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A Modified Unit-Circle Zero Location Test

Yuval Bistritz

***Abstract*—A modification to an efficient procedure to determine zero location with respect to the unit circle of polynomials with complex coefficients is presented. This form is more suitable to determine zero location constraints for polynomials with variable coefficients. It also bears a more direct relation to the Schur-Cohn (Marden-Jury tabular) test.**

I. Introduction

An algebraic unit-circle zero location test aims to determine the numbers, α, β, and γ, of zeros of a polynomial $P(z)$ inside, on, and outside the unit circle (IUC, UC, OUC)

$$C = \{z \mid z = e^{j\theta},\ \theta \in [0, 2\pi]\} \tag{1}$$

respectively, without their numerical calculation.

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The first direct algebraic solution to this zero location problem is the Schur-Cohn algorithm, with Schur obtaining a procedure for determining whether or not $\alpha = n$ ($P(z)$ is "stable") [1] and Cohn extending it to a full α, β, and γ determination for any polynomial [2]. This Schur-Cohn solution is more known today in a modified form obtained for it by Marden [3] and in several tabular forms advanced by Jury [4], [5]. Many variations on this test were published and were recently classified in [6]. A different approach to the unit circle zero location problem, has been obtained in [7] for real and in [8] for complex polynomials and it is the most efficient solution available today for this problem.

This brief contains a modification to the tests in [7] and [8] that is of equal strength and efficiency in ordinary use but one that offers a more suitable form for handling stability or zero location for polynomials whose coefficients are literal, depend on some parameters, or are function of further variables. Applications of this kind arise in determining stability constraints on parameters in feedback control of linear shift (discrete time) invariant (LSI) systems, in developing stability conditions for multidimensional LSI filters and other design problems of stable LSI systems. The current modification also bears a simpler relation to the Schur-Cohn and Marden-Jury tests.

The presentation here relies to some extent on [7] and [8]. We prove the current modified form by drawing an equivalence between it and a the treatment of a certain auxiliary polynomial by the test in its form in [8]. This provides a shorter proof than alternative independent proofs that could also be used. We also refer to [8] and [7] for the treatment of not regular cases after showing that their treatment in [8] and [7] is applicable also to the current modification.

II. The Algorithm

We shall currently consider the determination of the unit circle zero location distribution triplet (α, β, γ) for a polynomial

$$P(z) = p_0 + p_1 z + \cdots + p_n z^n, \quad \mathcal{R}e\{P(1)\} \neq 0,\ \ p_n \neq 0 \tag{2}$$

where the coefficients p_i are complex numbers. The assumption on $P(1)$ needs to be checked first. If $P(1) = 0$, then the algorithm described below is applied to the reduced degree polynomial obtained after zeros at $z = 1$ were removed (an operation that involves only addition arithmetics on the coefficients). Since a $P(1) = 0$ implies $\alpha \leq n$, this observation may already terminate a stability test. A case when $P(1) \neq 0$ but it is purely imaginary, may also be brought to terms with the assumption by without true multiplicative arithmetics, e.g., by $P(z) \Rightarrow jP(z)$ ($j := \sqrt{-1}$).

By difference, in the assumption in [8] is that $P(1) \neq 0$ and it is real. Thus when $P(1) = 0$, and further information on the zero location of this "not stable" polynomial is sought, both procedures involve a preliminary step of removing zeros at $z = 1$. The significant difference is in the next needed adjustments for a complex coefficient polynomial $P(z)$ with $P(1) \neq 0$. For a polynomial $P(z)$ to satisfy the requirement "$P(1)$ is real" it is necessary in general to multiply (or divide) it by a nontrivial complex number, e.g., by the complex conjugate $P(1)^*$ of $P(1)$. The operation $P(z) \Rightarrow P(1)^* P(z)$ means just n multiplications for a fixed coefficient polynomial but it has a more adverse impact on a polynomials with literal coefficients, or the coefficients depend on additional variables or parameters. In such cases that arise in generalized applications, the result of multiplying variable coefficients of the polynomial by a factor that is similarly a polynomial of secondary variables or a function of the parameters involved, complicates the coefficient dependencies of the polynomial that is passed to subsequent stages of the algorithm. In difference, the requirement in (2) may be satisfied without posing any further complicity on a generalized polynomials passed to subsequent steps, while, it will become apparent that, for fixed polynomial coefficients and in all other respects, the modified procedure is as simple and efficient as the original test. A manifest for the significance of the current modification is in that most of today's applications of algebraic stability tests involve the manipulation of generalized polynomials.

