Reflections on Schur-Cohn Matrices and Jury-Marden Tables and classification of related unit-circle zero location criteria*

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ABSTRACT

We use the so called *reflection coefficients* (RC) to examine, review, and classify the Schur-Cohn and Marden-Jury (SCMJ) class of tests for determining zero location of a discrete-time system polynomial with respect to the unit circle. These parameters are taken as a platform to propose a partition of the SCMJ class into four useful type of schemes. The four types differ in the sequence of polynomials (i.e. the 'table') they associate with the tested polynomials by scaling factors: (A) a sequence of monic polynomials, (B) a sequence of least arithmetic operations, (C) a sequence that produces the principal minors of the Schur-Cohn matrix and (D) a sequence that avoids division arithmetic. A direct derivation of a zero location rule in terms of the RC is first provided and then used to track a proper zero location rule in terms of the leading coefficients of the polynomials of the B, C and D scheme prototypes. We review many of the published stability tests in the SCMJ class and show that each can be sorted into one of these four types. This process is instrumental in extending some of the tests from stability conditions to zero location, from real to complex polynomial, in providing a proof to tests stated without a proof, or in correcting some inaccuracies. Another interesting outcome of the current approach is that a by product of the developing a zero location rule for the C-type test is one more proof for the relation between the zero location of a polynomial and the inertia of its Schur-Cohn matrix.

1. INTRODUCTION

The condition of stability for linear discrete shift invariant systems corresponds to necessary and sufficient conditions for the zeros of the system's polynomial (its characteristic equation) to lie inside the unit-circle in the z-plane. Consider a polynomial of degree n with complex coefficients

$$p(z) = \sum_{i=0}^{n} p_i z^i \qquad (1)$$

and call it *stable* if it has all its zeros inside the unit-circle. Algebraic stability tests for discrete systems are methods that determine in some finite number of arithmetic operations whether a characteristic polynomial of a system is stable. Zero location tests extend this problem to also

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counting the number of zeros of the polynomial inside the unit-circle (IUC), on the unit-circle (UC) and outside the unit-circle (OUC).

The first *direct* algebraic criterion for unit-circle stability is due to Schur [1], and its extension to zero location to Cohn [2]. Assume a polynomial $f_m(z)$ of degree m, its conjugate, reverse, and reciprocal (conjugate-reverse) polynomials will be denoted by

$$f_m(z) = \sum_{i=0}^{m} f_{m,i} z^i \ , \ \bar{f}_m(z) := \sum_{i=0}^{m} f^\star_{m,i} z^i \ , \ f^r_m(z) := \sum_{i=0}^{m} f_{m,m-i} z^i \ , \ f^\sharp_m(z) := \bar{f}^r_m(z) \ . \tag{2}$$

respectively (* will denote complex conjugation). Cohn provided an algorithm to implement his zero location criterion. Given a polynomial $p(z)$ he proposed to construct a sequence of polynomials $\{f_m(z)\}$ of descending degrees $m = n, n-1, \ldots, 1, 0$, starting with $f_n(z) = p(z)$ then for $m = n, \ldots, 1$ applying the following recursion:

$$f_{m-1}(z) = \begin{cases} \frac{1}{z}[f_m(z) - \frac{f_{m,0}}{f^\star_{m,m}} f^\sharp_m(z)] & \text{for} \quad |f_{m,m}| > |f_{m,0}| \\ f_m(z) - \frac{f_{m,m}}{f^\star_{m,0}} f^\sharp_m(z) & \text{for} \quad |f_{m,m}| < |f_{m,0}| \end{cases} \tag{3}$$

The information on the number of IUC and OUC may be obtained from the changes between the two types of recursions [2] (A recent account and a new proof is available in [3, Example 2.2]).

The Schur-Cohn test in the form it is more familiar today is the growth of a modification devised to Cohn's setting by Marden [4, 5]. Subsequently, Marden's approach has been advanced in several stability table forms by Jury and other system theory researchers [6-19]. Marden's scheme avoids switching between two modes as in (3) when the polynomial has both IUC and OUC zeros. Instead the recursion retains a *uniform* structure viz.,

$$f_{m-1}(z) = f^\star_{m,0} f_m(z) - f_{m,m} f^\sharp_m(z) \quad , \quad f_n(z) = p(z)$$

The information on the number of IUC and OUC zeros is obtained from the relative magnitudes of $f_{m,m}$ and $f_{m,0}$ in a manner that will be shortly detailed.

In digital signal processing applications, stability testing is better known by an approach that comes from its own yard and terminology. There, a polynomial is determined as stable if (and only if) its "reflection coefficients" are all with moduli less than unity. The "reflection coefficient" are parameters that celebrate in several digital signal processing models [20] that may be associated with solving a Toeplitz set of equations by the Levinson algorithm [21].

It will turn out that most of the tests available today that stem from the works of Schur, Cohn and Marden, can be regarded as associating the tested polynomial $p(z)$ by a sequence $\{f_m(z)\}$ formed by a recursion that obey the following form

$$z f_{m-1}(z) = \psi_m \{f_m(z) + k_m f^\sharp_m(z)\} \quad , \quad k_m = -\frac{f_{m,0}}{f^\star_{m,m}} \tag{4}$$

Here the k_m will be shortly identified as the aforementioned *reflection coefficients* (RC) and ψ_m's represent some non-zero (real or complex) numbers. Different tests may be the result of the choice of ψ_m, or in operating a reversion or conjugation (or both) at either or both sides of (4). Further differences may be caused by the initiation of the recursions. Namely, $f_n(z)$ may be taken to be $p(z)$, or $p^r(z)$, or $\bar{p}(z)$, or $p^\sharp(z)$ (where a demand to begin with a monic $p(z)/p_n$ may also imposed sometimes). We shall refer to the class of tests that may be described by (4) within all above listed variations as the Schur-Cohn Marden-Jury (SCMJ) class. A typical stability or zero location test consists of a specific sequence $\{f_m(z)\}$ in the SCMJ and a rule to determine zero location from a subset of n or $n+1$ *distinguished entries* from the array $f_{i,j}$ of coefficients. The distinguished can be conveniently arranged to be *real* even for complex polynomial. A long standing tradition has been to lay out the entries $f_{i,j}$ in a tabular array. We shall therefore interchangeable refer to a member of the SCMJ class also as a (stability-) 'table'. The zero location rules depend on the signs of the distinguished entries. The number of IUC or OUC zeros are usually given by either the count of the number of negative or positive entries or by the number of signs variations or sign consistencies in an (ordered) sequence of distinguished entries. The scaling factors ψ_m and the remaining possible variations that differentiate one test in the SCMJ class from another often have an intricate effect on the proper form of an accompanying stability and zero location rule for each individual test. A valid zero location rules turns out to be in particular prone to error because it is more sensitive to the accumulating effects of sign changes.

We would like to draw attention to a peculiar outcome of our definition of the SCMJ class by which Cohn's scheme does not belong actually to the SCMJ class (unless one's interest is restricted to just stability conditions). The main difference between Cohn's original scheme and the SCMJ class we defined may be described as follows. The "reflection coefficients" in Cohn's recursions (momentarily referring so to the fractions that appear in (3)) are always with moduli less than unity, while the recursion is switching between the indicated two modes. In difference, in the SCMJ recursions (4), the recursions retain a uniform structure and instead the k_m's are allowed to take both $k_m < 1$ and $k_m > 1$ values. The information on distribution of zeros with respect to the unit circle in the SCMJ class is held in the moduli of the k_m's rather than in Cohn's variations of the structure of the recursion.

It is worthwhile to clarify that other tests to determine zero location with respect to the unit circle are available today that do not conform to the SCMJ class. The recent class of "immittance" tests [22, 23, 24] is notable as alternative approach that also offers a lower count of arithmetic operations than the lowest count of operations that is achievable within the SCMJ class (the type B tests) by approximately a factor of two.

The set of reflection coefficients of a polynomial are the central theme around which this paper revolves. This choice is motivated beyond their popularity in signal processing, by also their following properties pertinent to our classification goals:

(1) The RC of a given polynomial can be easily determined and they normally contain all (actually more than) the necessary information on the number of zeros of the polynomial inside and outside the unit-circle.

(2) The RC are offer for the parametrization of the SCMJ class a set of parameters that is not affected by scaling and other possible twists that distinguish the many versions of tests in the SCMJ class.

