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A GOHBERG-SEMENCUL INVERSION FORMULA FOR ADMISSIBLE QUASI-TOEPLITZ MATRICES AND ITS GENERATION BY LEVINSON ALGORITHMS †

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Abstract. The inverse of certain close to Toeplitz matrices known as Quasi-Toeplitz (QT) matrices is totally defined by four vectors (two for symmetric matrices) and can be expressed as the sum of two lower-upper triangular matrix products known for Toeplitz matrices as the Gohberg-Semencul (G-S) inversion formula. Here we consider an important subclass of *admissible* QT matrices that arise in modeling signal propagation in lossless layered media when the assumption of "perfect reflection" at the surface (that leads to a Toeplitz matrix) is replaced by a more realistic assumption of partial reflectance. We derive a G-S type inversion formula for admissible QT matrices and show that their generating vectors are given by a Levinson algorithm for such matrices in just $3n^2$ $(1.5n^2)$ arithmetic operations for an $(n+1)$ sized non-Hermitian (Hermitian, resp.,) matrix. The new results may become useful in applications that employ such models (e.g., Speech, Seismology) and where low storage or fast computation is important.

I. Introduction

A non-Hermitian quasi-Toeplitz (QT) matrix is a matrix that can be written in the form

$$\mathbf{R}_n = L(\tilde{\mathbf{u}})\, L^t(\mathbf{u}) - L(\tilde{\mathbf{v}})\, L^t(\mathbf{v}) \tag{1}$$

where $L(\mathbf{a}_n)$ denotes the lower triangular Toeplitz matrix with first column $\mathbf{a}_n$. The matrix $\mathbf{R}_n = (r_{i,j})$ (say with $r_{0,0} = 1$), is defined by four *generating* vectors

$$\tilde{\mathbf{u}} = \begin{bmatrix} 1 \\ \tilde{u}_1 \\ \vdots \\ \tilde{u}_n \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \tilde{\mathbf{v}} = \begin{bmatrix} 0 \\ \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ v_1 \\ \vdots \\ v_n \end{bmatrix} \tag{2}$$

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The Toeplitz matrix is a special case of this structure for the choice $u_k = v_k$ and $\tilde{u}_k = \tilde{v}_k$ to yield

$$\mathbf{T}_n = [\, c_{i-j} \,] \quad , \quad c_0 = 1 \quad ; \quad c_k = v_k \quad , \quad c_{-k} = \tilde{v}_k \quad , \ k > 0 \tag{3}$$

A matrix $\mathbf{R}_n$ is calledQT if its displacement $\Delta\{\mathbf{R}_n\}$ defined below [1], is of rank two,

$$\Delta\{\mathbf{R}_n\} := \mathbf{R}_n - \mathbf{Z}_n \mathbf{R}_n \mathbf{Z}_n^t = \tilde{\mathbf{u}}\, \mathbf{u}^t \ - \ \tilde{\mathbf{v}}\, \mathbf{v}^t \tag{4}$$

Here, $\mathbf{Z}$ is the shift matrix with $1's$ on the main sub-diagonal and zeros elsewhere. Even when a matrix is known to be QT from the modeling or mathematical context its displacement matrix helps to obtain its generating vectors. Relation (4) also clarifies why there is no loss of generality (beyond the scaling to $\mathbf{R}_n$ with $r_{0,0} = 1$) in assuming the form (2),(i.e. with $\tilde{v}_0 = v_0 = 0$) for the generating vectors.

Another characterization of a QT matrix that we shall use is that $\mathbf{R}_n$ is close to a "hidden" Toeplitz matrix $\mathbf{T}_n$ through the similarity relation [2],

$$\mathbf{R}_n = L(\,\tilde{\mathbf{h}}_{0:n}\,)\mathbf{T}_n L^t(\,\mathbf{h}_{0:n}\,) \quad ; \quad \tilde{\mathbf{h}}_{0:n} := \tilde{\mathbf{u}} - \tilde{\mathbf{v}} \quad , \quad \mathbf{h}_{0:n} := \mathbf{u} - \mathbf{v} \tag{5}$$

This relation extends the congruency between a Hermitian QT class and a Toeplitz matrix via a lower triangular Toeplitz matrix [1-5].