Normal Algorithm: Given a polynomial $P(z)$ as in (2) assign to it $n+2$ conjugate-symmetric polynomials $\{F_m(z) = \sum_{i=0}^{m} f_{m,i} z^i, m = n+1, \cdots, 0\}$ by the following algorithm.

Start with

$$F_{n+1}(z) = (z-1)(P(z) - P^\sharp(z)) \quad (3a)$$

$$F_n(z) = P(z) + P^\sharp(z) \quad (3b)$$

where $P^\sharp(z)$ denotes the (conjugate-) reciprocal of $P(z)$, *viz.*, $P^\sharp(z) = \sum_{i=0}^{n} p_{n-i}^\star z^i$ and * denotes complex conjugation.

Then obtain polynomials of descending degrees by the recursion

$$zF_{m-2} = (\delta_m + \delta_m^\star z)F_{m-1}(z) - F_m(z) \quad m = n+1, n, \cdots, 2 \quad (4a)$$

where

$$\delta_m = \frac{F_m(0)}{F_{m-1}(0)}. \quad (4b)$$

This algorithm is referred to as regular if the *normal conditions* that $F_{m-1}(0) \neq 0$ for all $m = n, \cdots, 1$ hold. For normal conditions the recursion provides a sequence of $n+2$ conjugate-symmetric (self reciprocal) polynomials, $F_m^\sharp(z) = F_m(z)$, where each $F_m(z)$ is of (full) degree m.

The regular recursion needs extensions for cases when a $F_{m-1}(0) = 0$ occurs. Such singular situation may be treated in the same way as in [8]. A brief account follows.

Structural Singularities: The occurrence of an $F_{s-1}(z) \equiv 0$ is referred to as *a "structural singularity"* at degree s. A structural singularities occurs if and only if the zeros of $F_s(z)$ are also zeros of $P(z)$, or, equivalently, iff $P(z)$ has UC zeros or zeros in reciprocal pairs (RP), z_r, and $\frac{1}{z_r^\star}$. (A symmetric polynomial like $F_s(z)$, has either zeros on C or in pairs located in reciprocal symmetry to it.) This case is treated by deriving from $F_{s-1}(z)$, as described in [8], a new pair of symmetric polynomials $\{F_{s-1}(z), F_{s-2}(z)\}$ then using them to resume the regular three-term recursion. If $P(z)$ has UC or RP of zeros of multiplicity higher than one then structural singularity will recur.

Patternless Singularities: Patternless singularities refers to the remaining cases when an $F_h(0) = 0$ (but $F_h(z) \not\equiv 0$) occurs. This situation is not related to any particular pattern of the location of zeros except that it implies that not all zeros are IUC. The case may be treated by deriving from $F_h(z)$ a pair $\{F_h(z), F_{h-1}(z)\}$, as described in [8], and use them to resume the regular recursion. This substitution does not interfere with the occurrence of structural singularities, their treatment and conclusions on UC or RP zeros described below.

In regular, as well as after singular, treatments, the polynomials $F_m(z)$ are always symmetric $F_m(z) = F_m^\sharp(z)$. This symmetry may be used to actually compute only half of the coefficients for each polynomial. It also follows that $\sigma_m := F_m(1)$ in the subsequent theorems are real. Also note that an efficient way to compute them normally is by the next parallel recursion

$$\sigma_{n+1} = 0, \quad \sigma_n = 2\mathcal{R}e\{P(1)\},$$
$$\sigma_{m-2} = 2\mathcal{R}e\{\delta_m\}\sigma_{m-1} - \sigma_m \quad m = n+1, n, \cdots, 2 \quad (5)$$

III. Zero Location Rules

Zero location rules for the algorithm detailed in the previous section are presented in this section in the form of three theorems that consider the solution in ascending degree of generality.