It will soon become evident that for a polynomial's zero distribution the information of only the relative magnitudes of the RC with respect to unity of a given *ordered* sequence of RC is sufficient. The remaining information - knowing the actual numerical values of the RC - represent the equivalent of knowing the exact numerical values of the zeros of the polynomial.

First we state and bring a direct proof for the rule to determine the number of IUC and OUC zeros of a polynomial from its given sequence of RC. Then we focus on four choices of scaling parameters ψ_m that represent interesting schemes in the SCMJ class that we define as prototypes A, B, C, and D. We maintain a quite broad generality of treatment by always considering zero location rule, and not just stability conditions, and in always considering the polynomial to have complex coefficients. However as the paper uses the reflection coefficients to classify the SCMJ class, the scope of zero location generality is limited by the assumption that the set of RC is well defined. It should be said that it is always possible to determine the distribution of zeros of any polynomial with respect to the unit circle also in complementary cases that do not yield a well defined and unique set of RC. How this can be done has been too shown already by Cohn and some of the reviewed tests contain also treatment of these singular (or pseudo singular) cases. The assumption that the RC are well defined will be referred to as *strong regularity* and will be characterized in several related equivalent conditions.

The four types of SCMJ zero location schemes that are defined are as follows. The first scheme associates the tested polynomial with a sequence $\{a_m(z)\}$ of monic polynomial and is labeled "Scheme A". The sequence $\{a_m(z)\}$ serves as a reference to all other sequences. It is also used to derive the relations between RC and the principal minors of the Schur-Cohn matrix. The second "Prototype B" procedure produces a sequence $\{b_m(z)\}$ that corresponds to choosing the scalars in (4) $\psi_m = 1$ for $m \leq n-1$. It represent the lowest arithmetic count algorithms in the SCMJ class. An example of a "Type B" test is Raible's test [11]. The third "Type C" schemes generates n distinguished entries that produces (or relate up to sign) to the Schur-Cohn determinants. In particular, the C-prototype algorithm produces a sequence denoted by $\{c_m(z)\}$ whose leading coefficients are equal to the principal minors of the Schur-Cohn matrix. "Type C" schemes in the literature were advanced by Jury in several version [8, 7, 13, 14]. We presented this part of the current paper in a recent symposium dedicated to Professor Jury [25]. Type C schemes are considered important in obtaining stability tests for two-dimensional systems and they perform better in finite precision arithmetic. The fourth and last "prototype D" scheme generates a sequence $\{d_m(z)\}$ while avoiding the arithmetic operation of division. "Division-free" schemes of "Type D" available in the literature include the tests in [6, 10] and in [12, 16]. This form may be useful in applications that involve testing polynomials with coefficients that depend on parameters when it is of advantage to avoid divisions and remain with polynomial (rather than rational function) coefficients.

The uniform parametrization of the SCMJ class also turn to be helpful during a subsequent review of test published in the literature to sometimes support theorems stated without proof, or at times provide missing zero location rules or correct zero location rules when not stated properly. Since we consider polynomial with complex coefficient a complex version for tests proposed for only real polynomials also become at once apparent.

In the course of deriving the zero location rule for the scheme C prototype algorithm, we actually provide a yet another proof to the celebrated Schur-Cohn theorem on the relation between the number of positive and negative eigenvalues of the Schur-Cohn matrix and the number of zeros of the polynomial inside and outside the unit circle. The specialty of this proof is that its starting point is the zero location rule proved first on the RC is then used through relations between unit triangular (Cholesky) factorizations of the Schur-Cohn and Toeplitz matrices to achieve the sought proof (cf. [26]).

The outline of the paper is as follows. The zero location criterion in terms of the set of RC is derived in the next section. This section associates the tested polynomial with a monic sequence of polynomials that is labeled Algorithm A. Section 2 also contains the expression for the principal minors of the Schur-Cohn matrix in terms of the RC. The subsequent sections 3, 4, and 5 are devoted to the Type B, Type C and Type D schemes, respectively. The presentation of each prototype is followed by reviewing tests in the literature that belong to that type and is also accompanied by a simple numerical demonstration.

2. REFLECTION COEFFICIENTS AND ZERO COUNTING

Schur [1] and Cohn [2] associated with the polynomial $p(z)$ a Hermitian *form* such that, provided that all its *squares* are non-zero, then its number of positive and negative squares provide the number of IUC and OUC zeros (see e.g. [27]). Fujiwara [28] (see also [27]) showed that the Schur-Cohn conditions can be posed equivalently on the principal minors of the following $n \times n$ matrix

$$\mathbf{C} = \begin{bmatrix} p_n^\star & & & 0 \\ p_{n-1}^\star & & & \\ \vdots & & & \\ p_1^\star & \cdots & p_{n-1}^\star & p_n^\star \end{bmatrix} \begin{bmatrix} p_n & p_{n-1} & \cdots & p_1 \\ & & & \vdots \\ & & & p_{n-1} \\ 0 & & & p_n \end{bmatrix} - \begin{bmatrix} p_0 & & & 0 \\ p_1 & & & \\ \vdots & & & \\ p_{n-1} & \cdots & p_1 & p_0 \end{bmatrix} \begin{bmatrix} p_0^\star & p_1^\star & \cdots & p_{n-1}^\star \\ & & & \vdots \\ & & & p_1^\star \\ 0 & & & p_0^\star \end{bmatrix}$$

The Schur-Cohn zero location rule was also posed on the determinants of a sequence of n matrices of sizes $2m \times 2m$, $m = 1, \ldots, n$ (called the Schur-Cohn determinants) and was also given in terms of a (not Hermitian) $2n \times 2n$ matrix $\Delta_{1:2n,1:2n}$ whose sequence of centrally situated sub-matrices $\Delta_{m:2n-m,m:2n-m}$, $m = 1, \ldots, n$ have determinants that are equal to the principal minors of $\mathbf{C}$ [9] (see also [18, Theorem 2.6], [19, Theorem 3.11]). The matrix $\mathbf{C}$ is refered in the literature as the Schur-Cohn-Fujiwara matrix or the Hermitian Schur-Cohn matrix (see [18, Theorem 3.2] [19, Theorem 3.13] and [27, Theorem XVa]).

The term RC stems from their interpretation in modeling lossless layered media in certain digital signal processing applications that , e.g. in modeling the vocal tract in speech processing or the earth surface in geophysics (see e.g.[20, 29]). Another name for RC that originates from their statistical interpretation in approximating a stationary process by an auto-regressive (AR) model is *Partial Correlation (ParCor)* coefficients. Common to both interpretations is that they involve the solution of a set of equations that may be represented in the following normal form

$$\mathbf{T}_n[a_{n,0}, a_{n,1}, \ldots a_{n,n-1}, 1]^t = [0, \ldots, 0, d_m]^t \tag{6}$$

where $\mathbf{T}_n$ is a Hermitian positive definite Toeplitz matrix of size $(n+1) \times (n+1)$ and the set is to be solved for the $n+1$ unknowns $\{a_{n,i}, d_n\}$. An efficient solution to this set of equations is provided by Levinson's algorithm [21]. The Levinson algorithm is a recursive algorithm that includes (in a polynomial notation) the recursions

$$a_m(z) = za_{m-1}(z) - k_m a^\sharp_{m-1}(z) \tag{7}$$

and a formula (that need not concern us here) to compute the k_m's - the "RC" - that brings into the solution the entries of the Toeplitz matrix. The algorithm starts with $a_0(z) = 1$ and is carried out for $m = 1, ..., n$. It is seen that the polynomials $a_m(z) = \sum_0^n a_{m,i} z^i$ are all monic, $a_{m,m} = 1$.

Given a polynomial $p(z)$ it is possible to determine its RC (namely to find k_m's such that (7) will produce $a_n(z) = p(z)/p_n$) by reversing the recursion (7).

Algorithm A. Apply the algorithm

$$za_{m-1}(z) = \frac{a_m(z) + k_m a^\sharp_m(z)}{1 - |k_m|^2} \quad , \quad k_m = -\frac{a_{m,0}}{a^\star_{m,m}} \tag{8}$$

with the initiation $a_n(z) = p(z)/p_n$

Theorem 1. (Zero Location rule by RC.) *A polynomial $p(z)$ with a well-defined set of RC $\{k_m \ , \ m = 1, \ldots, n\}$ has ν OUC zeros (no UC zeros) and $n - \nu$ IUC zeros where ν is given by the number of negative terms in the sequence*

$$\nu = n_-\{q_n, q_{n-1}, \ldots, q_1\} \tag{9}$$

whose members are defined by

$$q_m := \prod_m^n (1 - |k_i|^2) \quad , \quad m = 1, \ldots, n \tag{10}$$

It is seen that the q_m parameters still hold enough information to determine zero location. These parameters will play an important role in the forthcoming developments. They can more simply obtained from the sequence of RC by the recursion:

$$q_m = q_{m+1}(1 - |k_m|^2) \ , \qquad q_{n+1} := 1 \qquad m = n, n-1, \ldots, 1. \tag{11}$$

We prove Theorem 1 in the appendix. The proof is obtained by invoking the *Principle of Argument* on the recursions (7).

