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A STABILITY TEST FOR CONTINUOUS-DISCRETE BIVARIATE POLYNOMIALS

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ABSTRACT

This paper addresses the problem of testing whether a bivariate polynomial does not vanish in the product of the closed exterior of the unit-circle times the right half-plane. This requirement presents stability conditions for certain mixed continuous-discrete systems. An algebraic method to solve the problem in polynomial order of complexity is developed from the Jury's modified stability test.

1. INTRODUCTION

Stability of linear systems poses restrictions on where in the complex plane $\mathbb{C}$ the system's characteristic polynomial may or may not vanish. A stability test for a linear systems has to determine whether a polynomial satisfies the relevant restriction. Denote the imaginary axis by

$$\mathbb{I} = \{s : \mathcal{R}e(s) = 0\} \tag{1}$$

and the open left half plane and its complimentary in $\mathbb{C}$ (the complex plane) by

$$\mathbb{L} = \{s : \mathcal{R}e(s) < 0\ ,\ |s| < \infty\},\ , \quad \overline{\mathbb{R}} = \mathbb{C} - \mathbb{L}\ . \tag{2}$$

Also, denote the unit circle by

$$\mathbb{T} = \{z :\ |z| = 1\} \tag{3}$$

its open interior, and its complimentary by

$$\mathbb{U} = \{z :\ |z| < 1\}\ , \quad \overline{\mathbb{V}} = \mathbb{C} - \mathbb{U}\ . \tag{4}$$

A continuous system is stable if the zeros of its characteristic polynomial lie in $\mathbb{L}$. A discrete system is stable if the zeros of its characteristic polynomial lie in $\mathbb{U}$. For a one-variable(1-V)polynomial $h(s)$ that is not a constant, the condition "$h(s_i) = 0$ implies $s_i \in \mathbb{L}$" is equivalent to

$$h(s) \neq 0 \quad \forall s \in \overline{\mathbb{R}}\ . \tag{5}$$

Similarly, for a 1-V polynomial $d(z)$ that is not a constant, the condition "$d(z_i) = 0$ implies $z_i \in U$" is equivalent to

$$d(z) \neq 0 \quad \forall z \in \overline{\mathbb{V}}\ . \tag{6}$$

A 1-V polynomial will be called "1-C stable" if it satisfies (5) and "1-D stable", if the property (6) holds for it.

Turning to stability of multidimensional systems, one has to consider zero location of multivariate (M-V) polynomials with $M > 1$ variables. In general, the zeros of a multivariate polynomial can not be confined to a compact subset of $\mathbb{C}^M$. However, the dual manner of characterization, like that used in (5) and (6), still makes sense. Namely, it is still possible to pose stability requirement for multidimensional systems as conditions on where the characteristic multivariate polynomials is *not allowed* to vanish. The common subsets of interest are cartesian products of $\overline{\mathbb{R}}$'s and $\overline{\mathbb{V}}$'s. In considering two-dimensional systems, the two variables of the characteristic polynomials may be either both s-type, both z-type, or one s- and one is z-type. The *s-s* type stability arises in continuous-time two-dimensional systems and hence will be referred as 2-C stability. It poses the constraint that the polynomial has no zeros in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$, see [1] and references there in. The *z-z* case arises in stability of two-dimensional discrete-time systems and is referred as *2-D stability*. It poses the requirement of no zeros in $\overline{\mathbb{V}} \times \overline{\mathbb{V}}$ and was studied most intensively till very recently, see [2] [3] [4] and references there in. For a survey of earlier results in two-dimensional continuous and discrete systems see [5]. References for the third case, the *s-z* stability will be enrolled in the following. An illuminating treatment of all these three cases in a common framework of condition posed on appropriate polynomial Bezoutian matrices appears in [6].