Theorem 1 (Stability Criterion): Necessary and sufficient conditions for $P(z)$ to have $\alpha = n$ IUC zeros are that all $F_m(1)$ are of a same sign (the sign of $\mathcal{R}e\{P(1)\}$), *viz.*,

$$\text{Sgn}\{F_m(1)\} = \text{Sgn}\{\mathcal{R}e\{P(1)\}\} \quad m = n, n-1, \cdots, 0. \quad (6)$$

The theorem also implicitly states that regularity is a necessary conditions for $P(z)$ to have only IUC zeros. Occurrence of a $f_{m-1,0} = 0$ implies $\alpha < n$. Similarly, a $\text{Sgn}\{F_m(1)\} \neq \text{Sgn}\mathcal{R}e\{P(1)\}$ condition implies too that the polynomial is not stable. For stability testing it is worthy to incorporate the evaluation of $F_m(1)$ into the recursion. This way, the procedure may be interrupted with the first indication of instability.

Theorem 2 (No Structural Singularity): When no structural singularity has been observed, $P(z)$ has $\alpha = n - \nu_n$ IUC zeros, no UC zeros and $\gamma = \nu_n$ OUC where ν_n denotes the number of sign variations ("Var") in the subsequent sequence of real numbers,

$$\nu_n = \text{Var}\{F_n(1), F_{n-1}(1), \cdots, F_1(1), F_0\}. \quad (7)$$

Theorem 3 (General Case): In the most general case $P(z)$ may have β UC zeros given by

$$\beta = 2\nu_s - s$$

where s is the degree that feature the first occurrence of a $F_{s-1}(z) \equiv 0$ ($s = 0$ stands for no structural singularities), and

$$\nu_s = \text{Var}\{F_s(1), F_{s-1}(1), \cdots, F_1(1), F_0\} \quad (8)$$

α IUC given by

$$\alpha = n - \nu_n$$

where

$$\nu_n = \text{Var}\{F_n(1), F_{n-1}(1), \cdots, F_1(1), F_0\} \quad (9)$$

and γ OUC zeros given by $\gamma = n - \alpha - \beta$.

As a proof for all the previous assertions it will suffice to prove the next lemma. It will relate them to their counterparts in [7] and [8] via an auxiliary polynomial

$$D_n(z) = \frac{1}{2}F_n(z) + \frac{1}{2}(z-1)F_{n-1}(z) \quad (10)$$

where $F_n(z)$ and $F_{n-1}(z)$ are the second and third polynomials produced by the algorithm of Section II for $P(z)$. Note that $D_n(1) = \frac{1}{2}[P(1) + P(1)^\star]$ is real and non zero as requested in [8]. Assume the algorithm in there yields for $D_n(z)$ the sequence $\{T_m(z) \quad m = n, \cdots, 0\}$.

Lemma 1:

a) $T_m(z) = F_m(z)$ for $m = n, \cdots, 0$
b) $D_n(z)$ and $P(z)$ have the same (α, β, γ) count of zero distribution with respect to the unit circle.

The rest of this section is devoted to proving this Lemma. First it is apparent that the method in [8] produces for $D_n(z)$ the next two first polynomials.

$$T_n(z) := [D_n(z) + D_n^\sharp(z)] = F_n(z) \quad (11a)$$

$$T_{n-1}(z) := [D_n(z) - D_n^\sharp(z)]/(z-1) = F_{n-1}(z). \quad (11b)$$

Part (a) of the Lemma follows from the fact that two consecutive degree polynomials of the three term recursion here and in [8] determines uniquely the remaining sequence. This fact holds also for the occurrence of singularities as they are treated in both cases identically.

To proceed, we shall express $D_n(z)$ in terms of $P(z)$. Using (3) and (4) one obtains

$$2zD_n(z) = (P(z) + P^\sharp(z))\big(z + \delta^\star_{n+1}z^2 + \delta_{n+1}z - \delta^\star_{n+1}z - \delta_{n+1}\big) + (P(z) - P^\sharp(z))(-z^2 + 2z - 1) \quad (12)$$

where $\delta_{n+1} = (p^\star_n - p_0)/(p^\star_n + p_0)$. Let us write the above as

$$zD(z) = P(z)A(z) + P^\sharp(z)B(z) \quad (13)$$

where $A(z)$ and $B(z)$ are two second degree polynomials defined by

$$2A(z) = -1 - \delta_{n+1} + \big(3 + \delta_{n+1} - \delta^\star_{n+1}\big)z + \big(-1 + \delta^\star_{n+1}\big)z^2 \quad (13a)$$

$$2B(z) = 1 - \delta_{n+1} + \big(-1 + \delta_{n+1} - \delta^\star_{n+1}\big)z + \big(1 + \delta^\star_{n+1}\big)z^2 \quad (13b)$$

Property 1: The polynomial $A(z)$ has one IUC and one OUC zeros.

