-1}(z)$ is the discrete tangent function of a stable polynomial.

The proofs for Corollaries 1 and 2 follow respectively from Theorems 1 and 2. The 'only if' parts are obvious from the nested structure of the expansions (12) and (14). The 'if' parts are verified by their repeated application on $\rho_{n-1}(z)$, $\rho_{n-2}(z)$, ... till the entire expansions (12) and (14) are revealed.

The two corollaries allow the generalization of the former two continued fraction expansions of $\rho_n(z)$ into mixed forms that contain both $(z-1)/(z+1)$ and $(z+1)/(z-1)$ terms. The generalized expansions are constructed by switching at will from one corollary to the other while applying them for $n-1$, $n-2$, ... on the 'remainders' in (15) or (16). It is then obvious that each combination of a CF expansion for $\rho_n(z)$, with $(z-1)/(z+1)$ and $(z+1)/(z-1)$ appearing in any properly constructed order (that is in consistency with the above two corollaries), corresponds to a stable polynomial if and only if its resulting δ_i and γ_i coefficients are all positive. Let us take the $n = 4$ degree case to illustrate the situation and to gain some insight. Given $D_4(z)$ its tangent function can be expanded in any of the mixed forms outlined below, where we conveniently use the notations

$$y = (z+1)/(z-1) \quad \text{and} \quad x = (z-1)/(z+1),$$

as follows:

$$\rho_4(z) = \frac{1}{\gamma_4 y} \, \frac{1}{+\, \gamma_3 y} \, \frac{1}{+\, \gamma_2 y + \delta_1 x}, \tag{17.1}$$

$$\rho_4(z) = \frac{1}{\gamma_4 y + \delta_3 x} \, \frac{1}{+\, \gamma_2 y} \, \frac{1}{+\, \gamma_1 y}, \tag{17.2}$$

$$\rho_4(z) = \frac{1}{\gamma_4 y} \, \frac{1}{+\, \gamma_3 y} \, \frac{1}{+\, \delta_2 x} \, \frac{1}{+\, \delta_1 x}, \tag{17.3}$$

$$\rho_4(z) = \frac{1}{\gamma_4 y + \delta_3 x} \, \frac{1}{+\, \gamma_2 y + \delta_1 x}, \tag{17.4}$$

$$\rho_4(z) = \frac{1}{\delta_4 x} \, \frac{1}{+\, \delta_3 x} \, \frac{1}{+\, \gamma_2 y} \, \frac{1}{+\, \gamma_1 y}, \tag{17.5}$$

$$\rho_4(z) = \frac{1}{\delta_4 x} \, \frac{1}{+\, \delta_3 x} \, \frac{1}{+\, \gamma_2 y + \delta_1 x}, \tag{17.6}$$

$$\rho_4(z) = \frac{1}{\gamma_4 y + \delta_3 x} \, \frac{1}{+\, \delta_2 x} \, \frac{1}{+\, \delta_1 x}, \tag{17.7}$$

The other combinations that may have been expected, are redundant because any two passages of the type

$$\rho_{2k}(z) = \frac{1}{\gamma_{2k} y + \delta_{2k-1} x + \rho_{2k-2}(z)} \tag{18}$$

and

$$\rho_{2k}(z) = \frac{1}{\hat{\delta}_{2k} x + \hat{\gamma}_{2k-1} y + \hat{\rho}_{2k-2}(z)} \tag{19}$$

satisfy $\gamma_{2k} = \hat{\gamma}_{2k-1}$, $\delta_{2k-1} = \hat{\delta}_{2k}$ and $\rho_{2k-2}(z) = \hat{\rho}_{2k-2}(z)$. The identity between (18) and (19) can be verified from the algorithm (25) presented in the next section to carry out the expansion (20). A crucial point in the construction of such mixed CF is a careful consideration of the distinct forms that Corollary 2 takes for the even and the odd degree tangent functions.

The above mixed CF expansions can be regarded as the bilinear transformation mapping of the mixed Hurwitz CF forms on which the generalization of the Routh–Padé model reduction method in [3] has been based. The reason why these mixed forms were not suggested in [1] to equally derive high frequency 'biased' discrete versions of the method in [3] is because the bilinear term $(z-1)/(z+1)$ is an adequate approximation of $z = e^{sT}$ only for low frequencies $(s \to 0)$. Most of the new mixed forms may similarly not present useful forms for a stability test because of their casual appearance (although they may all find other possible applications, for example in the design of sampled data networks [15]–[17]). One exceptional mixed form should however be pointed out. This form is presented by the mixed CF expansion that orderly alternates between $(z+1)/(z-1)$ and $(z-1)/(z+1)$ terms. For the above $n=4$ illustration this special form is presented by (17.4). This ordered mixed form is stated as our third theorem.