Remark 1. The immediate corollary of Theorem 1 that necessary and sufficient conditions for stability (i.e. $\nu = 0$) are

$$|k_m| < 1 \qquad , \qquad m = 1, \ldots, n \ ,$$

is very familiar result in digital signal processing applications where the k_m's are also serve as gains in a lattice realization of the all pole filter $1/a_n(z)$ (e.g. [20]).

We define Algorithm A together with the zero location rule given by Theorem 1 as our "Scheme A" in the SCMJ class .

The relation of Algorithm A to the Levinson recursions motivates the use of the term RC also in the context of the SCMJ class of stability and zero location. It is clear from (8) that the sequence of the RC of a polynomial is not affected by the possible scaling factors in (4). Consequently the RC in conjunction with Theorem 1 form convenient tools for tracking properly zero location rules for other algorithms in the SCMJ class.

Definition 1. **Strong Regularity.** A polynomial $p(z)$ is said to satisfy conditions of strong regularity if Algorithm A (or any other test in the SCMJ class) does not encounter a premature termination (and hence produces a well defined set of n RC)

Remark 2. The condition that causes the recursion to terminate prematurely is the occurrence of a k_m such that $|k_m| = 1$. The offense in general is not the division by $[1-|k_m|^2]$ apparent in Algorithm A but that a subsequent $k_{m-1} = -f_{m-1,0}/f^{\star}_{m-1,m-1}$ is not well defined when $f_{m-1,m-1} = 0$ and an $f_{m-1,m-1} = 0$ is necessarily preceded by, and always follows a, $|k_m| = 1$. This is seen by reading from (4) that

$$f_{m-1,m-1} = \psi_m(f_{m,m} - k_m f^{\star}_{m,0}) = \psi_m f_{m,m}(1 - |k_m|^2)\,.$$

Several equivalent conditions for strong regularity will be given in Theorem 2 below. Strong regularity will be presumed throughout in this paper.

Fujiwara who contributed the expression (5) for the Schur-Cohn matrix also showed that a generating function for $\mathbf{C} = [c_{i,j}]$ is given by

$$\mathcal{C}(z,w) = \frac{p^{\sharp}(z)p^{r}(w) - p(z)\bar{p}(w)}{1 - zw} = \sum_{i,j=0}^{n-1} c_{i,j} z^i w^j \tag{12}$$

In more modern terminology $\mathcal{C}(z,w)$ is sometimes referred to as a generating function for T-Bezoutian (a Bezoutian with respect to the unit circle). The T-Bezoutians and the former Bezoutians with respect to a line may be treated in a unified manner [3].

The relations between the principal minors of the Schur-Cohn matrix and the reflection coefficients will be obtained by using a couple of results known in the context of the Levinson algorithm and the inversion of Toeplitz matrices. First it may be noticed from the nested structure of the Levinson algorithm that the Levinson algorithm provides in fact a solutions to all normal sets of equations of the form (6) defined by all the submatrices $\mathbf{T}_m$, $m = 0, 1, \ldots, n$. This observation implies that the Levinson algorithm creates a UDL triangular factorization for the inverse of $\mathbf{T}_n$, viz.,

$$\mathbf{T}_n^{-1} = \mathbf{A}_n \mathbf{\Lambda}_n^{-1} \mathbf{A}_n^H \tag{13}$$

where

$$\mathbf{A}_n = \begin{bmatrix} 1 & a_{1,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & 1 & \cdots & a_{n-1,1} & a_{n,1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n,n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{13a}$$

the superscript ^{H} denotes conjugate transpose, and $\mathbf{\Lambda}_n$ is the diagonal matrix

$$\mathbf{\Lambda}_n \;=\; diag[1\;,\;\lambda_1\;,\;\ldots,\;\lambda_n] \quad ; \quad \lambda_m \;= \prod_{i=1}^{m}(1 - |k_i|^2) \;. \tag{13b}$$

The second result that we shall use is a formula due to Gohberg and Semencul [30] that allows the expression of the inverse of a Toeplitz matrix $\mathbf{T}_n^{-1}$ in terms of just the coefficients of $a_n(z)$.

$$\mathbf{T}_n^{-1} = \frac{1}{\lambda_n}\begin{bmatrix} a^{\star}_{n,n} & & & 0 \\ a^{\star}_{n,n-1} & & & \\ \vdots & & & \\ a^{\star}_{n,0} & \cdots & a^{\star}_{n,n-1} & a^{\star}_{n,n} \end{bmatrix}\begin{bmatrix} a_{n,n} & a_{n,n-1} & \cdots & a_{n,0} \\ & & & \vdots \\ & & & a_{n,n-1} \\ 0 & & & a_{n,n} \end{bmatrix} -$$

$$-\frac{1}{\lambda_n}\begin{bmatrix} 0 & & & 0 \\ a_{n,0} & & & \\ \vdots & & & \\ a_{n,n-1} & \cdots & a_{n,0} & 0 \end{bmatrix}\begin{bmatrix} 0 & a^{\star}_{n,0} & \cdots & a^{\star}_{n,n-1} \\ & & & \vdots \\ & & & a^{\star}_{n,0} \\ 0 & & & 0 \end{bmatrix} \tag{14}$$

This formula indicates that a generating function for $\mathbf{R}_n := \mathbf{T}_n^{-1}$ is given by (compare to (12) and (5))

$$\mathcal{R}_n(z,w) = \frac{1}{\lambda_n}\;\frac{a_n^{\sharp}(z)a_n^{r}(w) - zw a_n(z)\bar{a}_n(w)}{(1 - zw)} \tag{15}$$

The relation of the Schur-Cohn matrix for $p(z)$ and the Schur-Cohn matrix $\hat{\mathbf{C}}$ for the corresponding monic polynomial $a_n(z) = p(z)/p_n$ is seen from (5) to be

$$\mathbf{C} = |p_n|^2 \hat{\mathbf{C}} \tag{16}$$

and the generating function for $\hat{\mathbf{C}}$ is, (12),

$$\hat{\mathcal{C}}(z,w) = \frac{a_n^\sharp(z) a_n^r(w) - a_n(z)\bar{a}_n(w)}{1 - zw} \tag{17}$$

Next, the recursion (8) can be used to obtain the identity

$$a_n^\sharp(z) a_n^r(w) - a_n(z)\bar{a}_n(w) = (1 - |k_n|^2)[a_{n-1}^\sharp(z) a_{n-1}^r(w) - zw a_{n-1}(z)\bar{a}_{n-1}(w)] \,.$$

Set this identity in the generating functions (17) for $\mathbf{C}$ and compare the result with the generating function (15) for $\mathbf{R}_{n-1} = \mathbf{T}_{n-1}^{-1}$ to find that

$$\hat{\mathcal{C}}(z,w) = \lambda_n \mathcal{R}_{n-1}(z,w)$$

Thus the next relation between the monic SC matrix and inverse of the $n \times n$ Toeplitz matrix becomes evident.

$$\frac{1}{\lambda_n} \hat{\mathbf{C}} = \mathbf{T}_{n-1}^{-1} \tag{18}$$

We proceed to use this relation to connect the minors of the Schur-Cohn matrix with the RC. Toward this goal, first observe from (5) that the Schur-Cohn matrix is centro-Hermitian, i.e. has the property $\mathbf{JCJ} = \mathbf{C}$ where $\mathbf{J}$ denotes a matrix with 1's along the anti-diagonal and zeros elsewhere (the reversion matrix). The UDL triangular factorization (13) implies through (18) and the centro-Hermitian symmetry the following LDU triangular factorization for $\hat{\mathbf{C}}$,

$$\hat{\mathbf{C}} \;=\; \mathbf{B}_{n-1} \mathbf{Q}_{n-1} \mathbf{B}_{n-1}^H \tag{19}$$

where

$$\mathbf{B}_{n-1} \;=\; \mathbf{J}\mathbf{A}_{n-1}\mathbf{J} \;=\; \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_{n-1,n-2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1,1} & a_{n-2,1} & \cdots & 1 & 0 \\ a_{n-1,0} & a_{n-2,0} & \cdots & a_{1,0} & 1 \end{bmatrix} \tag{19a}$$

and

$$\mathbf{Q}_{n-1} \;=\; diag[q_n \;,\; q_{n-1} \;,\; \ldots,\; q_1] \quad ; \quad q_i \;=\; \frac{\lambda_n}{\lambda_{i-1}} \quad , \quad i = 1, \ldots, n \,. \tag{19b}$$

It is noticed that these q_m's are indeed identical to the q_m's as defined before in (10). The implicit assumption that the underlying Schur-Cohn and Toeplitz (sub-) matrices are invertible follows from the strong regularity assumption. In fact we may at this point summarize several equivalent conditions for strong regularity.