This paper considers the testing of the *s-z* type stability conditions as follows. Consider a bivariate polynomial of degree (n_1, n_2)

$$Q(s, z) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} q_{ij} s^i z^j \quad , \ q_{n_1,n_2} \neq 0 \tag{7}$$

for which the following condition has to be assessed.

$$Q(s, z) \neq 0 \qquad \forall (s, z) \in \overline{\mathbb{R}} \times \overline{\mathbb{V}} \tag{8}$$

A polynomial that satisfies the condition (8) will be called C-D (continuous-discrete)stable. The problem arises in testing stability of certain systems that can be described by a linear differential-difference equations, where $Q(s, z)$ presents the characteristic polynomial [7]. A closely related problem is the stability of certain differential delay equations with commensurate delay [8], [9]. It leads to a characteristic 2-V polynomial, say $A(s, z)$, in which the second variable is dependent on s via $z = e^{-sh}$, where $h \geq 0$ denotes the unit of delay duration. It is quite apparent that C-D stability of $A(s, z^{-1})z^{n_2}$ is sufficient for the stability of $A(s, e^{-sh})$. Thus, the algorithm to test C-D stability that will be presented here is relevant also for testing efficiently stability of $A(s, z)$. However, the dependency between the two variables makes C-D stability more than necessary for this latter stability problem. This observation requires a quite intricate tuning of the method presented here to turn it into minimal conditions that are both necessary and sufficient for this stability problem. The stability of continuous systems with commensurate delay will be left for another publication.

Papers published so far on C-D stability (and even more so on the stability of the $A(s, e^{-sh})$) were concerned mostly with obtaining adequate stability conditions and with replacing them by equivalent conditions of simpler form rather than with offering any specific algebraic algorithm for testing the stability conditions and attending its complexity. One difference between an algebraic and a numerical stability test is that an algebraic test requires a finite number of arithmetic operations. A second asset that is often mentioned is that an algebraic test provides an unambiguous answer for the validity of the stability condition it tests. The problem with the second property is that it holds in practice for only low degree polynomials especially when the finite cost increases rapidly with the degree of the tested polynomial. Until very recently, the available stability tests for two-dimensional systems were of complexity that increases in a rapid rate with the degree of the tested polynomial. The resulting problem is not the high amount of computation itself but some consequences, like bulkiness in trying to use it in symbolic computation environment and, in conventional manners of use, the implied accumulation of numerical inaccuracy that may hinder the correct decision about stability because at the end the decision is based on rules that depend on signs of some numbers. This problem was well conceived in the more intensively studied problem of testing 2-D stability, where only recent research activity produced reduction of the cost of computation from complexity that grows at a severe exponential rate with polynomial degree to an only polynomial order of complexity, see [2] [3] [4] and references there in. A reviewer of this paper has brought to our attention a recent paper [10] that proposes a procedure to test C-D stability. It can be shown that the sought solution there has a cost that increases exponentially with the degree of the tested polynomial. The current paper offers, apparently for the first time, a C-D stability test with only polynomial order of complexity.

2. NOTATION

Our notation convention will use a same letter for both a polynomial and its matrix (or vector) of coefficients. For example, the coefficients matrix of the polynomial (7) is denoted too by $Q = (q_{i,k})$. We use $\mathbf{z}$ to denote a vector whose entries are powers in ascending degrees of the variable, $\mathbf{z} = [1, z, \ldots, z^i, \ldots]^t$ (of length determined by context). It allows one to write $d(z) = \mathbf{z}^t d$ and $Q(s,z) = \mathbf{s}^t Q \mathbf{z}$. It will be instructive during derivation to regard $Q(s,z)$ as a 1-V polynomial over $\mathbb{C}[z]$ with coefficients over $\mathbb{C}[s]$ (where $\mathbb{C}[x]$ denotes the set of polynomials in indeterminate x with complex coefficients). To this end, the columns of Q will be denoted $Q = [q_0\,, q_1\,, \ldots, q_{n_2}]$. So that one may write $Q(s,z)$ as

$$Q(s,z) = \sum_{k=0}^{n_2} q_k(s) z^k = [q_0(s), q_1(s), \ldots, q_{n_2}(s)]\mathbf{z}\,. \qquad (9)$$

We define a "diamond" operation for a 2-V polynomials of degree (n_1, n_2) with mixed *s-z* indeterminates and for a matrix (of its coefficients) as follows.

