when written into a memory will result in a maximum of M logical transitions, hence providing an M fold reduction in power dissipation.

The orthogonal technique also provides a speed advantage compared to using conventional memories for multiplexing or demultiplexing. A possible approach in using conventional memories is to write into the memory as described in Fig. 2 and read from the memory via the normal outputs. In order to read the information contained in one column, the memory would have to be read M times for each column and only one out of N bits would be used from each read operation. Reading the entire memory would require $N \times M$ read operations vs only N read operations for the orthogonal RAM. For 8 bits of voice and 8 bits of data $M=16$, giving the orthogonal RAM a 16:1 speed advantage.

It can be concluded from the above discussion that architectural designs employing memory structures have a significant advantage in density, power dissipation, and speed–density product.

Chip Design and Simulation

The chip was designed hierarchically, with heavy emphasis on computer simulation. The chip design employs over 25 cell types, each customized electrically and topologically to optimize performance and area utilization. Each cell was simulated at a circuit level using the SPICE circuit simulation program. The architectural design was simulated on the Daisy Logician Workstation from a schematic entry. The layout was performed on the Calma Graphic System. In order to validate the design, the circuitry was extracted from the chip layout and compared to the netlist used in Daisy Logic simulator.

Conclusion

A new type of memory array has been described which can provide time–division multiplexing and demultiplexing with a regular and compact structure. The memory performs storage as well as serial-to-parallel or parallel-to-serial conversion. This is achieved by reading the RAM in an orthogonal direction to the write operation. By combining this RAM with a high-speed transmission media (i.e., optical fiber), many channels can be communicated over a single line. Due to its regular structure, this technique lends itself to efficient VLSI implementation and to redundancy techniques used with the regular structures of memory arrays.

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Stability Criterion for Continuous-Time System Polynomials with Uncertain Complex Coefficients

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***Abstract*—Let a polynomial $P_n(s)=p_0+p_1s+\cdots+p_ns^n$ have coefficients $p_i=a_i+jb_i$ that may vary, or only known to be, in intervals $\underline{a}_i \leqslant a_i \leqslant \bar{a}_i, \underline{b}_i \leqslant b_i \leqslant \bar{b}_i$. Kharitonov's criterion asserts that $P_n(s)$ has all its zeros in the left half of the complex plane (is Hurwitz) for all admissible values of the coefficients, if, and only if, some well-defined 8 complex *fixed* coefficient polynomials are Hurwitz. When the uncertain polynomial is real the criterion involves only four fixed real polynomials. We restate and give a simple proof for Kharitonov's criterion for both real and complex polynomials. Our derivation is based on evaluation of complex rational lossless positive real functions and their relation to Hurwitz polynomials.**

I. Introduction

Systems designs are based on mathematical models whose exact parameters are known only approximately or are subject to variations from their nominal values. The possibility arises that certain polynomials designed to have zeros in the stability region —the left half of the complex plane for continuous-time systems —may actually have coefficients in some interval around their assumed values. Consequently, it is of interest to have a *robust stability* criterion, one that guarantees stability (zeros in the left half of the complex plane) for all expected variations of the coefficients within assumed intervals of uncertainty. We shall refer to a polynomial whose zeros are all in the open left half of complex plane as Hurwitz polynomial (the term "strictly Hurwitz" is sometimes used for this case to emphasize that zeros on the imaginary axis are excluded)

The Robust Stability Problem

Given the infinite family of polynomials,

$$\Delta_n=\left\{h_n(s)\,|\,h_n(s)=\sum_{i=0}^{n}(a_i+jb_i)s^i,\ a_i\in[\underline{a}_i,\bar{a}_i],\ b_i\in[\underline{b}_i,\bar{b}_i],\ i=1,\cdots,n\right\} \quad (1)$$

find conditions under which Δ_n is a subset of S^n, the set of all Hurwitz polynomials of degree n.

Kharitonov gave a simple criterion to solve this problem. For real polynomials the necessary and sufficient condition for $\Delta_n \subset S^n$ is equivalent to the stability of 4 deterministic (fixed coeffi-

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cient) polynomials [1]. Complex polynomials require the inspection of 8 deterministic polynomials whose coefficients are composed from the boundary values $\underline{a}_i$, $\bar{a}_i$, $\underline{b}_i$ and $\bar{b}_i$ [2]. ([2] is available only in Russian, [1] has English translation). Kharitonov's theorem for real polynomial caught the attention of additional researchers who suggested alternative derivations, related forms, or applications [6]–[10]. The complex version of the criterion was not tackled with similar attention. Naturally, the derivation of Kharitonov's theorem for complex polynomials is slightly more complicated. In contrast to the real case, Hurwitz conditions for complex polynomials are less known and not as well supported by closely related and well-established mathematical theory of networks and systems. Furthermore, (and as also experienced by the author), the paper in which Kharitonov extends his interesting criterion to complex polynomials [2], and this may have escaped the notice of many researchers for whom it is relatively inaccessible and the Russian text presents a language obstacle. The original intention of the author in this research was to bring forth an extension of Kharitonov's (real) criterion to complex polynomials and deduce an alternative and simple proof for the real case. While being aware of [2], the emphasis of this paper is in providing an alternative and simple derivation for Kharitonov's theorem for both the complex and the real cases.

Our approach is based on extensions to complex polynomials of certain properties associated with Hurwitz polynomials that are well known for the real case in the context of passive network theory [3], [4]. First, in Section II, we define and characterize *lossless positive real* (LPR) complex rational functions. Next, in Section III, we associate them with (complex) Hurwitz polynomials. Based on these mathematical preliminaries, Kharitonov's stability criterion for uncertain complex polynomials is derived in Section IV. This last section also considers the special form of the criterion for real polynomials.

II. Lossless Positive Real Functions

We consider the division of the complex plane into two (left and right) halves and denote the left and right half *open* domains and their (common) boundary, respectively, by

$$\mathrm{LHP} = \{ s \mid \operatorname{Re} s < 0 \} \tag{2a}$$

$$\mathrm{RHP} = \{ s \mid \operatorname{Re} s > 0 \} \tag{2b}$$

$$\Gamma := \{ s \mid s = j\omega,\ \omega \in [-\infty, \infty] \}. \tag{2c}$$

Definition 1: A function $F(s)$ is positive real (PR, in the right half plane) if:

(a) $F(s)$ is analytic in RHP, and
(b) $\operatorname{Re} F(s) \geqslant 0$ in RHP.