The QT matrix $\mathbf{R}_n$ is said to be *admissible*, extending the term from Hermitian matrices [3], if

$$u(z) = 1 + \alpha_0 v(z) \quad , \quad \tilde{u}(z) = 1 + \beta_0 \tilde{v}(z) \quad . \tag{6a,b}$$

Here and throughout we associate vectors $\mathbf{a}_n$ with polynomials $a_n(z)$ by the convention

$$\mathbf{a}_m = [a_{m0}\,,\, a_{m,1}\,,\ \cdots\ ,\, a_{m,m}]^t \quad , \quad a_m(z) = [1\,,\, z\,,\ \cdots\ ,\, z^m]\mathbf{a}_m = \sum_{i=0}^{m} a_{m,i} z^i$$

Admissible QT matrices form a subclass of QT matrices by imposing a relation between the generating vectors (2) of $\mathbf{R}_n$ as in Figure 1. The Toeplitz matrix is a member in this subclass that corresponds to the choice $\alpha = \beta = 1$.

Some additional conventions of notation that we shall subsequently need are as follows. Let $\mathbf{J}$ be a square matrix with 1's on the anti-diagonal and zeros elsewhere and with size determined by context. Then $\overleftrightarrow{\mathbf{a}}_n = \mathbf{J}\mathbf{a}_n$ denotes the reverse of the vector $\mathbf{a}_n$ and $\overleftrightarrow{a}_n(z)$ is the revered polynomial associated with $\overleftrightarrow{\mathbf{a}}_n$, that is $\overleftrightarrow{a}_n(z) = z^n a_n(z)$. For a matrix $\mathbf{M}_n$ reversion is defined as $\overleftrightarrow{\mathbf{M}}_n$:=$\mathbf{J}\mathbf{M}_n\mathbf{J}$. We also use $\downarrow\mathbf{a}_n := \mathbf{Z}\mathbf{a}_n$ to denote down-shift of the vector $\mathbf{a}_n$.

If $\mathbf{R}_n$ is QT then $\overleftrightarrow{\mathbf{R}}_n^{-1}$ is also QT. This follows from the algebraic identity [1],

$$rank\,\{\mathbf{M}_n - \mathbf{Z}\mathbf{M}_n\mathbf{Z}^t\} = rank\,\{\mathbf{M}_n^{-1} - \mathbf{Z}^t\mathbf{M}_n^{-1}\mathbf{Z}\} \quad . \tag{7}$$

Generally the inverse of a QT $\mathbf{R}_n$ is not in itself QT. However, the inverse of a Toeplitz matrix is in itself QT due to its pery-symmetry, i.e the property $\overleftrightarrow{\mathbf{T}}_n = \mathbf{T}_n^t$, that holds for a Toeplitz matrix.

The fact that $\mathbf{T}_n^{-1}$ is QT renders the inversion formula

$$\mathbf{T}_n^{-1} = \frac{1}{D_n}\left\{L(\overleftrightarrow{\mathbf{b}}_n)L^t(\overleftrightarrow{\mathbf{a}}_n) - L(\downarrow\hat{\mathbf{a}}_n)L^t(\downarrow\hat{\mathbf{b}}_n)\right\} \tag{8}$$

obtained by Gohberg and Semencul (G-S) [6]. The generating vectors in (8) are the solution of the normal equations

$$\mathbf{b}_m^t\mathbf{R}_m = [0,...,0,D_m] \quad , \quad \mathbf{R}_m\mathbf{a}_m = [0,...,0,D_m]^t \quad , \tag{9a,b}$$

with $\mathbf{R}_n = \mathbf{T}_n$. An efficient way to derive these solutions is by the following Levinson algorithm for non-Hermitian Toeplitz matrices.