Theorem 3. *A real polynomial $D_n(z)$ is stable if and only if $\rho_n(z)$ has a CF expansion that for $n=2m$ is given by*

$$\rho_{2m}(z)=\cfrac{1}{\delta_{2m}\left(\frac{z-1}{z+1}\right)+\gamma_{2m-1}\left(\frac{z+1}{z-1}\right)+\cdots+\cfrac{1}{\delta_2\left(\frac{z-1}{z+1}\right)+\gamma_1\left(\frac{z+1}{z-1}\right)}} \tag{20a}$$

with δ_{2i}, $\gamma_{2i-1}>0$ for all $i=1,\dots,m$, and for $n=2m+1$ is given either by

$$\rho_{2m+1}(z)=\cfrac{1}{\gamma_{2m+1}\left(\frac{z+1}{z-1}\right)+\cfrac{1}{\delta_{2m}\left(\frac{z-1}{z+1}\right)+\gamma_{2m+1}\left(\frac{z+1}{z-1}\right)+\cdots+\cfrac{1}{\delta_2\left(\frac{z-1}{z+1}\right)+\gamma_1\left(\frac{z+1}{z-1}\right)}}} \tag{20b.1}$$

or by

$$\rho_{2m+1}(z)=\delta_{2m+1}\left(\frac{z-1}{z+1}\right)+\cfrac{1}{\delta_{2m}\left(\frac{z-1}{z+1}\right)+\gamma_{2m-1}\left(\frac{z+1}{z-1}\right)+\cdots+\cfrac{1}{\delta_2\left(\frac{z-1}{z+1}\right)+\gamma\left(\frac{z+1}{z-1}\right)}} \tag{20b.2}$$

with $\gamma_n>0$ or $\delta_n>0$ and δ_{2i}, $\gamma_{2i-1}>0$ for all $i=1,\dots,m$.

It is pointed out that in view of the identity between (18) and (19), in any of the above two term quotients, the term $(z+1)/(z-1)$ may equally be written (and derived) before the $(z-1)/(z+1)$ term. The last theorem deserves its particularization among the many other possible mixed forms because, as the next section will show, it can be implemented by a computational algorithm that compares favourably with the algorithms that can be obtained for the first two theorems.

3. Computation

Any of the three theorems of the previous section can be used to test stability by carrying out the successive invert and divide scheme implied by its CF expansion till all the γ_i or δ_i coefficients are obtained. This section presents refined procedures to obtain these coefficients. Each of the three algorithms to be presented suggests a possible procedure to test the stability of discrete system polynomials.

Let $\rho_i(z)$ denote the remainder tangent function at some intermediate step in the CF expansions of Theorems 1, 2 or 3. Write $\rho_i(z)$ as

$$\rho_i(z)=A_i(z)/M_i(z) \tag{21}$$

where, similar to the relations in (4), (5) and (11) for $i=n$, $M_i(z)$ and $A_i(z)$ are, respectively, mirror (7) and anti-mirror (8) polynomials of degree i such that $\rho_i(z)$ is the tangent function for

$$D_i(z)=A_i(z)+M_i(z). \tag{22}$$

We next provide three computational algorithms for each of the above three theorems. The derivation of the first algorithm is given in [1]. The second and third algorithms can be deduced from Theorem 2 and 3 following mostly a similar line of derivation. The polynomials $M_i(z)$ and $A_i(z)$ in each of the algorithms below are related by (21) to the intermediate remainders $\rho_i(z)$ of the CF expansion of the respective theorem.

Algorithm for Theorem 1. Given $D_n(z)$, use (4) and (5) to form $M_n(z)$ and $A_n(z)$. Then, to obtain $\gamma_n,\dots,\gamma_1$, repeat the next cycle for $k=n$, $n-1,\dots,1$:

$$M_k(z)=\frac{A_k(z)}{z-1}, \tag{23.1}$$

$$\gamma_k = \frac{1}{2}\frac{M_k(1)}{M_{k-1}(1)}, \tag{23.2}$$

$$A_{k-1}(z) = \frac{M_k(z) - \gamma_k(z+1)M_{k-1}(z)}{z-1}. \tag{23.3}$$

$D_n(z)$ is stable if and only if $\gamma_i > 0$ for all $i = 1,\ldots,n$.

Algorithm for Theorem 2. Given $D_n(z)$, form $M_n(z)$ and $A_n(z)$ according to (4) and (5). The algorithm has two cycles, (a) for $k = 2i+1$ and (b) for $k = 2i$. They are alternatingly performed for $k = n, n-1,\ldots,1$ and yield $\delta_n,\ldots,\delta_1$.