The definition extends an identical term used in classical network theory as a synonym to immittance functions, by dropping the extra requirement on $F(s)$ to be real for real s. The extended definition requires a careful revision of the properties of PR functions. Subsequently, we characterize the class of (complex) PR functions and its important subclass of lossless PR functions (to be defined below) in a series of comments mainly that emphasize those points that will be utilized later. Usually, these extended features can be proven by reasonings that are similar, or only slightly modify, proofs for corresponding real properties. A possibly resulting slightly casual approach to proofs in the sequel should not reflect on the credibility of the statements; fully detailed proofs can always be completed using the provided remarks and following the outlines of proofs for the real form of these properties as available in mathematical texts on network theory (e.g., [3], [4]).

Remarks

(1) It follows from the maximum modulus principle for $\operatorname{Re} F(s)$ asserts that $\operatorname{Re} F(s) = 0$ is possible only for $\operatorname{Re} s = 0$ (the imaginary axis, IA) or at $s = \infty$. More precisely, $\operatorname{Re} F(s)$ takes its minimum on the boundary Γ of the RHP (e.g., [3], [4]),

(2) Clearly, if $F(s)$ is PR then $F^{-1}(s)$ is PR and $rF(s)$ is PR for a real and positive scalar r. Furthermore, if $\hat{F}(s)$ is another PR function, then $F(s) + \hat{F}(s)$ is also PR.

(3) Poles of a PR function on Γ, if any, must be simple and with positive residues. A proof of this property for the complex case is essentially the same as for the real case (e.g., [4]).

(4) If $F(s)$ is a rational function, $\operatorname{Re} F(s) > 0$ in the RHP is a sufficient definition for its positive realness. (Analyticity follows by properties of harmonic functions.)

For subsequent purposes we may consider only rational functions of finite degree n, $F_n(s)$. It follows from Remark 4) that the two polynomials that compose a PR rational function of degree n must both have degree n or degree n and $n-1$ because a greater than 1 difference in degrees would imply pole at infinity of multiplicity greater than 1 for $F_n(s)$ or its inverse. Our interest will focus on *lossless* PR functions, a special subclass of finite rational PR functions.

Definition 2: A rational function $L_n(s)$ is *Lossless Positive Real* (LPR) if:

(a) $L_n(s)$ is PR, and
(b) $L_n(s)$ has all its zeros and poles on Γ.

Remarks

(5) If $L_n(s)$ is LPR and r is a real positive scalar then $L_n^{-1}(s)$ and $rL_n(s)$ are LPR of the same degree. If $\hat{L}_m(s)$ is another LPR function then $L_n(s) + \hat{L}_m(s)$ is also a LPR function (of degree $\leqslant m + n$).

(6) $L_n(s)$ is LPR if, and only if,

$$\operatorname{Re} L_n(s) > 0 \text{ in RHP and } \operatorname{Re} L_n(s) < 0 \text{ in LHP.}$$

($\operatorname{Re} L_n(s) = 0$ on Γ, though superfluous, can also be added to this characterization.) The statement can be proven as follows. Assume that the above property holds, then, by Remark 4), $L_n(s)$ is PR (in the RHP) and "negative real" (i.e., $-L_n(s)$ is positive real) in the LHP. Therefore, $L_n(s)$ is analytic in both the LHP and the RHP hence its poles are confined to Γ. Repeating the last argument for $L_n^{-1}(s)$ reveals that its zeros must be on Γ too. This proves that $L_n(s)$ is LPR. Conversely, if $L_n(s)$ is LPR its zeros and poles are confined to Γ and, therefore, it is analytic in both (open) half planes. Since the real part of a function analytic in an open domain takes its extremum values on the boundary (cf. Remark 1), the real part of $L_n(s)$ maintains a constant sign in each of the two open half planes. The sign has to be positive in the RHP and of opposite sign in the LHP.

Lemma 1: $L_n(s)$ is LPR if, and only if,

$$L_n(s) = jr_0 + \frac{r_1}{s - j\omega_1} + \cdots + \frac{r_n}{s - j\omega_n} \tag{3}$$

where r_0 is a real number and the residues are real and positive, $r_i > 0$, $i = 1, \cdots, n$. In wording, $L_n(s)$ if LPR, if, and only if,

(i) The poles of $L_n(s)$ are simple and on Γ.
(ii) Their residues are positive.

Proof: If $L_n(s)$ is LPR then by Definition 2, part (b), all its poles are on Γ. Part (a) of the same definition and Remark 3) imply (i) and (ii). For the converse, if (3) holds then, since each term in the sum is a PR function with zeros and poles on Γ their sum $L_n(s)$ is LPR by Remark 5).

Remarks

(7) $L_n(s)$ may have a zero at ∞ in which case $r_0 = 0$ in (3) (r_0 may otherwise take any real value). $L_n(s)$ may have a pole at ∞ in which case a term, say the nth term, takes the form $r_n s$. Note however, that in the complex case, in contrast to the real case, it is not necessary for a LPR $L_n(s)$ to have a zero or a pole at infinity or at zero. (See also Remark 11) in the sequel.)

(8) The partial fraction expansion (3) can be viewed as the extension to complex polynomials of a form known in network theory as Foster's expansion for LC functions [3], [4].

(9) A LPR $L_n(s)$ can alternatively be characterized by its zeros and poles. $L_n(s)$ is LPR if, and only if, its zeros and poles are simple, located on Γ, they interlace, and the residue at one of the poles (or the real part at any RHP point) is positive.[1] This theorem is less known for complex polynomials than for the real case, although a proof of the above generalization is not more difficult than its real special form. In contrast to the real case, in the complex case it is not required that the zeros and poles of $L_n(s)$ appear in pairs $\pm j\omega_i$, nor is it necessary that $L_n(s)$ have a zero or pole at $s = 0$ or a zero or pole at $s = \infty$.

(10) After establishing the relation of LPR functions and Hurwitz polynomials (in Lemma 4), we shall base our derivation of Kharitonov's criterion on the positivity of the residues of the LPR functions associated with the family Δ_n. Kharitonov's derivations in [1] and [2] are based on the characterization in Remark 9).[2] Thus rather than employing arguments on the variation of zeros, constrained to Γ, for varying polynomial coefficients, we shall be able to use more transparent convexity relations between the change of coefficients and retention of positivity of the residues.