We apply to $A(z)$ the zero location procedure [8] to prove this property.

First, since $A(1) = 1$, the preliminary requirement there is satisfied. Let the real part, imaginary part and the phase of δ_{n+1} be denoted by

$$\delta_{n+1} = \delta^R + j\delta^I = |\delta_{n+1}|e^{j\phi_\delta}. \quad (14)$$

The procedure in [8] produces for $A(z)$ the next sequence of symmetric polynomials

$$T_2(z) = -2 + 6z - 2z^2 \Rightarrow T_2(1) = 2 > 0$$
$$T_1(z) = 2\delta^R + j2\delta^I + (2\delta^R - j2\delta^I)z \Rightarrow T_1(1) = 4\delta^R$$
$$T_0(z) = -4\cos(\phi_\delta) - 6 \Rightarrow T_0(1) < 0 \quad \forall\phi_\delta$$

It is seen that $\mathrm{Var}\{T_2(1), T_1(1), T_0(1)\} = 1$ for any δ_{n+1}. From this it follows from the zero location rules in [8] that $A(z)$ has 1 OUC and 1 IUC zero.

Property 2:

$$|A(z)| > |B(z)| \quad \text{for all } 1 \neq z \in C, \quad A(1) = B(1). \quad (15)$$

The part $A(1) = B(1) = 1$ is apparent from (13). The inequality may be verified by an evaluation of $z^{-1}A(z)$ and $z^{-1}B(z)$ on $z \in C$. Let $z = \exp(j\psi)$ then

$$e^{-j\psi}A(e^{j\psi}) = 3 - 2\cos\psi - 2\delta^I\sin\psi + j2(\delta^R\sin\psi + \delta^I)$$
$$e^{-j\psi}B(e^{j\psi}) = -1 + 2\cos\psi - 2\delta^I\sin\psi + j2(\delta^R\sin\psi + \delta^I).$$

The two right-hand sides (RHS) have identical imaginary parts. Concurrently, the difference between the respective real parts is given by $4(1 - \cos\psi)$, an expression that is positive for all $\psi \in (0, 2\pi)$. The inequality in Property 2 follows.

Property 3: UC (as well as RP) zeros of $P(z)$ are also zero of $D_n(z)$ and vice versa.

Assume $P(z)$ has β UC zeros and denote by $U_\beta(z)$ the polynomial formed by the collection of these zeros. Since $U^\sharp_\beta(z) = U_\beta(z)$ these are also zeros of $P^\sharp(z)$. Thus $U_\beta(z)$ is a divisor for $F_{n+1}(z)$ $F_n(z)$ and by properties inherent to the underlying three-term recursion it is then also a factor of $F_{n-1}(z)$. Therefore $U_\beta(z)$ divides $D_n(z)$ by definition (10). Conversely, let $U_\beta(z)$ represent the polynomial of all UC zeros of $D_n(z)$. Then it is also a factor of $D^\sharp_n(z)$. Therefore, in view of (11), $U_\beta(z)$ is a factor common to both $F_n(z)$ and $F_{n-1}(z)$ and walking the three term recursion one step upward, $U_\beta(z)$ is a factor of also $F_{n+1}(z)$. Consequently by (3) $U_\beta(z)$ is a factor of $(z-1)F_n(z) + F_{n+1}(z) = (z-1)P(z)$, and as $D_n(z)$ is not allowed to vanish at $z = 1$, $U_\beta(z)$ is a factor of $P(z)$. Although the next observation will not be used, it is evident that the above reasoning equally holds to also show that $P(z)$ and $D_n(z)$ share RP of zeros. It is stressed that the above reasoning also holds for the extreme situations when $U_\beta(z)$ is of degree n or when $P(z) = P^\sharp(z)$.

In order the prove claim (b) of Lemma 1 we obtain from (13)

$$\frac{zD_n(z)}{P(z)A(z)} = 1 + \frac{P^\sharp(z)}{P(z)}\frac{B(z)}{A(z)} \quad (16)$$

and will apply to it the principle of argument with the using of the above properties to show that the number of IUC zeros of $D(z)$ and $P(z)$ is equal.