(a) Find δ_{2i+1}, $M_{2i}(z)$ and $A_{2i}(z)$ from $M_{2i+1}(z)$ and $A_{2i+1}(z)$:

$$M_{2i}(z) = \frac{M_{2i+1}(z)}{z+1}, \tag{24a.1}$$

$$\delta_{2i+1} = -\frac{1}{2}\frac{A_{2i+1}(-1)}{M_{2i}(-1)}, \tag{24a.2}$$

$$A_{2i}(z) = \frac{A_{2i+1}(z) - \delta_{2i+1}(z-1)M_{2i}(z)}{z+1}. \tag{24a.3}$$

(b) Find δ_{2i}, $M_{2i-1}(z)$ and $A_{2i-1}(z)$ from $M_{2i}(z)$ and $A_{2i}(z)$:

$$A_{2i-1}(z) = \frac{A_{2i}(z)}{z+1}, \tag{24b.1}$$

$$\delta_{2i} = -\frac{1}{2}\frac{M_{2i}(-1)}{A_{2i-1}(-1)}, \tag{24b.2}$$

$$M_{2i-1}(z) = \frac{M_{2i}(z) - \delta_{2i}(z-1)A_{2i-1}(z)}{z+1}. \tag{24b.3}$$

$D_n(z)$ is stable if and only if $\delta_i > 0$ for all $i = 1,\ldots,n$.

Algorithm for Theorem 3. Given $D_n(z)$, obtain $M_n(z)$ and $A_n(z)$ according to (4) and (5). If $n = 2m+1$ obtain γ_n or δ_n and $M_{2m}(z)$ and $A_{2m}(z)$ by performing one cycle of (23) or of (24) in correspondence to a free choice to follow either the CF form of (20b.1) or (20b.2), respectively. The rest of the γ_i and δ_i coefficients are next obtained by repeating the following cycle for $i = m, m-1,\ldots,1$:

$$M_{2i-1}(z) = \frac{A_{2i}(z)}{z^2-1}, \tag{25.1}$$

$$\delta_{2i} = \frac{1}{4}\frac{M_{2i}(-1)}{M_{2i-2}(-1)}, \tag{25.2}$$

$$\gamma_{2i-1} = \frac{1}{4}\frac{M_{2i}(1)}{M_{2i-2}(1)}, \tag{25.3}$$

$$A_{2i-2}(z) = \{M_{2i-1}(z) - [\gamma_{2i-1}(z+1) + \delta_{2i}(z-1)]M_{2i-2}(z)\}/(z^2-1). \tag{25.4}$$

$D_n(z)$ is stable if and only if $\gamma_n > 0$ or $\delta_n > 0$ and $\delta_{2i}, \gamma_{2i-1} > 0$ for all $i = 1,\ldots,m$.

Remark 1. A mirror polynomial $M_k(z)$ has a zero at $z = -1$ for k odd, an anti-mirror polynomial $A_k(z)$ has a zero at $z = -1$ for k even and a zero at $z = 1$ for all k. It is easy to show from these observations that in the above three algorithms all the polynomials which are divided by $(z-1)$, $(z+1)$ or both, have these terms as a factor.

Remark 2. The multiplication or division of a polynomial $P(z)$ by $(z+1)$ or $(z-1)$, having these terms, involves only additive elementary operations (additions or substractions), see [1], [2].

Remark 3. The first algorithm (Theorem 1) and the second algorithm (Theorem 2) involve an equal number of elementary multiplicative (multiplication and division) and additive operations, possibly implying a preference of algorithm 1 for not distinguishing between even and odd parities. The third algorithm involves half the number of iterations of its cycle (25) in comparison to the number of iterations in (23) or (24). An iteration of the cycle (25) is equivalent to an iteration of the second algorithm followed by one iteration of the first algorithm. However, in the third algorithm for a given

$$P(z) = \sum_{i=0}^{n} p_i z^i$$

the following scheme that can be used in (25.1) and (25.4),

$$P(z)/(z^2-1) = \sum_{i=0}^{n-2} g_i z^i, \quad g_0 = -p_0, \quad g_1 = -p_1, \quad g_i = g_{i-2} - p_i, \tag{26}$$

reduces the would be equal number of elementary additive operations to about one half (the number of multiplications is left unchanged). Therefore the third algorithm is computationaly the most economical among the three stability tests.

Remark 4. The mirror and anti-mirror properties of $M_k(z)$ and $A_k(z)$, by which their coefficients satisfy $m_i = m_{k-i}$ and $a_i = -a_{k-i}$, admit more computational saving by calculating only the first half of the polynomial coefficients.