III. Hurwitz Polynomials

Consider a complex polynomial

$$h_n(s) = \sum_{i=0}^{n} h_i s^i \qquad h_i = a_i + jb_i, \qquad h_n \neq 0. \tag{4}$$

It can always be decomposed as

$$h_n(s) = \frac{1}{2} e_n(s) + \frac{1}{2} o_n(s) \tag{5}$$

where

$$e_n(s) = h_n(s) + h_n^{\#}(s) \qquad o_n(s) = h_n(s) - h_n^{\#}(s) \tag{6a}$$

with $h_n^{\#}(s)$ denoting the (conjugate) reciprocal polynomial,

$$h_n^{\#}(s) := h_n(-s^*)^* = h_0^* - h_1^* s + \cdots + h_n^*(-s)^n \tag{6b}$$

and * denoting complex conjugate. We note that $e_n(s)$ is *even* and $o_n(s)$ is *odd*, that is,

$$e_n(s)^{\#} = e_n(s) \qquad o_n^{\#}(s) = -o_n(s). \tag{7}$$

[1] Zeros on Γ that are simple and interlacing admit both lossless *positive* real as well as lossless *negative* real function. Consequently, they characterize polynomials $h_n(s)$, see Section III whose zeros are all in the LHP or the RHP. The extra positivity condition is thus crucial as it sharpens the correspondence to only LPR and Hurwitz polynomials. In the real case the extra condition can be chosen to be positivity of the ratio of the leading coefficients of the numerator and the denominator polynomials.

[2] The extra positivity condition discussed in Remark 9 and the previous footnote seems to be overlooked in [1], [2], and [10].

The following Lemma relates Hurwitz polynomials and LPR functions.

Lemma 2: The polynomial $h_n(s)$ is Hurwitz (all its zeros are in LHP, (2a)) if, and only if, the ratio of its odd part divided by its even part, forms a LPR rational function of degree n.

Proof: Define

$$\rho_n(s) = \frac{o_n(s)}{e_n(s)} \qquad \phi_n(s) = \frac{h_n^{\#}(s)}{h_n(s)} \tag{8}$$

and observe that they are related by the Cayley transform

$$\rho_n(s) = \frac{1-\phi_n(s)}{1+\phi_n(s)} \qquad \phi_n(s) = \frac{1-\rho_n(s)}{1+\rho_n(s)}. \tag{9}$$

It is well known that the underling bilinear transformation maps RHP (LHP) and the interior (exterior) of the unit circle one onto the other. Assume $h_n(s) = \Pi(s + s_i)$ and $s_i = \sigma_i + j\omega_i$ then

$$\phi_n(s) = \prod_{i=1}^{n} \phi_n^{(i)}(s) \qquad \phi_n^{(i)} = \frac{-s + s_i^*}{s + s_i}$$

where $s = \sigma + j\omega$. Checking the magnitudes $|\phi_n^{(i)}(s)|^2$ reveals that $\phi_n^{(i)}$ are greater than, equal, or less than 1 iff $\sigma\sigma_i$ are positive, zero or negative, respectively. Therefore, by Remark 6, $\rho_n(s)$ is LPR iff all $\sigma_i > 0$.

We associate to a complex polynomial $h_n(s)$ (4), a pair of real polynomials $\{f_n(\omega), g_n(\omega)\}$ such that along $s = j\omega$,

$$h_n(j\omega) = f_n(\omega) + jg_n(\omega). \tag{10}$$

Comparison with the decomposition (5) reveals that

$$f_n(\omega) = \frac{1}{2} e_n(j\omega) \qquad jg_n(\omega) = \frac{1}{2} o_n(j\omega). \tag{11}$$

Conversely, the pair of real polynomials $f_n(\omega), g_n(\omega)$, define a complex polynomial $h_n(s)$ by the continuation of (10) to the entire plane, that is by the substitution

$$h_n(s) = f_n(s/j) + jg_n(s/j). \tag{12}$$

The coefficients of $f_n(\omega)$ and $g_n(\omega)$ are, therefore, related to the real and imaginary parts of the coefficients $h_n(s)$ in the following manner:

$$f_n(\omega) = \sum_{i=0}^{n} f_{ni}\omega^i = a_0 - b_1\omega - a_2\omega^2 + b_3\omega^3 + a_4\omega^4 - \cdots \tag{13a}$$

$$g_n(\omega) = \sum_{i=0}^{n} g_{ni}\omega^i = b_0 + a_1\omega - b_2\omega^2 - a_3\omega^3 + b_4\omega^4 + \cdots. \tag{13b}$$

The pattern indicated in (13a) and (13b) is as follows: a_i and b_i alternate, $\cdots a_i, b_{i+1}, a_{i+2}, b_{i+3}, \cdots$; with two plus and two minus signs period, $\cdots + + - - + + - - \cdots$.

Definition 3: We shall say that a pair of real polynomials $\{f_n(\omega), g_n(\omega)\}$ is *a stable pair*, if it corresponds, via (12), to a Hurwitz polynomial.

Necessary and sufficient conditions for the pair $\{f_n(\omega), g_n(\omega)\}$ to be stable can be drawn from combining Lemma 1 with Lemma 2. For this, we define

$$R_n(\omega) := \frac{g_n(\omega)}{f_n(\omega)} = \frac{1}{j}\rho_n(j\omega) \tag{14}$$

and deduce from Lemmas 1 and 2 the next lemma.

Lemma 3: The pair $\{f_n(\omega), g_n(\omega)\}$ is stable if and only if $R_n(\omega)$ has a partial fraction expansion

$$R_n(\omega) = \frac{g_n(\omega)}{f_n(\omega)} = r_0' + \frac{r_1'}{-\omega+\omega_1'} + \cdots + \frac{r_n'}{-\omega+\omega_n'} \quad (15a)$$

with real (arbitrary) r_0 and positive residues $r_i' > 0$, $i = 1, \cdots, n$. Equivalently (by Remark 5), the pair $\{f_n(\omega), g_n(\omega)\}$ is stable if and only if

$$R_n^{-1}(\omega) = \frac{f_n(\omega)}{g_n(\omega)} = r_0'' + \frac{r_1''}{-\omega+\omega_1''} + \cdots + \frac{r_n''}{-\omega+\omega_n''} \quad (15b)$$

with real (arbitrary) r_0'' and positive residues $r_i'' > 0$, $i = 1, \cdots, n$.