Assume first that $D(z)$ and $P(z)$ have no UC zeros. Then the mapping of the unit circle by $\frac{zD_n(z)}{P(z)A(z)}$ is well defined and we want to show that it does not encircle the origin. For this, it suffices to show that the real part in the RHS of (16) remains positive for all $z = \exp(j\psi) \in C$. For $1 \neq z \in C$ we have (using property 2)

$$\left|\frac{P^\sharp(z)}{P(z)}\frac{B(z)}{A(z)}\right| = \left|\frac{B(z)}{A(z)}\right| < 1$$

and the strict inequality guarantees that the real part in the RHS of (16) is positive. For the value $z = 1$ $A(1) = B(1)$ but a direct evaluation of the real part of the RHS of (16) yields

$$1 + \mathcal{R}e\left\{\frac{P^\sharp(1)}{P(1)}\right\} > 0 \quad \Leftrightarrow \quad \mathcal{R}e\{P(1)\} \neq 0$$

Consequently, adding the assumption in (2) on $P(z)$, it follows that the mapping of C by (16) never encircle the origin. Invoking the principle of argument it follows that $zD_n(z)$ has as many IUC zeros as $P(z)A(z)$. Therefore, $P(z)$ has α IUC zeros if and only if (iff) $P(z)A(z)$ has $\alpha + 1$ IUC zeros (using property 1), iff $zD_n(z)$ has $\alpha + 1$ IUC zeros, iff $D_n(z)$ has α IUC zeros. If there are no UC zeros than both $P(z)$ and $D_n(z)$ have $\gamma = n - \alpha$ OUC zeros.

Assume next that $\beta > 0$ UC zeros are present. Using Property 3, all UC zeros form a factor $U_\beta(z)$ that is common to $P(z)$, $D_n(z)$ and $P^\sharp(z)$. This factor cancels out from the rational function at both sides of (16). Let $P(z) = \tilde{P}(z)U_\beta(z)$ and $D_n(z) = \tilde{D}(z)U_\beta(z)$. It is possible to repeat the previous evaluation with $\tilde{P}(z)$ and $\tilde{D}(z)$ substituting $P(z)$ and $D_n(z)$, and show that ($\tilde{P}(z)$ and therefore) $P(z)$ has α IUC zeros, iff ($\tilde{D}(z)$ and therefore) $D_n(z)$ has α IUC zeros. This time because there are UC zeros, the resulting zero distribution for both $P(z)$ and $D_n(z)$ is: α IUC, β UC and $\gamma = n - \alpha - \beta$ OUC zeros. This completes the proof of Lemma 1.

The lemma provides at once a circuitous proof for all the assertions made for the modified test through corresponding results proven in [8] and [7].

IV. Additional Aspects

The current procedure may also be regarded as applying the method in [8] to the next polynomial

$$D_{n+1}(z) = (z - 1)P(z) \quad (17)$$

rather than to $P(z)$. This may be seen by applying the method in [8] to $D_{n+1}(z)$ to obtain that the two first polynomials there

$$T_{n+1}(z) := D_{n+1}(z) + D^\sharp_{n+1}(z) = (z - 1)(P(z) - P^\sharp(z))$$
$$T_n(z) := \big[D_{n+1}(z) - D^\sharp_{n+1}(z)\big]/(z - 1) = P(z) + P^\sharp(z)$$

coincide with the current initiation pair. The situation may be regarded as follows. After ensuring that $P(z)$ has no zeros at $z = 1$, the current procedure "deliberately" plants a zero at $z = 1$. Then the test in [8] is applied to this polynomial that is assured to have a zero of multiplicity 1 at $z = 1$. It is easily shown that the latter feature holds iff $T_{n+1}(1) = 0$ but $T_n(1) \neq 0$. Such a single UC zero, unlike other constellations of UC zeros, can not "drift" downward to show itself later as (part of) a structural singularity. With this property it also can not make all lower degree polynomials $T_m(z)$ vanish at $z = 1$, therefore no interference is caused for deducing information on zeros location from the signs of the $T_m(1)$'s.