Remark 5. The positivity of δ_i and γ_i being a necessary condition for stability in any of the above algorithms implies that $M_k(1)$, $A_k(1)$, $M_k(-1)$, $A_k(-1)$ do not vanish in any of the expressions (23.2), (24a.2), (24b.2) (25.2) and (25.3) for stable polynomials. More specifically it can be shown that the occurence of a vanishing or indefinite γ_i or δ_i implies (and is implied by) the fact that $D_n(z)$ either has (one or more) zeros on the unit circle or has (one or more) reciprocal pairs of zeros (z_i and z_i^{-1} are both zeros of $D_n(z)$).

4. Zeros outside the unit circle

Any of the introduced mixed bilinear z-plane expansions yields n coefficients δ_i or γ_i, $i = 1,\ldots,n$, the positivity of which is a necessary and sufficient condition for $D_n(z)$ to be stable. This is not however the most general information that can be drawn from these coefficients. It can be shown that if all the coefficients are well defined then the number of positive and negative coefficients is equal to the number of zeros of $D_n(z)$ inside and outside the unit circle, respectively. These extensions can all be proven from analogous known s-plane Routh stability conditions and the one to one s to z mapping properties of the bilinear transformation. We shall in the following restrict ourselves to the assumption that all the n coefficients are well defined, that is, assume that the mixed CF form does not terminate prematurely because of an indefinite γ_i or δ_i. This assumption, referred to as regular conditions, is equivalent, as already mentioned in Remark 5, to the assumption that $D_n(z)$ has no zeros on the unit circle or reciprocal pairs of zeros. It is possible (again by appropriately adopting corresponding s-plane procedures) to extend the method to encompass also the complementary singular conditions and in this way to present a method of full capacity to always determine the number of zeros of any $D_n(z)$ inside, on and outside the unit circle. However, for brevity we shall not discuss the singular conditions any further. We conclude this section by stating as a theorem the extension of Theorem 1 for zeros outside the unit circle. Extensions to the two other theorems (or for the rest of possible mixed forms) could equally be stated.

Theorem 4. *If the bilinear CF expansion* (12) *complies with regular conditions* (*if the first algorithm* (23) *does not terminate prematurely*) *then* $D_n(z)$ *does not have zeros on the unit circle* (*or reciprocal pairs of zeros*) *and the number of its zeros inside and outside the unit circle is given by the number of positive and negative terms, respectively, in the sequency* $\gamma_n,\ldots,\gamma_1$.

Conclusions

The paper has presented the class of all possible z-plane CF expansions and stability conditions that are obtainable from corresponding s-plane stability conditions by the bilinear transformation. Three stability theorems (that are related via the bilinear transformation to the first, second and third Cauer forms [11]), were particularized. These theorems were followed by three algorithms to carry out the implied stability tests efficiently. The algorithm for Theorem 1 has the simplest set up, whereas the algorithm for Theorem 3 requires the least computational effort. The method can be extended to obtain also the number of zeros of a polynomial on and outside the unit circle. It was shown that the provided algorithm yields in general also the number of zeros outside the unit circle.

Other stability tests that can be applied on the discrete system polynomial are the well known table of Jury [6] based on the early solutions to this problem by Schur and Cohn and Marden [7], and the new methodology that has recently been introduced by the author in [8]–[10]. The number of elementary multiplicative and additive operations required for the third algorithm of this paper is in general lower than the corresponding numbers for the stability table of Jury [6] but is higher than the corresponding numbers in the new stability table of [10] or [8] and [9]. Alternative stability theorems, not mentioned in this paper, that employ a matrix theorem framework rather than direct operations on the system polynomials are also known (e.g. the Schur–Cohn matrices, the discrete the Lyapunov equation, signature theorems and certain canonical forms). The approach of this paper (in particular the unmixed expansions) may be regarded as related conceptually to certain matricial methods that investigate the unit circle stability of the A matrix via the Hurwitz stability properties of the matrix $(A+1)(A-1)^{-1}$ (cf. [12]) but avoid the actual bilinear transformation of A, see [13], [14].

The s-plane expansions of (10) and (13) as well as mixed expansions of $\rho(s)$ about $s=0$ and $s=\infty$ play an important role in the synthesis of lossless (and consequently, also lossy) ladder networks [11]. The variety of mixed bilinear z-plane expansions presented in this paper may be similarly useful in the design of stable digital networks. For such synthesis purposes a reversed approach, one that recursively constructs stable polynomials $D_i(z)$ of successively higher degrees may be needed. A recursion formula related to (12) for the synthesis of a stable $D_k(z)$ for given k positive coefficients $\gamma_1,\ldots,\gamma_k$ can be found in [1]. Similar expressions can be obtained also for the Theorem 2 or for the other mixed expansions. The z-plane CF expansions in this paper may, for example, simplify the design of switched capacitor filters which are based on s-plane filter responses and the bilinear s to z transformation [15]–[17].

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