$$Q^{\diamondsuit}(s,z) = z^{n_2} Q^{\star}(-s, z^{-1}) \qquad , \qquad Q^{\diamondsuit} = K Q^{\star} J \qquad (10)$$

where J denotes the reversion matrix, a matrix with 1's on the main anti-diagonal and zeros elsewhere, K denotes a diagonal matrix with diagonal elements $(-1)^k$, $k = 0, 1 \ldots$, i.e $K = diag[1, -1,\ 1, -1, 1, \ldots]$ of size determined by context, and $\star$ denotes complex conjugation. Note that the definitions are such that the relation $Q^{\diamondsuit}(s,z) = \mathbf{s}^t Q^{\diamondsuit} \mathbf{z}$ remains true.

For an s-variable 1-V polynomial $h(s) = \mathbf{s}^t h$ and its coefficient vector we denote the para-conjugate operation by

$$h^{\natural}(s) = h^{\star}(-s) \qquad , \qquad h^{\natural} = K h^{\star} \qquad (11)$$

So that $h^{\natural}(s) = \mathbf{s}^t h^{\natural}$. For a z-variable 1-V polynomial $d(z) = \mathbf{z}^t d$ of degree n and its coefficient vector we denote the operation of conjugate reciprocation by

$$d^{\sharp}(z) = z^n d^{\star}(z^{-1}) \qquad , \qquad d^{\sharp} = J d^{\star} \qquad (12)$$

Again the reciprocated polynomial may be written as $d^{\sharp}(z) = \mathbf{z}^t d^{\sharp}$. Note that the diamond operation amounts to the combination of a pre para-conjugate and a post reciprocal (or a pre para a post conjugate-reciprocal) operations.

3. CONTINUOUS-DISCRETE STABILITY

In difference from all previous works that considered the stability condition (8) for only real coefficient polynomial, we shall admit Q to be complex, i.e. $Q(s,z) \in \mathbb{C}[s,z]$. The need for a stability test procedure for complex polynomials arises when they are embedded in testing stability of higher dimensional systems. It will become apparent that treating the complex case requires almost no extra effort. To this end, a polynomial $h(s) \in C[s]$ will too be called 1-C stable if it satisfies (5) and a $d(z) \in C[z]$ - 1-D stable if it satisfies (6).

Consider a $Q(s,z) \in \mathbb{C}[s,z]$ as in (7). The assumption that the leading coefficient $q_{n_1,n_2} \neq 0$ is an easily visible and not restrictive requirement because $q_{n_1,n_2} = 0$ implies that $Q(s,z)$ is already not stable. (Note that $s = \infty$ is belongs to $\overline{\mathbb{R}}$ so that a zero there implies an unstable 1-C polynomial and hence also C-D instability.) It should ne noted that failing to be stable because of a $q_{n_1,n_2} = 0$ does not infer anything on whether or not a polynomial that is equal to $Q(s,z)$ but is *defined* as one with lower degree and has a non-vanishing leading coefficient is stable.

The following simplifying condition for C-D stability are pertinent to our derivation.

Lemma 1. *The polynomial $Q(s,z)$ (7) satisfies (8) (it is C-D stable) if and only the following two conditions hold:*

(i) *$Q(s,b)$ is 1-C stable for some $b \in \overline{\mathbb{V}}$.*

(ii) *$Q(s,z) \neq 0\ \forall (s,z) \in \mathbb{I} \times \overline{\mathbb{V}}$.*

The above lemma first appeared in [6] with the values $b = \infty$. It next appeared with value $b = 1$ in [8, Theorem 2:(2)], with $b \in \mathbb{T}$ in [11], and with its above generality in [9]. The latter two references contain more simplifying conditions for C-D stability not covered by Lemma 1.

Our starting point will be Jury's modified 1-D stability test for complex polynomials [12]. The following form is the version called in [13] the main form for "C-type" 1-D stability tests.