$$\begin{bmatrix} \hat{a}_m(z) \\ \overleftrightarrow{b}_m(z) \end{bmatrix} = \begin{bmatrix} 1 & -k_m \\ -\xi_m & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_{m-1}(z) \\ \overleftrightarrow{b}_{m-1}(z) \end{bmatrix} , \quad \begin{matrix} \hat{a}_0(z) = 1 \\ \overleftrightarrow{b}_0(z) = 1 \end{matrix} \tag{10a}$$

and two inner products

$$k_m = \frac{[u_1, \ldots, u_m]\hat{\mathbf{a}}_{m-1}}{D_{m-1}} \quad , \quad \xi_m = \frac{[\tilde{u}_1, \ldots, \tilde{u}_m]\hat{\mathbf{b}}_{m-1}}{D_{m-1}} \tag{10b,c}$$

$$D_m = (1 - \xi_m k_m)D_{m-1} \quad , \quad D_0 = 1 \tag{10d}$$

This Levinson algorithm was obtained in [2]. It is closely related to several other recursive explicit derivation of the inverse $\mathbf{T}_n^{-1}$ [7-11]. The recursion (10) can be traced in these references but they miss the inner product formulae that complements the recursions to an executable Levinson type algorithm. It reduces to the classical Levinson algorithm for Hermitian Toeplitz matrices when $b_m(z) = \overleftrightarrow{a}_m(z)$ and $\xi_m = k_m^*$. We note that the G-S formula (8) was derived originally for non-Hermitian Toeplitz matrices.

Admissible QT matrices represent a relatively simple deviation from Toeplitz matrices that deserves special attention due to its significant physical interpretation. In modeling a layered media (see [5] and [2] for Figures and details), the Toeplitz matrix correspond to an ideal lossless media while the admissible QT matrix incorporates into the model allowance for partial surface reflection without, as shown, a substantial increase in complexity of the associated inversion formulae and computational procedures. Modeling problems of this type often involve matrices of large sizes where fast Levinson-like algorithms and inversion formulae with low storage requirements like the G-S formula are desirable or even necessary.

An extension of the Levinson algorithm to solve efficiently the normal equations (9) for admissible QT matrices was obtained in [2] and is given as follows. The recursions become

$$\begin{bmatrix} a_m(z) \\ \alpha_m(z) \end{bmatrix} = \begin{bmatrix} 1 & -k_m \\ -\xi_m & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{m-1}(z) \\ \alpha_{m-1}(z) \end{bmatrix} , \quad \begin{matrix} a_0(z) = 1 \\ \alpha_0(z) = \alpha_0 \end{matrix} \tag{11a}$$

$$\begin{bmatrix} b_m(z) \\ \beta_m(z) \end{bmatrix} = \begin{bmatrix} 1 & -\xi_m \\ -k_m & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_{m-1}(z) \\ \beta_{m-1}(z) \end{bmatrix}, \quad \begin{matrix} b_0(z) = 1 \\ \beta_0(z) = \beta_0 \end{matrix} \tag{11b}$$

The *reflection coefficients* are to be calculated by *two* inner products

$$k_m = \frac{[\nu_1, \ldots, \nu_m]\mathbf{a}_{m-1}}{D_{m-1}}, \quad \xi_m = \frac{[\tilde{\nu}_1, \ldots, \tilde{\nu}_m]\mathbf{b}_{m-1}}{D_{m-1}} \tag{11c,d}$$

$$D_m = (1 - \xi_m k_m)D_{m-1}, \quad D_0 = 1 \tag{11e}$$

II. Main Result

The existence of a G-S formula for the revered inverse of a QT matrix is evident from its being a QT matrix. Our subsequent theorem manifests the existence of a G-S formula for admissible QT with generating vectors that can be derived efficiently by the above (11) Levinson algorithm.

Theorem. Let $\mathbf{R}_m$ be the the $(m+1)\times(m+1)$ upper left sub-matrix of a n admissible QT matrix $\mathbf{R}_n$. Then, for all $m=1,\ldots,n$,

$$\overleftrightarrow{\mathbf{R}}_m^{-1} = \frac{1}{D_m}\left\{L(\overleftrightarrow{\mathbf{a}}_m)L^t(\overleftrightarrow{\mathbf{b}}_m) - L(\downarrow\overleftrightarrow{\boldsymbol{\alpha}}_m)L^t(\downarrow\overleftrightarrow{\boldsymbol{\beta}}_m)\right\} \tag{12}$$

where $\mathbf{a}_m$, $\boldsymbol{\alpha}_m$, $\mathbf{b}_m$ and $\boldsymbol{\beta}_m$ are the vectors at step m of the algorithm (11).