Remark

(11) Some modifications occur in (15) if either f_{nn} or g_{nn} are equal to zero. (They can not be both zeros if $h_n(s)$ is assumed to be of full degree). $f_{nn} = 0$ corresponds to $R_n(\omega)$ having a pole at ∞ and $R_n^{-1}(\omega)$ having there a zero. In this case, one of the partial fractions in (15a), say the last, is replaced by $r_n'\omega$ and

$$r_n' = \frac{g_{nn}}{f_{n,n-1}} > 0, \quad r_0'' = 0, \quad r_0' = \frac{g_{n,n-1}}{f_{n,n-1}} - \frac{f_{n,n-2}g_{n,n}}{f_{n,n-1}^2}.$$

Similarly, $g_{n,n} = 0$ corresponds to $R_n^{-1}(\omega)$ having a pole at ∞ and $R_n(\omega)$ having there a zero. In this case (say) the last partial fraction in (15b) is replaced by $r_n''\omega$ and

$$r_n'' = \frac{g_{nn}}{f_{n,n-1}} > 0, \quad r_0' = 0, \quad r_0'' = \frac{f_{n,n-1}}{g_{n,-1}} - \frac{g_{n,n-2}f_{n,n}}{g_{n,n-1}^2}.$$

Recalling Remark 7, a stable pair need not necessarily have either of the above two forms. When $g_n(\omega)$ and $f_n(\omega)$ both have full degrees, the partial fractions are exactly as written in (15) with nonzero

$$r_0' = \frac{g_{n,n}}{f_{n,n}}, \qquad r_0'' = \frac{f_{n,n}}{g_{n,n}}. \quad (16)$$

We shall need some more explicit expressions for the residues r_i. For this we denote

$$\phi_i = \frac{f_n(\omega)}{-\omega+\omega_i'|_{\omega=\omega_i'}} \quad (16a)$$

$$\gamma_i = \frac{g_n(\omega)}{-\omega+\omega_i''|_{\omega=\omega_i''}}. \quad (16b)$$

where ϕ_i, $\gamma_i \neq 0$ but otherwise may be either positive or negative. The residues in (15a) and (15b) are then given by

$$r_i' = \frac{g_n(\omega_i')}{\phi_i}, \qquad i = 1, \cdots, n \quad (17a)$$

$$r_i'' = \frac{f_n(\omega_i'')}{\gamma_i}, \qquad i = 1, \cdots, n. \quad (17b)$$

IV. Robust Stability Criteria

Consider now Δ_n, the family of polynomials (1) in the robust stability problem stated in Section I. If $h_n(s) \in \Delta_n$ then the range of admissible values of the coefficients of $h_n(s)$ induce admissible ranges on the values of $f_n(\omega)$ and $g_n(\omega)$, say

$$f_{ni} \in [\underline{f}_{ni}, \bar{f}_{ni}] \quad (18a)$$

$$g_{ni} \in [\underline{g}_{ni}, \bar{g}_{ni}], \qquad i = 0, 1, \cdots, n. \quad (18b)$$

The limits of these intervals can be determined from (13)

$$f_{n0} \in [\underline{a}_0, \bar{a}_0], f_{n1} \in [-\bar{b}_1, -\underline{b}_1], f_{n2} \in [-\bar{a}_2, -\underline{a}_2], \text{etc.} \quad (19a)$$

$$g_{n0} \in [\underline{b}_0, \bar{b}_0], g_{n1} \in [\underline{a}_1, \bar{a}_1], g_{n2} \in [-\bar{b}_2, -\underline{b}_2], \text{etc.} \quad (19b)$$

We now define four boundary polynomials, $\underline{G}_n^+(\omega)$, $\bar{G}_n^+(\omega)$, $\underline{G}_n^-(\omega)$, and $\bar{G}_n^-(\omega)$, for the family of admissible polynomials $g_n(s)$. The first two are defined by

$$\underline{G}_n^+(\omega) := \sum_{i=0}^{n} \underline{g}_{ni}\omega^i \quad (20a)$$

$$\bar{G}_n^+(\omega) := \sum_{i=0}^{n} \bar{g}_{ni}\omega^i \quad (20b)$$

and we immediately observe that if $\omega_k' > 0$ then for all admissible $g_{ni} \in [\underline{g}_{ni}, \bar{g}_{ni}]$,

$$\underline{G}_n^+(\omega_k') \leqslant g_n(\omega_k') \leqslant \bar{G}_n^+(\omega_k'). \quad (21)$$

The second two polynomials are defined by

$$\underline{G}_n^-(\omega) := \underline{g}_{n0} + \bar{g}_{n1}\omega + \underline{g}_{n2}\omega^2 + \bar{g}_{n3}\omega^3 + \cdots \{\underline{g}_{ni} i \text{ even}, \bar{g}_{ni} i \text{ odd}\} \quad (22a)$$

$$\bar{G}_n^-(\omega) := \bar{g}_{n0} + \underline{g}_{n1}\omega + \bar{g}_{n2}\omega^2 + \underline{g}_{n3}\omega^3 + \cdots \{\bar{g}_{ni} i \text{ even}, \underline{g}_{ni} i \text{ odd}\} \quad (22b)$$

and it is then seen that for gu$_k' < 0$ and all admissible g_{ni},

$$\underline{G}_n^-(-\omega_k') \leqslant g_n(\omega_k') \leqslant \bar{G}_n^-(-\omega_k'). \quad (23)$$

Similarly, we define for the family of admissible polynomials $f_n(\omega)$, four boundary polynomial, $\underline{F}_n^+(\omega)$, $\bar{F}_n^+(\omega)$, $\underline{F}_n^-(\omega)$, and $\bar{F}_n^-(\omega)$. The first two are defined by

$$\underline{F}_n^+(\omega) = \sum_{i=0}^{n} \underline{f}_{ni}\omega^i \quad (24a)$$

$$\bar{F}_n^+(\omega) = \sum_{i=0}^{n} \bar{f}_{ni}\omega^i \quad (24b)$$

and satisfy for $\omega_k'' > 0$, and all admissible $f_{ni} \in [\underline{f}_{ni}, \bar{f}_{ni}]$,

$$\underline{F}_n^+(\omega_k'') \leqslant f_n(\omega_k'') \leqslant \bar{F}_n^+(\omega_k''). \quad (25)$$