Algorithm 1. Assign to a complex 1-D polynomial $p(z) = \sum_0^n p_i z^i$ a sequence of polynomials $\{c_m(z) = \sum_{i=0}^m c_{m,i} z^i; m = n-1, \ldots, 0\}$, as follows.

$$z c_{n-1}(z) = p_n^{\star} p(z) - p_0 p^{\sharp}(z)\,;\ d_{n-1} = 1 \qquad (13a)$$

For $m = n-1, \ldots, 1$ do:

$$z c_{m-1}(z) = \frac{c_{m,m} c_m(z) - c_{m,0} c_m^{\sharp}(z)}{d_m}\,;\ d_{m-1} = c_{m,m} \qquad (13b)$$

The Schur-Cohn-Fujiwara matrix is an $n \times n$ matrix, say B_T (also known as the unit-circle Bezoutian), that is associated with $p(z)$ by the expression,

$$B_T = L(p^\star_{n:1})L^t(p_{n:1}) - L(p_{0:n-1})L^t(p^\star_{0:n-1}) \qquad (14)$$

where $p_{0:n-1} = [p_0, \ldots, p_{n-1}]^t$, $p_{n:1} = [p_n, \ldots, p_1]^t$ and $L(a)$ denotes the lower triangular Toeplitz matrix whose first column is the vector a.

Stability conditions for Algorithm 1 and some pertinent relations with B_T are summarized in the next theorem (see [**?**] for a proof).

Theorem 1.(a) *If Algorithm 1 becomes singular (a $c_{m,m} = 0$ occurs) then $p(z)$ is not stable. Else, $p(z)$ is stable if, and only if, $c_{m,m} > 0$ for all $m = 1, \ldots, n$.*
(b) *Algorithm 1 does not become singular if, and only if, its SC matrix B_T is strongly regular (all its leading principal minors are non-zero).*
(c) *If B_T is strongly regular then the principal minors of B_T are given by the leading coefficients of the polynomials created by Algorithm 1, $\det\{B_{T\,1:k}\} = c_{n-k,n-k}$ $k = 1, \ldots, n$ (where $B_{T\,1:k}$ denotes the $k \times k$ upper-left submatrix of B_T).*

Algorithm 1 and Theorem 1 can be used to implement the testing of condition (ii) in Lemma 1

$$Q(s,z) \neq 0 \quad \forall (s,z) \in \mathbb{I} \times \overline{\mathbb{V}} \qquad (15)$$

by regarding $Q(s,z)$ as a 1-D polynomial in z with coefficients dependent on $s \in I$ as in (9).

Algorithm 2 [The C-D stability 'table']. Assign to $Q(s,z)$ written as in (9) with $n := n_2$ a sequence
$\{C_m(s,z) = \sum_{k=0}^{m} c_{[m]k}(s)z^k, m = n-1, \ldots, 0\}$,
obtained as follows. Start with

$$zC_{n-1}(s,z) = q_n^\natural(s)Q(s,z) - q_0(s)Q^\diamondsuit(s,z) \qquad (16a)$$

and $d_{n-1}(s) = 1$. Then for $m = n-1, \ldots, 1$ do:

$$zC_{m-1}(s,z) = \frac{c_{[m]m}(s)C_m(s,z) - c_{[m]0}(s)C_m^\diamondsuit(s,z)}{d_m(s)}$$

$$d_{m-1}(s) = c_{[m]m}(s) \qquad (16b)$$

The algorithm produces for a real/complex $Q(s,z)$ a sequence of real/complex 2-V polynomials (respectively) $C_m(s,z)$ of degree $(2(n_2-m)n_1, m)$, $m = n_2-1, \ldots, 0$. The fact that it produces polynomial is because the divisor polynomial $d_m(s)$ is always a factor of the numerator 2-V polynomial that it divides, as can be shown similar to the proof in [2]. The polynomials $c_{[m]\,m}(s)$ are even, i.e. $c^\natural_{[m]\,m}(s) = c_{[m]\,m}(s)$ or $c^\natural_{[m]\,m} = c_{[m]\,m}$. So that they take real values for $s \in \mathbb{I}$. This also implies that, the coefficients of $c_{[m]\,m}(s)$ start with a real free term and are alternatingly purely real and purely imaginary. In particular, when Q is real it follows that the second, fourth and so forth entries $c_{[m]\,m}$ are zero. Since Algorithm 2 is, per each $s \in \mathbb{I}$, an implementation of Algorithm 1 for $p_s(z) = Q(s,z)$, it follows from Theorem 1 that the condition (15) holds if and only if all the $c_{[m],m}(s)$ polynomials that Algorithm 2 creates satisfy

$$c_{[m]\,m}(s) > 0 \quad \forall s \in \mathbb{I} \quad , \quad m = n-1, \ldots, 0\,. \qquad (17)$$