Theorem 2. (Strong Regularity.)
The following conditions are equivalent

- **(i)** *The set of RC are well defined (strong regularity according to Definition 1).*
- **(ii)** *All the leading principal minors of* $\mathbf{C}$ *are different from zero.*
- **(iii)** *All the leading principal minors of* $\mathbf{T}_n$ *are different from zero.*
- **(iv)** *All* $|k_m| \neq 1$ $m = 1, \ldots, n$

Proof. The equivalencies $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ are apparent from factorizations (13) and (19). The equivalence $(i) \Leftrightarrow (iv)$ was explained already in Remark 1. Q.E.D.

Remark 3. Strong regularity implies that the degree η of the greatest common divisor of $p(z)$ and $p^\sharp(z)$ is $\eta = 0$. In particular it implies that $p(z)$ has no UC zeros. However, although $\eta = 0$ is sufficient for no UC zeros, and for non-singularity of the matrices $\mathbf{T}_n$ and $\mathbf{C}$, it is still not a sufficient condition for strong regularity.

It is sufficient to recognize the factorization (19) as a congruency relation between $\hat{\mathbf{C}}$ and $\mathbf{Q}$ in order to conclude that the two matrices have the same number of positive and negative eigenvalues (the same *inertia*). The eigenvalues of the diagonal matrix $\mathbf{Q}$ are the q_m's. The number of positive and negative q_m's were shown in Theorem 1 to give the number of IUC and OUC zeros of $p(z)$. Thus proof for Theorem 1 in combination with the factorization (19) provides a new proof to the Schur-Cohn criterion.

Theorem 3. (Schur-Cohn.) *If all the leading principal minors of the Schur-Cohn matrix for $p(z)$, $\mathbf{C}$, are different from zero then the number OUC and IUC zeros of polynomial $p(z)$, ν and $n-\nu$, are equal to the number of negative and positive eigenvalues of $\mathbf{C}$, respectively.*

We proceed to examine in finer detail the factorization (19) in order to obtain explicit expression for the principle minors of $\mathbf{C}$ in terms of the RC. Denote the principal minors of $\mathbf{C}$ by

$$\mu_m := det\{\mathbf{C}_{0:m-1}\} \quad ,m = 1,\dots,n \tag{20}$$

where the $\mathbf{C}_{0:l-1}$ represents the $l \times l$ leading submatrix of $\mathbf{C}$ (our count of rows and columns of matrices begin with $i = j = 0$).

Theorem 4. (Minors of C in terms of $\{k_m\}$.) *Consider $p(z)$ its Schur-Cohn matrix $\mathbf{C}$ and assume strong regularity.*

(i) The principal minors of $\mathbf{C}$ are related to the set of RC by

$$\mu_m = |p_n|^{2m} \prod_{i=n+1-m}^{n} q_i \quad , \quad m = 1,\dots,n \tag{21}$$

(ii) The polynomial $p(z)$ has ν OUC and $n-\nu$ IUC zeros with

$$\nu = Var\{1, \mu_1, \mu_2, \dots, \mu_n\} \tag{22}$$

where Var denotes the number of sign variations in the indicated sequence.

Proof. The lower upper *unit* triangular factorization (19) implies that the principal minors $\hat{\mu}_m$ of $\hat{\mathbf{C}}$ are given by

$$\hat{\mu}_m = \prod_{i=n+1-m}^{n} q_i \quad , \quad m = 1,\dots,n \tag{23}$$

Consequently (21) follows from (16). Noticing that $\mu_m/\mu_{m+1} = q_{n-m}$, the rule (9) of Theorem 1 can be written as

$$\nu = n_-\{\mu_1, \frac{\mu_2}{\mu_1}, \dots, \frac{\mu_n}{\mu_{n-1}}\} \tag{24}$$

It remains to realize that (22) and (24) are equivalent. Q.E.D.

Example 1. Consider the polynomial

$$p(z) = p(z) = 4 + 12.5z + 5z^2 + z^3$$

that can be checked numerically to have 2 OUC and 1 IUC zero. The sequence $\{a_m(z)\}$ consists of $a_3(z) = p(z)$

$$a_2(z) = 0.5 + 3z + z^2 \quad , \quad a_1(z) = 2 + z \quad , \quad a_0(z) = 1$$

The RC are

$$k_1 = -2 \quad , \quad k_2 = -0.5 \quad , \quad k_3 = -4$$

Thus the q_m parameters (11)

$$q_3 = -15 \quad , \quad q_2 = -11.25 \quad , \quad q_1 = 33.75$$

Then according to the rule (9)

$$\#OUC = n_-\{q_3, q_2, q_1\} = 2$$

The Schur-Cohn matrix is

$$\mathbf{C} = \begin{bmatrix} -15 & -45 & -7.5 \\ -45 & -146.25 & -45 \\ -7.5 & -45 & -15 \end{bmatrix}$$

The principal minors of $\mathbf{C}$ are indeed $\mu_1 = -15 \ \ (= q_3)$, $\mu_2 = 168.75 \ \ (= q_3 q_2)$ and $\mu_3 = 5695.3 \ \ (= q_3 q_2 q_1)$. According to the rule in the Theorem 4,

$$\#OUC = Var\{1, -15, 168.75, 5695.3\} = 2$$

In difference from some concluding remarks in [26] the number of OUC zeros *is not* given by the number of sign variation of the principal minors of $\mathbf{C}$ ($Var\{-15, 168.75, 5695.3\} = 1$ in this example) nor by the principal minors of $\mathbf{T}_{n-1}$ or its inverse . In fact the signs of the principle minors of $\mathbf{C}$ and $\mathbf{T}_{n-1}^{-1}$ is affected by the appearance of λ_n in (18) and λ_n may be negative.

3. Schemes of Type B

It is possible to restrict somewhat the range of relevant scalars ψ_m in the search after tests of merit in the SCMJ class without a danger of missing interesting cases by a cautious inspection on its structure. For example, when testing a real polynomial there is clearly no point in allowing complex ψ_m to complicate the recursion. More generally, the structure of (4) possesses the property that if $f_{n-1,n-1}$ is real then all subsequent $f_{m,m}$, $m \leq n-1$ stay real as long as the ψ_m's , $m \leq n-1$ are real. It is possible to exploit this property by insisting on $\psi_n = f^\star_{n,n}$,

$$z f_{n-1}(z) = f^\star_{n,n}[f_n(z) + k_n f^\sharp_n(z)] \quad , \quad f_n(z) = p(z) \ . \tag{25}$$

so that $f_{n-1,n-1} = |f_{n,n}|^2(1 - |k_n|^2)$ is indeed real (and perceives part of the information on k_n that is acute for finding zero location) and afterwards by restricting the subsequent ψ_m's to be real. As will be seen the $f_{m,m}$ form the distinguished entries in terms of which the zero location rules are given so that being real simplify these rules. All the tests of interest in the SCMJ class are featured by distinguished entries that are real.

3.1. Type B: Basic Form

This form correspond to choosing $\psi_n = p^\star_n$ and $\psi_m = 1$ for all $m \leq n-1$.

Algorithm B *Initiation:*

$$z b_{n-1}(z) = p^\star_n p(z) - p_0 p(z) \tag{B1}$$

For $m = n-1, \ldots, 1$ do:

$$z b_{m-1}(z) = b_m(z) + k_m b^\sharp_m(z) \quad , \quad k_n = -\frac{b_{m,0}}{b^\star_{m,m}} \tag{26}$$

[Alternative possible initiations are (1) to first normalize to monic the tested polynomial,

$$b_n(z) = \frac{1}{p_n} p(z) \tag{B2}$$

and (2) when the leading coefficient p_n is real it is also possible to start the recursions with

$$b_n(z) = \begin{cases} p(z) & \text{if } p_n > 0 \\ -p(z) & \text{if } p_n < 0 \end{cases} \qquad (B3)$$

For these two alternative initiations (26) may be used for already $m = n$.]