The last two boundary polynomials are defined by

$$\underline{F}_n^-(\omega) = \underline{f}_{n0} + \bar{f}_{n1}\omega + \underline{f}_{n2}\omega^2 + \bar{f}_{n3}\omega^3 + \cdots \{\underline{f}_{ni} i \text{ even}, \bar{f}_{ni} i \text{ odd}\} \quad (26a)$$

$$\bar{F}_n^-(\omega) = \bar{f}_{n0} + \underline{f}_{n1}\omega + \bar{f}_{n2}\omega^2 + \underline{f}_{n3}\omega^3 + \cdots \{\bar{f}_{ni} i \text{ even}, \underline{f}_{ni} i \text{ odd}\} \quad (26b)$$

and they satisfy for $\omega_k'' < 0$, and all admissible values of f_{ni}, the inequalities

$$\underline{F}_n^-(-\omega_k'') \leqslant f_n(\omega_k'') \leqslant \bar{F}_n^-(-\omega_k''). \quad (27)$$

Lemma 4:

(a) If the two pairs of polynomials

$$\{f_n(\omega), \underline{G}_n^+(\omega)\}, \quad \{f_n(\omega), \bar{G}_n^+(\omega)\} \quad (28)$$

are stable then $R_n(\omega)$ has positive residues $r_k' > 0$ at the positive zeros, $\omega_k' > 0$, of (the fixed) $f_n(\omega)$, for all admissible $\{g_n(\omega)\}$.

(b) If the two pairs of polynomials

$$\{\underline{F}_n^+(\omega), g_n(\omega)\}, \quad \{\bar{F}_n^+(\omega), g_n(\omega)\} \quad (29)$$

are stable then $R_n^{-1}(\omega)$ has positive residues $r_k'' > 0$ at the positive zeros, $\omega_k'' > 0$, of (the fixed) $g_n(\omega)$ for all admissible $\{f_n(\omega)\}$.

(c) If the two pairs of polynomials

$$\{f_n(\omega), \underline{G}_n^-(\omega)\}, \quad \{f_n(\omega), \bar{G}_n^-(\omega)\} \tag{30}$$

are stable then $R_n(\omega)$ has positive residues $r_k' > 0$ at the negative zeros, $\omega_k' < 0$, of (the fixed) $f_n(\omega)$, for all admissible $\{g_n(\omega)\}$.

(d) If the two pairs of polynomials

$$\{\underline{F}_n^-(\omega), g_n(\omega)\}, \quad \{\bar{F}_n^-(\omega), g_n(\omega)\} \tag{31}$$

are stable then $R_n^{-1}(\omega)$ has positive residues $r_k'' > 0$ at the negative zeros, $\omega_k'' < 0$, of (the fixed) $g_n(\omega)$ for all admissible $\{f_n(\omega)\}$.

Proof: We prove (a) using (21) and (17a) as follows. If $\phi_k > 0$ then $r_k' \geqslant \underline{r}_k'^{+} := \underline{G}_n^+(\omega_k')/\phi_k$ by the left side inequality in (21), while if $\phi_k < 0$ then $r_k' \geqslant \bar{r}_k'^{+} := \bar{G}_n^+(\omega_k')/\phi_k$ by the right side inequality in (21). In both situations, $r_k' > 0$ because $\underline{r}_k'^{+}$, $\bar{r}_k'^{+} > 0$ by the stability assumption on the pairs in (28).

To prove (b) we use (25) and (17b) to observe that

$$r_k'' \geqslant \underline{r}''^{+} := \underline{F}_n^+(\omega_k'')/\gamma_k > 0 \qquad \text{if } \gamma_k > 0$$

$$r_k'' \geqslant \bar{r}''^{+} := \bar{F}_n^+(\omega_k'')/\gamma_k > 0 \qquad \text{if } \gamma_k < 0.$$

Similarly, (c) is obtained from (23) and (17a)

$$r_k' \geqslant \underline{r}'^{-} := \underline{G}_n^-(-\omega_k')/\phi_k > 0 \qquad \text{if } \phi_k < 0$$

$$r_k' \geqslant \bar{r}'^{-} := \bar{G}_n^-(-\omega_k')/\phi_k > 0 \qquad \text{if } \phi_k > 0.$$

Finally, (d) is seen from

$$r_k'' \geqslant \underline{r}''^{-} := \underline{F}_n^+(-\omega_k'')/\gamma_k > 0 \qquad \text{if } \gamma_k > 0$$

$$r_k'' \geqslant \bar{r}''^{-} := \bar{F}_n^-(-\omega_k'')/\gamma_k > 0 \qquad \text{if } \gamma_k < 0$$

using (27) and (17b).

A criterion for the robust stability problem posed in Section I can now be obtained from the next couple of observations.

Observation 1. If the four pairs of polynomials of (28) and (30) are stable then, by the conditions on (15a) in Lemma 3, $\{f_n(\omega), g_n(\omega)\}$ is stable for $f_n(\omega)$ and *all admissible* $\{g_n(s)\}$.

Observation 2. If the four pair of polynomials in (29) and (31) are stable then, by the conditions on (15b) in Lemma 3, $\{f_n(\omega), g_n(\omega)\}$ is stable for $g_n(\omega)$ and *all admissible* $\{f_n(\omega)\}$.

Taking for the fixed polynomial in Observation 1, the four boundary polynomials $\underline{F}_n^+(\omega)$, $\bar{F}_n^+(\omega)$, $\underline{F}_n^-(\omega)$ and $\bar{F}_n^-(\omega)$, and for the fixed polynomial in Observation 2, the four boundary polynomials $\underline{G}_n^+(\omega)$, $\bar{G}_n^+(\omega)$, $\underline{G}_n^-(\omega)$ and $\bar{G}_n^-(\omega)$ yields immediately

Theorem 1. A necessary and sufficient condition for all polynomials in the infinite family of polynomials Δ_n, (1), to be stable, $\Delta_n \subset S^n$, is that all pairs $\{f_n(\omega), g_n(\omega)\}$, with $f_n(\omega)$ taken from the set

$$\{\underline{F}_n^+(\omega), \bar{F}_n^+(\omega), \underline{F}_n^-(\omega), \bar{F}_n(\omega)\} \tag{32a}$$

and $g_n(\omega)$ taken from the set

$$\{\underline{G}_n^+(\omega), \bar{G}_n^+(\omega), \underline{G}_n^-(\omega), \bar{G}_n^-(\omega)\} \tag{32b}$$

are stable.