Now, according to Theorem 1, the $c_{[m]\,m}(s)$ are the principal minors of B_T for the 1-V polynomial $p_s(z) = D(s,z)$, where now B_T becomes $B_T(s)$, a matrix with entries dependent on s. The necessary and sufficient conditions for positive definiteness of $B_T(s)$, depicted in (17) (positivity of all principal minors) holds if and only if $B_T(s)$ is positive definite at some single point $a \in \mathbb{I}$ plus the determinant of $B_T(s)$ is positive for all $s \in \mathbb{T}$ (repeating a key argument used before in [6]). Thus conditions (17) are equivalent to the next condition (a) & (b):

(a) $Q(a,z)$ is 1-D stable for some $a \in \mathbb{I}$ (because positive definiteness of $B_T(s)$ at $s = a$ is equivalent, again according to Theorems 1 to this 1-D stability condition).

(b) $c_{[0]0}(s) > 0$ $\forall s \in \mathbb{I}$ (because $c_{[0]0}(s)$ is equal to the determinant of $B_T(s)$).
But (17) together with the 1-C condition (i) of Lemma 1 form necessary and sufficient conditions for C-D stability of $Q(s,z)$. Finally, we can lessen the requirement in (b) from positivity to just (b':) $c_{[0]0}(s) \neq 0$ $\forall s \in \mathbb{I}$. Evidently $(b) \Rightarrow (b')$. For the converse, assume (b') & (a), we have from Theorem 1 that (17) holds at $s = a$, thus in particular $c_{[0]0}(a) > 0$. So that $(b')\&(a) \Rightarrow (b)$. In summary, we have proved the next theorem.

Theorem 2 [Stability conditions for Algorithm 2]. Assume algorithm 2 is applied to $Q(s,z)$ (7) and denote by $\epsilon(s) := C_0(s,z)$ the last polynomial that it produces. $Q(s,z)$ is stable if, and only if, the following three conditions hold.
(i) $Q(s,b)$ is 1-C stable for some $b \in \overline{\mathbb{V}}$.
(ii) $Q(a,z)$ is 1-D stable for some $a \in \mathbb{I}$.
(iii) $\epsilon(s) \neq 0 \quad \forall s \in \mathbb{I}$.

The polynomial $\epsilon(s)$ is the last polynomial that Algorithm 2 produces has degree $(2n_1n_2, 0)$. Namely, it is dependent on s only and has degree $2n_1n_2$. It is an even polynomial $\epsilon(s)$, like all the $c_{[m]m}(s)$'s. Thus condition (iii) holds if and only if $\epsilon(s)$ has n_1n_2 zeros in $\mathbb{L}$ and n_1n_2 zeros in $\mathbb{R}$. When Q is real then $\epsilon(s)$ is real and is has only powers of $x = s^2$. In this case condition (iii) amounts to no zeros of the polynomial in the variable x on the negative real axis, a condition that can be examined also by a conventional Sturm sequence. A general algebraic tool to examine the condition (iii) is an adequate extension of the Routh stability test to zero location with respect to the imaginary axis of complex polynomials, e.g. [14] and its cost is roughly n^4 (say $n_1 = n_2 = n$). Condition (i) and (ii) are tasks of relatively negligible cost of $O(n^2)$ operations by any 1-C and 1-D test (say a standard Routh test any the 1-D stability test that consists of Algorithm & Theorem 1). A convenient value for b is $b = 1$ (presented by the sum of columns of Q as coefficients). Condition (ii) is a 1-C stability test that can again be carried out by the Routh test. Two convenient choices of a for here are $a = 0$ (the first row of Q) and $a = \infty$ (the last row of Q).