In other words, the inverse of $\mathbf{R}_m$ is given by

$$\mathbf{R}_m^{-1} = \frac{1}{D_m}\left\{L^t(\overleftrightarrow{\mathbf{a}}_m)L(\overleftrightarrow{\mathbf{b}}_m) - L^t(\downarrow\overleftrightarrow{\boldsymbol{\alpha}}_m)L(\downarrow\overleftrightarrow{\boldsymbol{\beta}}_m)\right\} \tag{13}$$

where the four generating vectors are reverse of the vectors at the m-th step in the Levinson algorithm (11).

The particularization of this result for the Hermitian admissible QT case is also considered to be new. If $\mathbf{R}_n$ is Hermitian, (12b) and (12d) are redundant conjugate replica of (12a) and (12c) yielding the admissible Hermitian QT algorithm [4] for which we have (with ^{H} denoting conjugate transpose)

$$\overleftrightarrow{\mathbf{R}}_m^{-1} = \frac{1}{D_m}\left\{L(\overleftrightarrow{\mathbf{a}}_m)L^H(\overleftrightarrow{\mathbf{a}}_m) - L(\downarrow\overleftrightarrow{\boldsymbol{\alpha}}_m)L^H(\downarrow\overleftrightarrow{\boldsymbol{\alpha}}_m)\right\} \tag{14}$$

III. Alternative Algorithms

Another procedure to derive generating vectors for non-admissible QT matrix using a so called extended QT factorization algorithm was developed in [2]. It involves a recursive convolution algorithm procedure that combines the Schur and Levinson algorithms. It is not limited to admissible QT matrices and involves $7n^2$ (or $3n^2$) for obtaining generating vectors for the G-S inversion formula of a non-Hermitian (Hermitian, resp.) QT matrix, whether or not admissible. That is apparently the most efficient algorithm for a general QT matrix but since it does not exploit the admissibility constraints it involves more computation then the current result in the case of admissibility when it also involves more lines of programming then with the current Levinson algorithm. We also note that the generating vectors that the recursive convolution in [2] produces for the GS inversion formula there are not fully identical to the four vectors in (12). Due to space limitations in this conference paper we shall discuss the differences elsewhere.

There also exist alternative ways to derive the vectors for (12). For example, using the technique of moving from two-term to three-term recursions algorithms one could replace the (11) with the scheme

$$a_{m+1}(z) = (z - \frac{k_{m+1}}{k_m})a_m(z) - \frac{k_{m+1}}{k_m}(1 - \xi_m k_m)za_{m-1}(z) \tag{15a}$$

$$b_{m+1}(z) = (z - \frac{\xi_{m+1}}{\xi_m})b_m(z) - \frac{\xi_{m+1}}{\xi_m}(1 - \xi_m k_m)zb_{m-1}(z) \tag{15b}$$

$$k_{m+1} = \frac{[v_{01}, \ldots, v_{0,m+1}]\mathbf{a}_m}{D_m} \quad , \quad \xi_{m+1} = \frac{[\tilde{v}_{01}, \ldots, \tilde{v}_{0,m+1}]\mathbf{b}_m}{D_m} \tag{15c}$$

$$D_m = (1 - \xi_m k_m)D_{m-1} \quad , \quad D_0 = 1 \tag{15d}$$

The algorithm is initiated by

$$a_{-1}(z) = b_{-1}(z) = 0 \quad , \quad a_0(z) = b_0(z) = 1 \quad , \quad k_0 = \frac{1}{\alpha_0} \quad , \quad \xi_0 = \frac{1}{\beta_0} \tag{15e}$$

To obtain the generating vectors for (12) the recursions (15a-d) are carried out for m=0,1,...,$m-1$, then for once the suppressed two other variables are recovered using

$$\alpha_m(z) = \frac{1}{k_m}[\ a_m(z) + (1 - \xi_m k_m)\ a_{m-1}(z)\ ] \tag{15f}$$

$$\beta_m(z) = \frac{1}{\xi_m}[\ b_m(z) + (1 - \xi_m k_m)\ b_{m-1}(z)\ ] \tag{15g}$$

These variation requests the same amount of computation as (11).