This (preliminary) version of Kharitonov's criterion, requires the inspection of 16 polynomials. However, it is possible to decide whether $\Delta_n \subset S^n$ from a lower number of deterministic polynomials. To see this, we return to Lemma 4 and notice that the conditions there can be grouped in an interesting different way; Parts (a) and (b) in Lemma 4 can be combined and rephrased to state that if the four pairs,

$$\{\bar{F}_n^+(\omega), \bar{G}_n^+(\omega)\}, \{\bar{F}_n^+(\omega), \underline{G}_n^+(\omega)\},$$
$$\{\underline{F}_n^+(\omega), \bar{G}_n^+(\omega)\}, \{\underline{F}_n^+(\omega), \underline{G}_n^+(\omega)\} \tag{33}$$

are stable then the "lossless behavior" is guaranteed for $\omega \geqslant 0$, for *all admissible pairs* $\{f_n(\omega), g_n(\omega)\}$. Here, "lossless behavior" is taken to mean real, simple poles with positive residues of $R_n(\omega)$ or its inverse. Similarly, the combination of parts (c) and (d) in Lemma 4 implies that if the four pairs,

$$\{\bar{F}_n^-(\omega), \bar{G}_n^-(\omega)\}, \{\bar{F}_n^-(\omega), \underline{G}_n(\omega)\},$$
$$\{\underline{F}_n^-(\omega), \bar{G}_n^-(\omega)\}, \{\underline{F}_n(\omega), \underline{G}_n(\omega)\} \tag{34}$$

are stable then the "lossless behavior" is guaranteed for $\omega \leqslant 0$, for *all admissible pairs* $\{f_n(\omega), g_n(\omega)\}$. It remains to show that the stability of the eight pairs of polynomials in (33) and (34) is a sufficient condition for $\Delta_n \subset S^n$ (they are trivially also necessary, see Remark 11 in the sequel) and can replace the 16 pairs in Theorem 1. This can be seen as follows. Since $R_n(\omega)$ corresponds via (14) to $\rho_n(s)$, if it is "lossless" along all $\omega \in [-\infty, +\infty]$ then $\rho_n(s)$ is LPR. But, if both (33) and (34) represent stable pairs then clearly $R_n(\omega)$ is "lossless" along $[-\infty, 0]$ and $[0, \infty]$. The "lossless" behavior passes smoothly through the joint point $\omega = 0$, since each respective *plus* and *minus* superscripted two boundary polynomials, $\underline{G}_n^{\pm}$, $\bar{G}_n^{+}$, $\underline{F}_n^{+}$ and $\bar{F}_n^{+}$, take a common value at $\omega = 0$. The other joint point, viewing Γ as a closed curve, is at infinity, where the two points $\omega = \pm\infty$ coincide and if $R_n(\omega)$ has a pole there, the positivity of its residue is guaranteed either by the limit at $+\infty$ or $-\infty$. This completes the proof for the following complex version of Kharitonov's theorem.

Theorem 2. Necessary and sufficient conditions for all polynomials in the infinite family of polynomials Δ_n, defined in (1), to be Hurwitz is that the eight pairs in (33) and (34) are stable. Equivalently, $\Delta_n \subset S^n$ if, and only, if, the following 8 polynomials are Hurwitz

$$\begin{aligned}
H_n^{(1)}(s) &= (\bar{a}_0 + j\bar{b}_0) + (\bar{a}_1 + j\underline{b}_1)s + (\underline{a}_2 + j\underline{b}_2)s^2 + (\underline{a}_3 + j\bar{b}_3)s^3 + (\bar{a}_4 + j\bar{b}_4)s^4 + \cdots \\
H_n^{(2)}(s) &= (\bar{a}_0 + j\underline{b}_0) + (\underline{a}_e 1 + j\underline{b}_1)s + (\underline{a}_2 + j\bar{b}_2)s^2 + (\bar{a}_3 + j\bar{b}_3)s^3 + (\bar{a}_4 + j\underline{b}_4)s^4 + \cdots \\
H_n^{(3)}(s) &= (\underline{a}_0 + j\bar{b}_0) + (\bar{a}_1 + j\bar{b}_1)s + (\bar{a}_2 + j\underline{b}_2)s^2 + (\underline{a}_3 + j\underline{b}_3)s^3 + (\underline{a}_4 + j\bar{b}_4)s^4 + \cdots \\
H_n^{(4)}(s) &= (\underline{a}_0 + j\underline{b}_0) + (\underline{a}_1 + j\bar{b}_1)s + (\bar{a}_2 + j\bar{b}_2)s^2 + (\bar{a}_3 + j\underline{b}_3)s^3 + (\underline{a}_4 + j\underline{b}_4)s^4 + \cdots \\
H_n^{(5)}(s) &= (\bar{a}_0 + j\bar{b}_0) + (\underline{a}_1 + j\bar{b}_1)s + (\underline{a}_2 + j\underline{b}_2)s^2 + (\bar{a}_3 + j\underline{b}_3)s^3 + (\bar{a}_4 + j\bar{b}_4)s^4 + \cdots \\
H_n^{(6)}(s) &= (\bar{a}_0 + j\underline{b}_0) + (\bar{a}_1 + j\bar{b}_1)s + (\underline{a}_2 + j\bar{b}_2)s^2 + (\underline{a}_3 + j\underline{b}_3)s^3 + (\bar{a}_4 + j\underline{b}_4)s^4 + \cdots \\
H_n^{(7)}(s) &= (\underline{a}_0 + j\bar{b}_0) + (\underline{a}_1 + j\underline{b}_1)s + (\bar{a}_2 + j\underline{b}_2)s^2 + (\bar{a}_3 + j\bar{b}_3)s^3 + (\underline{a}_4 + j\bar{b}_4)s^4 + \cdots \\
H_n^{(8)}(s) &= (\underline{a}_0 + j\underline{b}_0) + (\bar{a}_1 + j\underline{b}_1)s + (\bar{a}_2 + j\bar{b}_2)s^2 + (\underline{a}_3 + j\bar{b}_3)s^3 + (\underline{a}_4 + j\underline{b}_4)s^4 - \cdots .
\end{aligned} \tag{35}$$

The equivalence of the eight real pairs in (33) and (34) and the eight complex polynomials in (35) follows from (10) to (12).