4. COMPUTATION AND ILLUSTRATION

Although the presented procedure is applicable for also a $Q(s,z) \in \mathbb{C}(s,z)$ it will be illustrated here for the more common (and less spacious) case of a real polynomial Consider the following polynomial of degree (3,3).

$$Q(s,z) = [1\ s\ s^2\ s^3] \begin{bmatrix} 6 & 6 & -10 & 15 \\ 5 & 8 & -15 & 25 \\ 2 & 2 & -4 & 7 \\ 1 & 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{bmatrix}$$

To test whether or not it is C-D stable we carry out the test that consists of Algorithm 2 and Theorem 2 through the following four steps.

Step 1: Test condition (i) in Theorem 2 for an appropriate $Q(s,b)$. For example, take $Q(s,1) = [17\,,\, 23\,,\, 7\,,\, 3]\mathbf{s}$ (coefficients formed by sum of columns of Q) and verify that it is 1-C stable.

Step 2: Test condition (ii) in Theorem 2 for an appropriate $Q(a,z)$. For example it can be verified that $Q(0,z) = [6\,,\, 6\,,\, -10\,,\, 15]\mathbf{z}$ (whose coefficients are the first row of Q) is 1-D stable.

Step 3: Apply algorithm 2. First note that

$$Q^{\diamond} = \begin{bmatrix} 15 & -10 & 6 & 6 \\ -25 & 15 & -8 & -5 \\ 7 & -4 & 2 & 2 \\ -3 & 2 & -1 & -1 \end{bmatrix}$$

The algorithm produces a sequence of 2-V polynomials $\{\mathbf{s}^t C_m \mathbf{z},\ m = 2,1,0\}$ whose coefficient matrices are:

$$C_2 = \begin{bmatrix} 150 & -186 & 189 \\ -70 & 43 & 0 \\ -159 & 261 & -414 \\ -9 & 1 & 0 \\ -52 & 76 & -95 \\ 1 & -2 & 0 \\ -5 & 7 & -8 \end{bmatrix}, C_1 = \begin{bmatrix} -7254 & 13221 \\ 1557 & 0 \\ 54599 & -103892 \\ -7704 & 0 \\ -56050 & 127065 \\ 283 & 0 \\ -29598 & 60541 \\ 1353 & 0 \\ -5817 & 11337 \\ 268 & 0 \\ -531 & 1001 \\ 19 & 0 \\ -21 & 39 \end{bmatrix}$$

$C_0^t = [646425, 0, -8915057, 0,\ 35480226, 0,\ -27528155,\ 0,\ -22357775,\ 0, -6569912,\ 0, -1050718, 0,\ -99997, 0, -5414,\ 0, -135]$

Step 4: Examine $\epsilon(s) = \mathbf{s}^t C_0$ and show that it has no zeros on $\mathbb{I}$. Thus the tested polynomial is C-D stable.

The remaining available space will be used for a brief account on the cost of computation and a note on its further possible reduction. For simplicity, assume in the following that $n_1 = n_2 = n$. As already mentioned the testing of the condition (i) and (ii) require $O(n^2)$ operations and the testing of condition (iii) requires $O(n^4)$ operations. What dominates the overall cost is Algorithm 2 that can be shown to have an $O(n^6)$ cost. Within this $O(n^6)$, the actual cost (in terms real operations) for the complex case is of course higher than for the real case (a more accurate counts will appear in some forthcoming publication). The presented method can be readily simplified by "telepolation", an approach introduced in [3] and [4]. As a consequence, a simplified test can be obtained that carries out the test by a finite collection of 1-D stability tests and has a cost of only $O(n^4)$ complexity. Another issue is the extension of the method to testing the related stability problem of a continuous system with commensurate delay [8] [9]. These topics will be addressed in future paper.

5. CONCLUSION

This ISCAS paper has presented a method for testing the condition that a bivariate polynomial does not vanish in the closed right half plane times the closed exterior of the unit circle. This condition arises in testing stability of differential-difference equations that describe certain filters and industrial processes. The proposed test is algebraic, "tabular" and has $O(n^6)$ complexity. A way to simplify it further into a solution of only $O(n^4)$ complexity was indicated. Some omitted issues and new results on extending the method to also test the stability of continuous systems with commensurate delay are underway.

6. REFERENCES

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