Theorem 5. (Zero location for Algorithm B) *If Algorithm B does not terminate prematurely then $p(z)$ has ν OUC and $n-\nu$ IUC zeros where ν is given by*

$$\nu = n_-\{b_{n-1,n-1}, b_{n-2,n-2}, \ldots, b_{1,1}, b_{0,0}\} \qquad (27)$$

Proof. From (26)

$$b_{m-1,m-1} = b_{m,m} + k_m b^\star_{m,0} = b_{m,m}(1 - |k_m|^2) \ , \qquad m \leq n-1$$

By comparison with (11) obtain that

$$b_{m-1,m-1} = b_{m,m} q_m$$

in which for (B1) assume $b_{n,n} = |p_n|^2$, for (B2) $b_{n,n} = 1$, and for (B3) $b_{n,n} = |p_n|$. Thus for any of the three initiations the stated rule follows from (9) of Theorem 1. Q.E.D.

Example 1 (cont'd) [part b]. Consider again $p(z) = 4 + 12.5z + 5z^2 + z^3$. Algorithm B produces a sequence that consists of $b_3(z) = p(z)$,

$$b_2(z) = -7.5 - 45z - 15z^2 \quad , \quad b_1(z) = -22.5 - 11.25z \quad , \quad b_0(z) = 33.75$$

Indeed,

$$\#OUC = n_-\{-15, -11.25, 33.75\} = 2$$

Type B schemes are special in offering the least number of operations in the SCMJ class - just $n^2 + O(n)$ operations as compared to $2n^2 + O(n)$ for tests of type A or for the later type D, and $3n^2 + O(n)$ for the next type C tests.

Since in the rest of the paper we shall review many tests published in the literature we at this point define the convention that we shall follow in converting tests that have appeared in a 'table' form to the more compact polynomial notation that we use.

Remark 4. (**Conversion Convention for Tables.**) SCMJ tests often were published in tabular forms. Usually the format is (format a:) $n+1$ *pairs* of rows (the second row in each pair being the reversed conjugate of the first) Else, (format b:) the table consists of just $n+1$ rows (omitting the reverted rows). We shall regard these tables as forming the coefficients of a sequence of polynomials $\{f_m(z)\}$ by the following conversion convention. We shall associate the first, third, fifth, ... in format (a) or the first, second, third, ... in format (b) with the polynomials $f_n(z)$, $f_{n-1}(z)$, $f_{n-2}(z)$ by post multiplying each row by a vector of powers $[1, z, z^2, \ldots]^t$ of a corresponding equal length.

3.2. Type B: Raible's version

Raible proposed in [11] a test for real polynomials $p(z)$ that may be presented by the algorithm:

$$f_{m-1}(z) = f_m(z) - \xi_m f^r_m(z) \ \ , \ \ \xi_m = \frac{f_{m,m}}{f_{m,0}} \quad , \quad ; \ f_n(z) = p^r(z) \qquad (28)$$

Assuming $p_n > 0$, Raible provided the rule that the number of OUC zeros is given by

$$n_-\{f_{n-1,0}, f_{n-2,0}, \cdots f_{0,0}\} \qquad (29)$$

The above recursion is comparable with the reversion (or the "reciprocation" see Remark 5 below) of the recursion in Algorithm B (26):

$$b^r_{m-1}(z) = b^r_m(z) + k_m \bar{b}_m(z)$$

by the identifications

$$f_m(z) = b^r_m(z) \quad , \quad \xi_m = -k_m$$

Consequently as a corollary from Theorem 5 one has the following extension to complex polynomial of the test.

Corollary 1. (Complex form for Raible's version) *. Apply to a polynomial $p(z)$ the next algorithm. Initiation:*

$$f_{n-1}(z) = p_n^\star p^r(z) - p_0 \bar{p}(z) \tag{R0}$$

For $m = n-1, \ldots, 1$ do:

$$f_{m-1}(z) = f_m(z) - \xi_m f^\sharp_m(z) \quad , \quad \xi_m = \frac{f_{m,m}}{f_{m,0}} \tag{30}$$

[Alternative valid initiations include $f_n(z) = \frac{1}{p_n} p^\sharp(z)$ or, when p_n is real, also 2)

$$f_n(z) = \begin{cases} p^r(z) & \text{if} \quad p_n > 0 \\ -p^r(z) & \text{if} \quad p_n < 0 \end{cases}$$

and for these alternative the recursion holds also for $m = n$.]

Provided the algorithm does not terminate prematurely, the number of OUC and IUC zeros are ν and $n - \nu$ where

$$\nu = n_-\{f_{n-1,0}, f_{n-2,0}, \ldots, f_{0,0}\}$$

Remark 5. We "extrapolate" Raible's test from a real $p(z)$ to a complex $p(z)$. In any situation of this kind there is more than one way to do so. A dual extension form could identify (28) with the reciprocation of (26) such that $f_m(z) = b^\sharp_m(z)$ and $\xi_m = -k^\star_m$ and consider the initiations to correspond to $f_n(z) = p^\sharp(z)$. The zero location rule for this dual form is the same since the $f_{m,0}$'s remain real.

4. Schemes of Type C

The goal in this section is to obtain an algorithm in the SCMJ class that produces a sequence whose leading coefficients are the principal minors of the Schur-Cohn matrix. For the derivation process it is convenient to define one more set of auxiliary parameters that is related to the RC through the q_m's (10) by

$$e_m := \prod_{i=m}^{n} q_i \quad ; \quad e_m = e_{m+1} q_m \ , \ e_{n+1} := 1 \quad , \quad m = n, \ldots, 1 \tag{31}$$

The sequence $\{e_m\}$ may be seen from (23) to represent the principal minors of the $\hat{\mathbf{C}}$ in reversed order,

$$\hat{\mu}_m = e_{n+1-m} \tag{32}$$

Assume there exists a recursible set of scalars ψ_m for which (4), or (25) produces a sequence $\{c_m(z)\}$ with the property $c_m(z) = e_{m+1} a_m(z)$ where $\{a_m(z)\}$ is the monic sequence. Assume, momentarily, that the algorithm starts with $c_n(z) = a_n(z)$. Then $c_{n-1,n-1} = \psi_n(1 - |k_n|^2)$ and for the choice $\psi_n = 1$ we get $c_{n-1,n-1} = e_n$. At the next step, $c_{n-2,n-2} = \psi_{n-1}(1 - |k_n|^2)(1 - |k_{n-1}|^2)$ and $c_{n-2,n-2} = e_{n-1}$ is again possible for the choice $\psi_{n-1} = (1 - |k_n|^2) = c_{n-1,n-1}$. Seemingly the pattern is that the choice

$$\psi_{n+1-i} = c_{n+1-i,n+1-i} / c_{n+2-i,n+2-i} \ \text{ yields } \ c_{n-i,n-i} = e_{n+1-i} \tag{33}$$

We verify this pattern by an induction step. Suppose the assertion in (33) holds for $c_{n-i,n-i}$'s till $i = 1, \ldots, l$. Then at the next step, $c_{n-l-1,n-l-1} = \psi_{n-l} q_{n-l} c_{n-l,n-l}$, indeed reduces for the choice $\psi_{n-l} = c_{n-l,n-l} / c_{n+1-l,n+1-l}$, to $c_{n-l-1,n-l-1} = e_{n-l}$.

This completes the proof that the choosing

$$\psi_n = 1 \ ; \ \psi_{n-1} = c_{n-1,n-1} \ ; \ \psi_m = \frac{c_{m,m}}{c_{m+1,m+1}} \quad ; \quad m \leq n-2 \tag{34}$$

yields, for the initiation $c_n(z) = p(z)/p_n$, a sequence of polynomials $\{c_m(z)\}$ such that

$$c_m(z) = e_{m+1}a_m(z) \quad \text{and} \quad c_{m,m} = e_{m+1} = \hat{\mu}_{n-m} \tag{35}$$

The sought algorithm emerges after removing the restriction to monic initiation to be as follows.

4.1. Type C : Basic Form

The basic algorithm for Type C schemes is as follows.