Remarks

(12) We did not comment so far on the necessity that has been added to the conditions in Theorems 1, 2, and before that, to related statements. The necessity of the conditions in each of these cases is trivial because the problem is formulated such that the boundary polynomials (or pairs) are themselves required to belong to the stable family.

(13) Without the requirement that the bounds of the uncertainty intervals be attained, the criterion provides (least restrictive but just) sufficient conditions. Thus in practice it is important to remember that an indication that Δ_n is not in S^n does not exclude the existence of tighter bounds on the coefficients with a corresponding subset $\tilde{\Delta}_n \subset \Delta_n$ such that $\tilde{\Delta}_n \subset S^n$. Kharitonov's theorem can be applied in fact to find maximal vicinity intervals around nominal coefficient values of a (complex or real) Hurwitz polynomial such that the corresponding family of polynomials Δ_n is in S^n [5]–[9].

In considering the use of the robust stability criterion of Theorem 2, one immediately thinks of testing the stability of each of the deterministic complex polynomials by an appropriate test. Appropriate tests are the (complex) continued fraction expansion test of Frank [11] and Routh table type tests [12], [13]. By a formulation due to Bilharz [13], the stability test of a complex polynomial of degree n can be carried out by examining a Routh table for a real polynomial of degree $2n$ [12]. With a slight modification of the latter procedure, it can be shown that the initial first and second rows for each of the 8 modified Routh tables needed for Kharitonov's criterion, are given precisely by the coefficient vectors of the first and second polynomials in the pairs (33) and (34). Therefore, the robust stability criterion can be carried out by testing directly the stability of the eight pairs of (33) and (34) without a further actual formation of the eight polynomials. At the same time, it is important to realize that, compared to the difficulty of the robust stability problem at first sight, Kharitonov's criterion is of merit even if used with numerical calculation of the zeros of the 8 polynomials. In fact, with the accessibility of computers (and advance of calculators), today, this latter approach may be convenient and the most immediate way of using the criterion in unrepeated applications (at least for polynomials of not too high degrees).

Finally, we consider Kharitonov's theorem for special case of real polynomials. The zeros of a real polynomial are located in symmetry with respect to the real axis of the complex plane. Reviewing the line of arguments from Lemma 4 to Theorem 2, the main changes (simplifications) in the more general derivation outlined so far would be the dropping everywhere of the separate consideration of the $\omega < 0$ cases together with the corresponding boundary functions that have *minus* superscripts, because positivity of residues (or other "lossless" properties) for $\omega > 0$ induces, by reflection with respect to the real axis, similar properties also for the lower half plane. Therefore, in the real case, it is sufficient to require the stability of the pairs in (33) only, or equivalently, to require only that the first four polynomials in (35) be Hurwitz. It is noticed also that the even and odd parts (6) of a real polynomial $h_n(s) = \sum_{i=0}^{n} a_i s^i$ involve only even and odd powers of s.

$$e_n(s) = \sum_{i=0}^{m} a_{2i} s^{2i}, \qquad o_n(s) = \sum_{i=0}^{m} a_{2i+1} s^{2i+1} \tag{36}$$

where $n = 2m$ (and $a_{2m+1} = 0$) or $n = 2m+1$. Similarly, the real polynomials $f_n(\omega)$ and $g_n(\omega)$ in the decomposition (10) have only even and odd powers of ω, respectively,

$$f_n(\omega) = \sum_{i=0}^{m} (-1)^i a_{2i} \omega^{2i} \qquad g_n(\omega) = \sum_{i=0}^{m} (-1)^i a_{2i+1} \omega^{2i+1}. \tag{37}$$

Thus the boundary real polynomials that participate in (33) take now the forms

$$\begin{aligned} \overline{F}_n(\omega) &= \bar{a}_0 - \underline{a}_2\omega^2 + \bar{a}_4\omega^4 - \underline{a}_6\omega^6 + \cdots \\ \underline{F}_n(\omega) &= \underline{a}_0 - \bar{a}_2\omega^2 + \underline{a}_4\omega^4 - \bar{a}_6\omega^6 + \cdots \\ \overline{G}_n(\omega) &= \bar{a}_1\omega - \underline{a}_3\omega^3 + \bar{a}_5\omega^5 - \underline{a}_7\omega^7 + \cdots \\ \underline{G}_n(\omega) &= \underline{a}_1\omega - \bar{a}_3\omega^3 + \underline{a}_5\omega^5 - \bar{a}_7\omega^7 + \cdots . \end{aligned} \tag{38}$$

and the criterion for real polynomials becomes

Corollary 1. All the polynomials in the infinite family of real polynomials

$$\Delta_n = \left\{ h_n(s) \mid h_n(s) = \sum_{i=0}^{n} a_i s^i, a_i \in [\underline{a}_i, \bar{a}_i], i = 1, \cdots, n \right\} \tag{39}$$

are Hurwitz if, and only if, the four pairs in (38) are stable, or equivalently, if and only if, the following four polynomials are Hurwitz:

$$H_n^{(1)}(s) = \underline{a}_0 + \underline{a}_1 s + \bar{a}_2 s^2 + \bar{a}_3 s^3 + \underline{a}_4 s^4 + \underline{a}_5 s^5 + \bar{a}_6 s^6 + \bar{a}_7 s^7 + \cdots \tag{40.1}$$

$$H_n^{(2)}(s) = \underline{a}_0 + \bar{a}_1 s + \bar{a}_2 s^2 + \underline{a}_3 s^3 + \underline{a}_4 s^4 + \bar{a}_5 s^5 + \bar{a}_6 s^6 + \underline{a}_7 s^7 + \cdots \tag{40.2}$$

$$H_n^{(3)}(s) = \bar{a}_0 + \bar{a}_1 s + \underline{a}_2 s^2 + \underline{a}_3 s^3 + \bar{a}_4 s^4 + \bar{a}_5 s^5 + \underline{a}_6 s^6 + \underline{a}_7 s^7 + \cdots \tag{40.3}$$

$$H_n^{(4)}(s) = \bar{a}_0 + \underline{a}_1 s + \underline{a}_2 s^2 + \bar{a}_3 s^3 + \bar{a}_4 s^4 + \underline{a}_5 s^5 + \underline{a}_6 s^6 + \bar{a}_7 s^7 + \cdots . \tag{40.4}$$

We obtained Corollary 1 by indicating what should be the main differences in an independent derivation based on LPR functions. Of course, this corollary can be deduced from the complex criterion simply by setting everywhere the imaginary parts b_i of the coefficients to zero. It is then clear that (34) repeats (33) when the boundary polynomials are just (38). Similarly, the last and first four polynomials in (35) become identical and simplify into the four in (40).