Another interesting feature of the current sequence $\{F_m(z)\}$ is its closer tie to the sequence of polynomials that are formed for $P(z)$ by the Schur-Cohn and Marden-Jury unit-circle zero location procedure [3]–[6]. To exhibit this relation, assume a polynomial $P(z)$ for which a regular recursion obtains $\{F_m(z), m = n, \cdots, 0\}$ and let $\{a_m(z), m = n, \cdots, 0\}$ be the Schur-Cohn sequence of *monic* polynomials which are obtained for $P(z)$ starting with $a_n(z) = P(z)/p_n$ and proceeding with the recursion

$$za_{m-1}(z) = \frac{a_m(z) + k_m a_m^\sharp(z)}{1 - |k_m|^2} \quad k_m = \frac{a_m(0)}{a_m^\sharp(0)}.$$

Regularity implies no UC zeros and the number of OUC and IUC zeros can be determined by certain rules from the reflection coefficient parameters k_m (or by other means for a not monic sequence see, e.g., [6]). The polynomials in the two sequences $\{F_m(z)\}$ and $\{a_m(z)\}$ are related as follows:

$$F_m(z) = \psi_m a_m(z) + \psi_m^\star a_m^\sharp(z) \tag{18}$$

where ψ_m are a sequence of complex numbers defined by the recursion

$$\psi_{m-1} = \frac{\psi_m\left(\psi_{m+1}^\star - \psi_{m+1}k_{m+1}\right)\left(1 - |k_m|^2\right)}{\psi_m^\star - \psi_m k_m}$$
$$\psi_n = 1, \ \psi_{n-1} = 2\frac{1 - |k_n|^2}{1 - k_n.} \tag{19}$$

This relation follows from [9] (after using in there the recursion for $\eta_m = \nu_m/\psi_m$ to show by induction that if $\nu_m = \psi_m^\star$ for $m = n, n-1$ then the same holds for all m). Thus each $F_m(z)$ forms the symmetric part of a corresponding scaled Schur-Cohn polynomial $\psi_m a_m(z)$. (In general the zeros of $F_m(z)$ are *not* those of $a_m(z) + a_m^\sharp(z)$.) When $P(z)$ is real the relation may be seen to simplify and the $F_m(z)$ become proportional to $a_m(z) + a_m^\sharp(z)$. (It is shown in [9] that a single-multiplier three-term recursion of polynomials proportional to the $a_m(z) + a_m^\sharp(z)$ is not possible for a complex $P(z)$.)

One reviewer drew our attention to our reference to [10]. It considers a stability test for real polynomials (only) that too propagates polynomials proportional to $a_m(z) + a_m^\sharp(z)$ but uses a three-term recursion with two multipliers per recursion step. This test and a previous stability test referenced in it that it modifies requests twice the amount of computation of [7] or the current test used for a real polynomial because our tests have a single-multiplier per recursion step. In addition they are not truly original because they form two of several possible choices of two-multiplier three-term recursions of immittance ("symmetric") type Schur recursions obtained before in [9] (for complex polynomials, and in earlier references given in there for real polynomials) and discarded during the systematic search in there for the more efficient single-multiplier recursions. In fact, neither the test in [10], nor the previous test it modifies, are computationally more efficient than a Schur recursion of polynomials proportional to $a_m(z)$ that uses a single-multiplier per recursion step (e.g., Raible's test and tests in the "B scheme" category in [6]). This observation is already enough to deny from [10] its expected advantage even by comparison to certain Schur recursion as a candidate for generalization to multidimensional stability testing. While the fact that, unlike the Schur recursions, [10] is limited to real polynomials makes it actually an inferior candidate for multidimensional stability testing by comparison to an appropriate Schur recursion. Tractable relation of a stability test to the principal minors of the Schur matrix has been assumed during the last two decades as an important asset for using it as a basis for an efficient multidimensional stability tests because it allows the immediate adoption of a simplification that Siljak introduced to the field for testing positivity of a Schur polynomial matrix [11]. However by developing new solutions directly to the needs of the problem under consideration instead of forcing on it older available results, it can be shown that any of the immittance stability tests in [7] and [8] and the current test, renders a simple 2-D stability test with inherent Siljak-type simplification without a reference to Schur minors or to Siljak's result [12]–[14].

The current and the original forms of the test involve a comparable number of elementary arithmetic operations. There is an approximate trade-off between the current extra recursion step and the pre-scaling in the original form. For real polynomials the one recursion step less represent a slight advantage for the original form. The current form is preferable for handling certain generalized applications. For example, it may be simpler to test a polynomial with some literal coefficients or coefficients dependent on more parameters in order to obtain stability constrains on the free parameters. Situations of this type of applications arise e.g., in feedback control of linear shift invariant systems. It may also simplify stability testing that involves polynomials whose coefficients are function of secondary variables. An application of this type is obtaining stability conditions for multidimensional systems [14].

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