Algorithm C. *Initiation:*

$$zc_{n-1}(z) = p_n^{\star}p(z) - p_0 p^{\sharp}(z) \tag{C1}$$

For $m = n-1, \ldots, 1$ do:

$$zc_{m-1}(z) = \phi_m\{c_{m,m}c_m(z) - c_{m,0}c_m^{\sharp}(z)\} \tag{36}$$

with ϕ_m given by

$$\phi_{n-1} = 1 \quad ; \quad \phi_m = \frac{1}{c_{m+1,m+1}} \quad , \ m = n-2, n-3, \ldots, 1$$

Theorem 6. (Zero location for Algorithm C.) *If Algorithm C does not terminate prematurely then $p(z)$ has ν OUC and $n-\nu$ IUC zeros where ν is given by*

$$\nu = Var\{1, c_{n-1,n-1}, c_{n-2,n-2} \ , \ \ldots \ , \ c_{0,0}\} \tag{37}$$

Proof. For the initiation with a monic polynomial we obtained the relation (35). The definition of the e_m's in combination with the rule (9) implies that

$$\nu = Var\{1, e_n, e_{n-1} \ , \ \ldots \ , \ e_1\} \tag{38}$$

For the more general initiation (C1) the relation (35) starts instead with $c_{n-1}(z) = |p_n|^2 a_{n-1}(z)$ and subsequently

$$c_{m-1,m-1} = (|p_n|^2)^{n+1-m}e_m \quad , \quad m = 1, \ldots, n \tag{39}$$

Thus (38) implies the stated rule. Q.E.D.

Algorithm C also achieves the goal of producing the principal minors of the Schur-Cohn matrix. (Initiations other then (C1) similar to those proposed before for Algorithm B will also lead to the rule (37) but will miss the identification of $c_{m,m}$ with the minors of $\mathbf{C}$ stated below.) The property is summarized as follows.

Theorem 7. (Algorithm C and the Minors of the matrix C.) *The leading coefficients of the sequence of polynomials produced by Algorithm C (with initiation (C1)) form the principal minors of the Schur-Cohn matrix $\mathbf{C}$ of $p(z)$, as follows.*

$$c_{m,m} = \mu_{n-m} \quad \text{where} \quad \mu_{m+1} := det\{\mathbf{C}_{0:m}\} \quad , \quad m = 0, \ldots, n-1 \tag{40}$$

Proof. Immediate from (31), (39) and (21). Q.E.D.

Example 1 (cont'd) [part c] Consider again $p(z) = 4 + 12.5z + 5z^2 + z^3$. The algorithm constructs

$$c_3(z) = p(z) \ ; \ c_2(z) = -7.5 - 45z - 15z^2 \ ;$$

$$c_1(z) = -33.75 + 168.75z \ , \ (\phi_2 := 1) \ ; \ c_0(z) = 5695.3 \ , \ (\phi_1 = -1/15)$$

Evaluate (37) with the requested values

$$\#OUC = Var\{1, -15, 168.75, 5695.3\} = 2$$

Theorem 6 implies for $p(z)$ 2 OUC and 1 IUC zero. The principal minors of the Schur-Cohn matrix were computed in part (a) of the example. The comparison shows that the principal minors are indeed given by : $\mu_1 = c_{2,2} = -15$, $\mu_2 = c_{1,1} = 168.75$ and $\mu_3 = c_{2,2} = 5695.3$, as claimed in Theorem 7.

4.2. Type C: Jury's versions

Scheme of type C were advanced by Jury in several occasions. They are considered to be of advantage in algorithms to test stability of two-dimensional system [13] and to offer better accuracy in finite arithmetic precision [31]. Two versions are available. An earlier version [17, p. 104] and [7] that considers only real polynomials and a modified version [8, 9, 13, 14] that in [13] also treats complex polynomials.

4.2.1. Jury's modified test

The construction rules for the table in [13] may be described in polynomial notation following the conversion convention of Remark 4 by the algorithm

$$f_{m-1}(z) = \frac{1}{\eta_m}\{f_{m,0}\bar{f}_m - f^{\star}_{m,m}f^r_m(z)\} \quad , \quad f_n(z) = p^r(z) \tag{41}$$

where

$$\eta_n = \eta_{n-1} = 1 \quad , \quad \eta_m = f_{m+1,0} \text{ for } m = n-2, n-3, \dots$$

A careful step by step comparison of this recursion with Algorithm C (36) reveals the following relations with the C-prototype polynomials

$$f_{n-2i}(z) = c^r_{n-2i}(z) \quad , \quad f_{n-2i+1}(z) = c^{\sharp}_{n-2i+1}(z) \quad , \quad i = 0, 1, 2, \dots$$

In the above use is made of the fact that $f_{m,0} = c_{m,m}$ are real for $m \leq n-1$. The distinguished entries that in [13, 14] are denoted by Δ_m correspond in the current notation to $f_{n-m,0}$. and therefore

$$\Delta_{n-m} = f_{m,0} = c_{m,m} = det\{\mathbf{C}_{0:m-1}\}, \quad , \ m = 1, 2, \dots, n \tag{42}$$

where the last equality follows from Theorem 7. This provides a proof for the statement in [13] that the Δ_m form the principal minors of the Schur-Cohn matrix.

Furthermore, Theorem 6 provides a zero location rule for the algorithm (41).

Corollary 2. (Jury's modified version) *Consider the algorithm (41). Provided (41) does not terminate prematurely, the number of OUC and IUC zeros are ν and $n-\nu$ where*

$$\nu = Var\{1, f_{n-1,0}, f_{n-2,0}, \dots, f_{0,0}\}$$

References [13] and [14] state without a proof only the stability conditions (that all $\Delta_m = f_{m,0} > 0$) for this table form. The current proof is different from a proof in an unpublished report [15] available from the author. Also the zero location rule we deduce for this table here is in disagreement with the rule proposes in [15] (Part c of example 1 is a ready counterexample that the number of OUC zeros is not $\nu = n_-\{f_{n-1,0}, f_{n-2,0} \dots, f_{0,0}\}$).

4.2.2. The earlier version.

The modified version is an enhancement to an earlier version of this table in [17, p. 104] and [7]. Since theses version consider only real polynomials the extension to complex polynomial is not unique (recall the earlier remark 5). A possible extension of it to complex polynomials is as follows.

$$f_{m-1}(z) = \frac{1}{\eta_m}\{f_{m,0}\bar{f}_m - f^{\star}_{m,m}f^r_m(z)\} \quad , \quad f_n(z) = p(z) \tag{43}$$

The seemingly minute change of the initiation by comparison with (41) has a remarkable impact on the accompanying stability and zero location rules and on the relations of the distinguished entries to the Schur-Cohn determinants μ_m. Currently the sequence$\{f_m(z)\}$ can be shown to relate to the Algorithm C prototype sequence (36) as follows.

$$f_{n-2i}(z) = c^{\sharp}_{n-2i}(z) \quad , \quad f_{n-2i+1}(z) = -c^r_{n-2i+1}(z) \quad , \quad i = 1, 2, \dots \tag{44}$$

Consequently, the relations with the Schur-Cohn determinants are now

$$f_{n-m,0} = (-1)^m det\{\mathbf{C}_{0:m-1}\} \quad , m = 1, \dots n$$

The next corollary follow at once.

Corollary 3. (Jury's earlier version.) *Consider the algorithm (43). Provided it does not terminate prematurely, the number of OUC and IUC zeros are ν and $n - \nu$ where*

$$\nu = Var\{1, -f_{n-1,0}, f_{n-2,0}, -f_{n-3,0}, \dots, (-1)^n f_{0,0}\}$$

These results support the stability conditions

$$f_{n-2i,0} > 0 \quad , \quad f_{n-2i+1,0} < 0 \quad , \quad i = 1, 2, \dots$$

(see e.g. [17, p. 105]) and the relations to the Schur-Cohn determinant provided by Jury [17, p. 105]. The enhancement of the modified version of §4.2.1 over this earlier version is in simpler forms for the stability conditions and the relations to the principal minors of $\mathbf{C}$.

5. Schemes of Type D

In the first paragraph of section 3 we explained that it suffices to search interesting tests that obey the recursion (4) with $\psi_n = f^\star_{n,n}$ and real ψ_m, $m < n$. It is in particular interesting in this subclass to examine the association of zero location rule to a recursion like (35),

$$zf_{m-1}(z) = \phi_m\{f_{m,m}f_m(z) - f_{m,0}f^\sharp_m(z)\} \ ,$$

in which with all $\phi_m = 1$. The specialty of this scheme is that it avoids the operation of division.