Another option is to replace (11) by its immittance version as shown in [12-14]. There using a physically meaningful change from the current variables that describe moving forward and scattered backward waves to variables that rather present their volume-velocity and pressure in conjunction with the above technique of moving to three-term recursions new Levinson algorithms for admissible QT matrices that request $3n^2$ for non-Hermitian and (just) n^2 for Hermitian matrices were obtained. They too can be used to produce the generating vectors for the G-S formula in (12). Full details are found in [14].

Appendix - Proof of Theorem

A QT matrix $\mathbf{R}_n$ shares with the Toeplitz matrix $\mathbf{T}_n$ to which it is related by the "congruency" relation (5) its "reflection coefficients" ξ_m , k_m, and hence also D_m, for all m = 1,...,n [2]. The solutions to the normal equations (9), $\mathbf{a}_m$ and $\mathbf{b}_m$ for $\mathbf{R}_m$ and

$\hat{\mathbf{a}}_m$ and $\hat{\mathbf{b}}_m$ for $\mathbf{T}_m$ must be related by

$$L(\mathbf{h}_{0:m})\mathbf{a}_m = \hat{\mathbf{a}}_m \quad , \quad L(\tilde{\mathbf{h}}_{0:m})\mathbf{b}_m = \hat{\mathbf{b}}_m \tag{A.1}$$

This follows from reading the first column and row after taking the inverse on both sides of (5).

Next, invert and reverse the two sides of (5) and substitute for $\mathbf{T}_m^{-1}$ the G-S formula (8) to obtain

$$D_m \overleftrightarrow{\mathbf{R}}_m^{-1} = L^{-1}(\mathbf{h}_{0:m})L(\overleftrightarrow{\mathbf{a}}_m)L^t(\overleftrightarrow{\mathbf{b}}_m)L^{-t}(\tilde{\mathbf{h}}_{0:m}) - L^{-1}(\mathbf{h}_{0:m})L(\downarrow\hat{\mathbf{b}}_m)L^t(\downarrow\hat{\mathbf{a}}_m)L^{-t}(\tilde{\mathbf{h}}_{0:m})$$

$$= L(\overleftrightarrow{\mathbf{a}}_m)L^t(\overleftrightarrow{\mathbf{b}}_m) - L^{-1}(\mathbf{h}_{0:m})L(\downarrow\hat{\mathbf{b}}_m)L^t(\downarrow\hat{\mathbf{a}}_m)L^{-t}(\tilde{\mathbf{h}}_{0:m}) \tag{A.2}$$

where we first used the fact that for Toeplitz matrices (including the $L(\mathbf{a}_m)$ type terms) reversion is equal to transposition and then we applied (A.1).

Up to this point the derivation was not specific to admissible QT matrices. For an admissible matrix the normal solutions $\mathbf{a}_m$ and $\mathbf{b}_m$ can be derived by the algorithm (11). It follows from a comrarison of (A.2) with (12) that it remains to prove that the following relations also hold.

$$L(\mathbf{h}_{0:m})(\downarrow\overleftrightarrow{\boldsymbol{\alpha}}_m) = \downarrow\hat{\mathbf{b}}_m \quad , \quad L(\tilde{\mathbf{h}}_{0:m})(\downarrow\overleftrightarrow{\boldsymbol{\beta}}_m) = \hat{\mathbf{a}}_m \quad . \tag{A.3a,b}$$