Numerical Illustration. We illustrate Theorem 2 by a simple numerical example. Suppose the coefficients of a third-degree polynomial

$$h_3(s) = \sum_{i=0}^{3} (a_i + jb_i) s_i$$

may take values in the following intervals:

$$a_0 \in [25,36],\ b_0 \in [42,56],\ a_1 \in [5,8],\ b_1 \in [20,25]$$
$$a_2 \in [1,4],\ b_2 \in [7,10],\ a_3 \in [0.6,1],\ b_3 \in [0.7,1.1].$$

To determine whether $h_3(s)$ is Hurwitz for all the above admissible coefficients values, one has to test the stability of eight fixed polynomials which are given for this case, see (35), by

$$H_3^{(1)}(s) = (36 + j56) + (8 + j20)s + (1 + j7)s^2 + (0.6 + j1.1)s^3$$
$$H_3^{(2)}(s) = (36 + j42) + (5 + j20)s + (1 + j10)s^2 + (1 + j1.1)s^3$$
$$H_3^{(3)}(s) = (25 + j56) + (8 + j25)s + (4 + j7)s^2 + (0.6 + j0.7)s^3$$
$$H_3^{(4)}(s) = (25 + j42) + (5 + j25)s + (4 + j10)s^2 + (1 + j0.7)s^3$$
$$H_3^{(5)}(s) = (36 + j56) + (5 + j25)s + (1 + j7)s^2 + (1 + j0.7)s^3$$
$$H_3^{(6)}(s) = (36 + j42) + (8 + j25)s + (1 + j10)s^2 + (0.6 + j0.7)s^3$$
$$H_3^{(7)}(s) = (25 + j56) + (5 + j20)s + (4 + j7)s^2 + (1 + j1.1)s^3$$
$$H_3^{(8)}(s) = (25 + j42) + (8 + j20)s + (4 + j10)s^2 + (0.6 + j1.1)s^3.$$

Applying an appropriate stability test (or computing the zeros) for each of these 8 polynomials shows that they are all Hurwitz. Therefore, $h_3(s)$ is Hurwitz for all the admissible variations of its coefficients.

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The Lyapunov Equation for *n*-Dimensional Discrete Systems

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Abstract **—The necessary and sufficient conditions for the existence of positive definite solutions to the *n*-D Lyapunov equation are presented. It is shown that such an existence is sufficient but not necessary for *n*-D stability. Furthermore, the extension from the 2-D to the *n*-D Lyapunov equation ($n \geqq 3$) is discussed and it is shown that some properties of the 2-D Lyapunov equation cannot be extended to the *n*-D ($n \geqq 3$) case.**

I. Introduction

The study of stability of multidimensional systems in state-space description is important for both, the design and implementation of such systems. An approach to this problem has been the extension of the Lyapunov equation to the *n*-dimensional case. This was first noted in [1] for the *n*-dimensional continuous case and extended in [2] to the *n*-dimensional discrete case using the *n*-dimensional bilinear transformation. In [3] both the necessary and sufficient conditions for the existence of positive definite solutions to the 2-D Lyapunov equation were developed based on strictly bounded real functions. These strictly bounded real conditions were compared with the 2-D stability conditions and it was shown that they are stronger than the 2-D stability conditions. This was demonstrated with an example of a stable 2-D system realization for which no positive definite solution to the 2-D Lyapunov equation exists.

This paper deals with the extension of these results on the 2-D Lyapunov equation to the *n*-D case as indicated in [4]. In Section II the *n*-D discrete state-space model is presented and some results on the 2-D Lyapunov equation are outlined. In Section III the necessary and sufficient conditions for the existence of positive definite solutions to the *n*-D Lyapunov equation are presented, and it is shown that they are stronger than the *n*-D stability condition. Finally, in Section IV some special cases are discussed and it is shown that some results from 2-D case cannot be extended to the *n*-D case.

Notation: $\bar{U}^n$ denotes the closed unit *n*-disk

$$\bar{U}^n = \{(z_1,\cdots,z_n) \mid |z_1| \leqq 1 \cdots |z_n| \leqq 1\}.$$

T^n the distinguished boundary of $\bar{U}^n$

$$T^n = \{(z_1,\cdots,z_n) \mid |z_1 = 1 \cdots |z_n| = 1\}.$$

I_n the $n \times n$ unit matrix, $\oplus$ the direct sum of matrices and for a symmetric matrix W, $W > 0$ indicates that W is positive definite.

II. Preliminaries

Linear shift invariant *n*-D systems can be represented by a state-space model of the following form [5], [7]:

$$\begin{bmatrix} x_1(i_1+1, i_2,\cdots,i_n) \\ x_2(i_1, i_2+1,\cdots,i_n) \\ \vdots \\ x_n(i_1,i_2,\cdots,i_n+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{in} \\ A_{21} & & & \vdots \\ \vdots & & & \vdots \\ A_{n1} & \cdots & \cdots & A_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1(i_1,i_2,\cdots,i_n) \\ \vdots \\ x_n(i_1,i_2,\cdots,i_n) \end{bmatrix} + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u(i_1,\cdots,i_n) \quad (1)$$

$$y(i_1,i_2,\cdots,i_n) = [C_1 \cdots C_n] \begin{bmatrix} x_1(i_1,\cdots,i_n) \\ \vdots \\ x_n(i_1,\cdots,i_n) \end{bmatrix} \quad (2)$$

where $x_j \in \mathbb{R}^{m_j}$, $j = 1,\cdots,n$ represent the states, u is the input and y the output. The stability of a *n*-D system described by such a model depends on the locations of the zeros of the

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