5.1. Scheme D : Basic Form

Algorithm D. For $m = n-1, \dots, 1$ do:

$$zd_{m-1}(z) = d^\star_{m,m}d_m(z) - d_{m,0}d^\sharp_m(z) \quad , \quad d_n(z) = p(z) \qquad (45)$$

($d_{m,m}$ are real for $m \le n-1$.)

We try to obtain a zero location rule for this algorithm. Comparing the highest power coefficients gives

$$d_{m-1,m-1} = |d_{m,m}|^2(1 - |k_m|^2) \quad , \ m = n, n-1, \dots \ 1 \qquad (46)$$

Therefore, the leading coefficients $\{d_{m,m}\}$ are real for all $m \le n-1$. Furthermore, a step by step comparison of the $d_m(z)$ with $a_m(z)$ yields

$$d_{n-1}(z) = |d_{n,n}|^2(1 - |k_n|^2)a_{n-1}(z) \,,$$
$$d_{n-2}(z) = |d_{n-1,n-1}|^2(1 - |k_{n-1}|^2)a_{n-2}(z) \,, \text{etc.}$$

The following relations become apparent

$$\begin{aligned}
q_n &= (1 - |k_n|^2) = \frac{d_{n-1,n-1}}{|d_{n,n}|^2} \\
q_{n-1} &= (1 - |k_n|^2)(1 - |k_{n-1}|^2) = \frac{d_{n-1,n-1}}{|d_{n,n}|^2}\frac{d_{n-2,n-2}}{d^2_{n-1,n-1}} = \frac{d_{n-2,n-2}}{|d_{n,n}|^2 d_{n-1,n-1}} \\
q_{n-2} &= q_{n-1}(1 - |k_{n-2}|^2) = \frac{d_{n-3,n-3}}{d^2_{n-2,n-2}}\frac{d_{n-2,n-2}}{|d_{n,n}|^2 d_{n-1,n-1}} = \frac{d_{n-3,n-3}}{|d_{n,n}|^2 d_{n-1,n-1} d_{n-2,n-2}} \\
&\vdots \\
q_{n-i} &= \frac{d_{n-i-1,n-i-1}}{|d_{n,n}|^2 \prod_{l=1}^{i} d_{n-l,n-l}}
\end{aligned}$$

The above relation provide the key to posing the zero location rule for Algorithm D in terms of its leading coefficients.

Theorem 8. (Zero Location for Algorithm D) *If Algorithm D does not terminate prematurely then $p(z)$ has ν OUC and $n-\nu$ IUC zeros where ν is given by*

$$\nu = n_-\{g_n, g_{n-1}, \ldots, g_1\} \quad , \quad g_m := \prod_{i=m}^{n} d_{i-1,i-1} \tag{47}$$

where the auxiliary parameter g_m's may also be calculated recursively by

$$g_m = d_{m-1,m-1}g_{m+1} \quad , \; g_{n+1} := 1 \;\; , \;\; m = n, n-1, \; \ldots$$

Proof. The rule follows from the basic rule (9) after developing further the expressions obtained for $d_{m,m}$ in terms of q_m's into the following relations.

$$g_n = |d_{n,n}|^2 q_n \; , \; g_{n-1} = [|d_{n,n}|^2]^2 q_n^2 q_{n-1} \qquad \text{etc.}$$

that is seen to lead to

$$g_{n-i} = [|d_{n,n}|^2]^{2^i} q_n^{2^i} q_{n-1}^{2^{i-1}} \cdots q_{n-i+1}^2 q_{n-i}$$

It is possible to obtain from here either a direct expression for g_m's in terms of q_m's

$$g_m = q_m q_{m+1}^2 q_{m+2}^{2^2} \cdots q_n^{2^{n-m}} [|d_{n,n}|^2]^{2^{n-m}} \quad , \quad m = 1, \ldots \; n \; . \tag{48}$$

or alternatively a more compact but recursive relation between these parameters, viz.

$$g_m = g_{m+1}^2 q_m$$

Either of these relations makes it clear that $sgn\{g_m\} = sgn\{q_m\}$. Q.E.D.

Example 1 (cont'd) [part d]. The sequence $\{d_m(z)\}$ produced for $p(z) = 4 + 12.5z + 5z^2 + z^3$ consists of $d_3(z) = p(z)$ and

$$d_2(z) = -7.5 - 45z - 15z^2 \quad , \quad d_1(z) = -33.75 + 168.75z \quad , \quad d_0(z) = -85430$$

Therefore

$$g_3 = d_{2,2} = -15 \quad , \quad g_2 = d_{2,2}d_{1,1} = -2531.2 \quad , \quad g_1 = d_{2,2}d_{1,1}d_{0,0} = 2.1624 \cdot 10^8$$

Then

$$\#OUC = n_-\{-15, -2531.2, 2.1624 \cdot 10^8\} = 2$$

5.2. Type D: Marden's version

Marden's original test is a Type D scheme devoid of division[4] (see also [5, Theorem (42,1)] and [18, Theorem 5.6]).

Marden considers the algorithm

$$f_{m-1}(z) = f_{m,0}^{\star} f_m(z) - f_{m,m} f_m^{\sharp}(z) \quad , \quad f_n(z) = p(z) \tag{49}$$

which in a step by step identification procedure of Marden's sequence $\{f_m(z)\}$ with the sequence $\{d_m(z)\}$ of Algorithm D shows that

$$f_n(z) = d_n(z) \;\; ; \;\; f_{n-1}(z) = -d_{n-1}^{\sharp}(z) \;\; ; \;\; f_m(z) = d_m^{\sharp}(z) \text{ for } m = n-2, n-3, \ldots, 0$$

Marden then defines the auxiliary parameters

$$p_m = \prod_{i=1}^{m} f_{n-i,n-i} \quad , \quad m = 1, \ldots, n \tag{50}$$

which, in our notation (47) corresponds to

$$p_m = -\prod_{i=1}^{m} d_{n-i,n-i} = -g_{n+1-m} \quad , \quad m = 1, \ldots, n$$

Consequently Marden's theorem is reproducible from Theorem 8.

Corollary 4. (Marden) *Consider for a polynomial $p(z)$ the algorithm (51) and define the parameters (50) If all the products p_k are nonzero then $p(z)$ has π IUC and $n-\pi$ OUC zeros where*

$$\pi = n_-\{p_1, p_2, \ldots .p_n\} \tag{51}$$

5.3. Type D: The Maria-Fahmy version

Maria and Fahmy proposed in [10] a table for complex polynomials that relied on the test for real polynomials that was proposed by Jury and Blanchard in [6] (see same in [17, p. 98]) The table in [10] converts by our standard convention of Remark 4 to the algorithm

$$f_{m-1}(z) = f_{m0}\bar{f}_m(z) - f^{\star}_{m,m} f^r_m(z) \quad , \quad f_n(z) = p(z) \tag{52}$$

The following relation to the polynomials in the main D-type algorithm may be detected.

$$f_n(z) = d_n(z) \quad ; \quad f_{n-1}(z) = -d^r_{n-1}(z)$$

$$f_{n-2i}(z) = d^{\sharp}_{n-2i}(z) \quad ; \quad f_{n-(2i+1)} = d^r_{n-(2i+1)}(z) \quad \text{for} \quad i = 1, 2, \ldots$$

For these relations Theorem 8 can be invoked to extend the stability conditions in [10],

$$f_{n-1,0} < 0 \quad , \quad f_{m,m} > 0 \quad , \quad m = n-2, n-3, \ldots, 0$$

into the next zero location rule.

Corollary 5. (Maria and Fahmy's version) *Consider algorithm (52). If it does not terminate prematurely define*

$$p_m = \prod_{i=1}^{m} f_{n-i,n-i} \quad , \quad m = 1, \ldots, n \ .$$

Then $p(z)$ has π IUC and $n-\pi$ OUC zeros where

$$\pi = n_-\{p_1, p_2, \ldots .p_n\} \tag{53}$$

The fact that the generalization to complex polynomials that Maria and Fahmy obtained using the real tests in [6] is not identical to Marden's test although [6] was derived from Marden's test is again a demonstration to the mentioned nonunique way that a real test procedure can be extended to complex polynomials.