For this we first write the recursions (11a,b) in reversed polynomials as

$$\begin{bmatrix} \overleftrightarrow{\alpha}_m(z) \\ \overleftrightarrow{a}_m(z) \end{bmatrix} = \begin{bmatrix} 1 & -\xi_m \\ -k_m & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overleftrightarrow{\alpha}_{m-1}(z) \\ \overleftrightarrow{a}_{m-1}(z) \end{bmatrix} , \quad \begin{matrix} \overleftrightarrow{\alpha}_0(z) = \alpha_0 \\ \overleftrightarrow{a}_0(z) = 1 \end{matrix} \tag{A.4a}$$

$$\begin{bmatrix} \overleftrightarrow{\beta}_m(z) \\ \overleftrightarrow{b}_m(z) \end{bmatrix} = \begin{bmatrix} 1 & -k_m \\ -\xi_m & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overleftrightarrow{\beta}_{m-1}(z) \\ \overleftrightarrow{b}_{m-1}(z) \end{bmatrix} , \quad \begin{matrix} \overleftrightarrow{\beta}_0(z) = \beta_0 \\ b_0(z) = 1 \end{matrix} \tag{A.4b}$$

Then we write the first row of polynomial equation in (A.4a) in vectorial form and pre-mutiply its both sides by $L(\mathbf{h}_{0:m})$ to obtain

$$L(h_{0:m})\overleftrightarrow{\boldsymbol{\alpha}}_m = -\xi_m \begin{bmatrix} \overleftrightarrow{\mathbf{a}}_{m-1} \\ x_m \end{bmatrix} + \begin{bmatrix} 0 \\ L(h_{0:m-1})\overleftrightarrow{\boldsymbol{\alpha}}_{m-1} \end{bmatrix} \tag{A.5a}$$

where we already used (A.1a) and where we defined the scalar $x_m := \mathbf{h}_{1:m}^t \mathbf{a}_{m-1}$. Now we want to compare the above update with the second row of the Toeplitz recursion (11) which in vectorial form and after reversion is equivalent to

$$\hat{\mathbf{b}}_m = -\xi_m \begin{bmatrix} \overleftrightarrow{\mathbf{a}}_{m-1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{\mathbf{b}}_{m-1} \end{bmatrix} \tag{A.6a}$$

It is seen that the first m of its $m+1$ rows are not affected by the difference in the last row due to the scalar x_m (A.5a) that is missing in (A.6a). So that an induction step $m-1 \rightarrow m$ may be invoked to prove (A.3a). A dual and similar induction step to verify (A.3b)starts with the first row in (A.4b) rewritten in vectorial form and pre-multiplied by $L(\tilde{\mathbf{h}}_{0:m})$ to give

$$L(\tilde{h}_{0:m})\overleftrightarrow{\boldsymbol{\beta}}_m = \begin{bmatrix} 0 \\ L(\tilde{h}_{0:m-1})\overleftrightarrow{\boldsymbol{\beta}}_{m-1} \end{bmatrix} - k_m \begin{bmatrix} \overleftrightarrow{\mathbf{b}}_{m-1} \\ y_m \end{bmatrix} \qquad \text{(A.5b)}$$

using (A.1b) and defining a second scalar $y_m = = \tilde{h}^t_{1:m}\mathbf{b}_{m-1}$. The comparison of this update with the one that follows from the first row of (10a),

$$\hat{\mathbf{a}}_m = \begin{bmatrix} 0 \\ \hat{\mathbf{a}}_{m-1} \end{bmatrix} - k_m \begin{bmatrix} \overleftrightarrow{\mathbf{b}}_{m-1} \\ 0 \end{bmatrix} \qquad \text{(A.6b)}$$

shows that if (A.3b) holds till $(m-1)$ then, since the first m entries in (A.5b) and (A.6b) are not affected by the existence or the missing of the scalar y_m in the last entry, therefore (A.3b) holds for m as well.

Since (A.3a,b) holds trivialy for $m = 0$, this completes the proof for the main theorem. We note that, using several identities derived in ([2], section 3), it is possible to obtain the following expressions for the the scalars x_m and y_m,

$$x_m = k_m(\alpha_0 - 1)D_{m-1} \quad , \quad y_m = \xi_m(\beta_0 - 1)D_{m-1} \quad ,$$

As expected they become zero for all m if and only if $\alpha_0 = \beta_0 = 1$, namely for the Toeplitz case.

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