5.4. Type D : Chen's version

The table of Chen with Shan in [12] and with Shiao in [16] translates into the following polynomial recursion,

$$zf_{m-1}(z) = f_{m,m} f_m(z) - f_{m,0} f^{\sharp}_m(z) \quad , \quad f_n(z) = p(z)/p_n \ ,$$

They considered only real polynomials and assumed the tested polynomial is first scaled to be monic. With these constraints the recursion coincides with Algorithm D. They provide the right stability conditions but their rule for the number of OUC zeros

$$\nu = n_-\{f_{n-1,n-1}, f_{n-2,n-2}, \ldots, f_{0,0}\}$$

is wrong as the next counterexample may illustrate.

Example 2. The polynomial $p(z) = 0.5 + 9z + 12z^2 + z^3$ has 1 OUC and 2 IUC zeros. It is real and monic so that both Algorithm D and Chen's test associate to it the same sequence of polynomials: $d_3 = p(z)$ and

$$d_2(z) = 3 + 7.5z + 0.75z^2 \quad , \quad d_1(z) = -16.875 - 8.4375z \quad , \quad d_0(z) = -213.57$$

Using (47)

$$g_3 = d_{2,2} = 0.75 \quad , \quad g_2 = d_{2,2}d_{1,1} = -6.3281 \quad , \quad g_1 = d_{2,2}d_{1,1}d_{0,0} = 1351.5$$

Theorem 8 obtains the correct number of OUC zeros

$$\nu = n_{-}\{g_3, g_2, g_1\} = n_{-}\{0.75, -6.3281, 1351.5\} = 1$$

Instead the Chen's rule suggests that the number of OUC zeros is

$$n_{-}\{d_{2,2}, d_{1,1}, d_{0,0}\} = n_{-}\{0.75, -8.4375, -213.57\} = 2$$

6. Conclusion

The paper used the reflection coefficient to parametrize the tests in the Schur-Cohn and Marden-Jury tests class of methods for determining zero location of a polynomial with respect to the unit circle The SCMJ class was classified into four useful types of recursions. Although the polynomials in the four prototype sequences that were defined differ only in scaling factors from each other, this difference has at times quite intricate effect on expressing the zero location rule in terms of the "native" polynomial coefficients. The invariance of the set of reflection coefficients for all tests in the SCMJ class facilitated the derivation of zero location rules for the B, C, D types in terms of the leading coefficients of the polynomials in the sequence. The current systematic approach makes it possible to classify essentially any test published in this class into one of the defined types in spite of the masking effect of operations such as difference in initiation, reversion, conjugation or reciprocation and sign variation of polynomials in corresponding sequences. The classification process led at times corrections to wrongly stated zero location rules, support results stated with no accessible proof, or to some extent of generalization for the reviewed tests.

In the course of pursuing the zero location rule for the C-type scheme we in fact obtained a yet another proof to the relation between zero location of a polynomial and the principal minors (hence the inertia) of the Schur-Cohn matrix generated for that polynomial.

Appendix A: Proof for Theorem 1

Let (π_m, ν_m) denote the number of (OUC,IUC) zeros of $a_m(z)$ (strong regularity implies no UC zeros). Obtain from the recursions (4) and its reciprocation the two equations

$$\frac{za_{m-1}(z)}{a_m(z)} = 1 + k_m \frac{a^\sharp_m(z)}{a_m(z)} \quad , \quad \frac{a^\sharp_{m-1}(z)}{k^\star_m a_m(z)} = 1 + \frac{1}{k^\star_m}\frac{a^\sharp_m(z)}{a_m(z)} \qquad (a.1, a.2)$$

Observe that $\frac{|a^\sharp_m(z)|}{|a_m(z)|} = 1$ for $|z| = 1$. Applying the Principle of Argument on (a.1) for $|k_m| < 1$ and on (a.2) for $|k_m| > 1$ proves, respectively, that

$$\begin{array}{ll} \text{if } \ |k_m| < 1 & (\pi_m, \nu_m) = (\pi_{m-1} + 1, \nu_{m-1}) \\ \text{if } \ |k_m| > 1 & (\pi_m, \nu_m) = (\nu_{m-1} + 1, \pi_{m-1}) \end{array} \qquad (a.3)$$

Define for each fixed m, $m \le n$, its own sequence (10) of q_l's, $\{q^{(m)}_l, l = 1 \dots m\}$ by

$$q^{(m)}_l = \prod_{i=l}^{m}(1 - |k_i|^2) \quad , \ l = 1, \dots, m \ \ , \ \ q^{(m)}_{m+1} := 1$$

Note that in this new notation the original sequence of q_m's in (10) corresponds to $\{q_m = q^{(n)}_m$, $m = 1, \dots, n\}$.

Normalize each such sequence $\{q^{(m)}_i\}$ by dividing each of its entries by its minimal indexed member $q^{(m)}_1$

$$\frac{q^{(m)}_{l+1}}{q^{(m)}_1} = \prod_{i=1}^{l} \frac{1}{(1 - |k_i|^2)} := \hat{q}_l \quad , l = 1, \dots, n$$

The resulting new parameters $\hat{q}_l$ are independent of the index m (therefore superscripts (m) were dropped) and they can be computed recursively as follows.

$$\hat{q}_l = \hat{q}_{l-1}\frac{1}{(1 - |k_l|^2)} \ \ , \ \hat{q}_0 := 1 \quad , l = 1, \dots, n \qquad (a.4)$$

Thus, the sequence in (9) is given by $\{q_1, q_2, \dots, q_n\} = \{[\hat{q}^{(n)}_1, \hat{q}^{(n)}_2, \dots, \hat{q}^{(n)}_n]q^{(n)}_1\}$ where the used notional convention is defined by

$$\{[\hat{q}_1, \hat{q}_2, \dots, \hat{q}_n]q_1\} := \{\hat{q}_1 q_1, \hat{q}_2 q_1, \dots, \hat{q}_n q_1\}$$

Lemma 1. *The number of (IUC,OUC) zeros of the $m-$th degree polynomial $a_m(z)$, $(m - \nu_m, \nu_m)$, are given by*

$$\nu_m = n_-\{q^{(m)}_1, q^{(m)}_2, \dots, q^{(m)}_m\} = n_-\{[\hat{q}_1, \hat{q}_2, \dots, \hat{q}_m]q^{(m)}_1\} \qquad (a.5)$$

Proof. The proof of the Lemma is by induction over $m = 1, 2, \dots, n$. The $m = 1$ is clear from (a.3). Assume the assertion (a.5) holds till $m = l - 1$ so that the number of IUC and OUC zeros of $a_{l-1}(z)$ are given, respectively, by

$$\nu_{l-1} = n_-\{[\hat{q}_1, \hat{q}_2, \dots, \hat{q}_{l-1}]q^{(l-1)}_1\} \ \text{ and } \ \pi_{l-1} = n_+\{[\hat{q}_1, \hat{q}_2, \dots, \hat{q}_{l-1}]q^{(l-1)}_1\}$$

We have to show that the above implies that the (a.5) gives the correct number of IUC and OUC zeros also for $a_l(z)$.

There are two possibilities; either $|k_l| < 1$ or $|k_l| > 1$. For the case $|k_l| < 1$ (a.3) implies that $\nu_l = \nu_{l-1}$ (and $\pi_l = \pi_{l-1} + 1$). It is needed to check whether (a.5) provides the same result. For $|k_l| < 1$ (a.4) shows that $sgn(q^{(l)}_1) = sgn(q^{(l-1)}_1)$. Therefore

$$\nu_l = n_-\{[\hat{q}_1, \hat{q}_2, \dots, \hat{q}_{l-1}, \hat{q}_l]q^{(l)}_1\} = n_-\{[\hat{q}_1, \hat{q}_2, \dots, \hat{q}_{l-1}]q^{(l)}_1\} = n_-\{[\hat{q}_1, \hat{q}_2, \dots, \hat{q}_{l-1}]q^{(l-1)}_1\} = \nu_{l-1}$$

In the above we dropped a by definition positive term $q_1^{(l)}\hat{q}_l = 1 > 1$ from the n_- count. In the case $|k_l| > 1$ (a.3) implies that $\nu_l = \pi_{l-1}$ and again it has to be verified that the rule (a.5) is consistent with this case too. For $|k_l| > 1$ (a.4) indicates that $sgn(q_1^{(l)}) = -sgn(q_1^{(l-1)})$. Therefore

$$\nu_l = n_-\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{l-1}, \hat{q}_l]q_1^{(l)}\} = n_-\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{l-1}]q_1^l\} = n_+\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{l-1}]q_1^{(l-1)}\} = \pi_{l-1}$$

Q.E.D.

Theorem 1 corresponds to $m = n$ in this lemma.

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