# The counterpropagating Rossby wave perspective on Kelvin Helmholtz instability as a limiting case of a Rayleigh shear layer with zero width

Eyal Heifetz and Yuval Reuveni
*Department of Geophysics and Planetary Sciences, Tel-Aviv University, Israel*

Alexander Gelfgat and Eliezer Kit
*School of Mechanical Engineering, Faculty of Engineering, Tel-Aviv University, Israel*

John Methven
*Department of Meteorology, University of Reading, United Kingdom*

(Received 22 August 2005; accepted 1 December 2005; published online 13 January 2006)

The Kelvin Helmholtz (KH) problem, with zero stratification, is examined as a limiting case of the Rayleigh model of a single shear layer whose width tends to zero. The transition of the Rayleigh modal dispersion relation to the KH one, as well as the disappearance of the supermodal transient growth in the KH limit, are both rationalized from the counterpropagating Rossby wave perspective. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2166450]

Heifetz and Methven[1] (HM) have recently showed how the modal and nonmodal growth of the Rayleigh[2] model of a single shear layer [Fig. 1(a)] can be fully explained in terms of the interaction between two Rossby edge waves. Especially, they focused on the mechanism of the optimal transient growth, in both enstrophy and energy norms.

Interestingly, the classical horizontal Kelvin Helmholtz (KH) problem (with zero stratification) can be regarded as a limiting case of the Rayleigh model when the shear layer width tends to zero [Fig. 1(a)]. However, as shown here, in this limit the KH problem becomes normal and hence no supermodal transient growth is allowed. Furthermore, while the Rayleigh modal dispersion relation has a short wave cutoff [Fig. 1(b)] and its growth rate is maximized for wavenumbers in between zero and the cutoff, the KH growth rate vanishes only for zero wavenumber and increases linearly with wavenumber [Fig. 1(b)]. The transition between the Rayleigh and the KH problems must be continuous as the shear layer width goes to zero. Here, we wish to rationalize this transition by implementing the counterpropagating Rossby wave (CRW) perspective, suggested by HM. We focus on the disappearance of non-normality in the KH limit and on the fundamental change in the modal dispersion relations.

HM showed, after Heifetz *et al.*[3] (and after Davies and Bishop,[4] for the analogous baroclinic Eady[5] model), that the discrete spectrum dynamics of the Rayleigh model can be described in terms of the interaction between two vorticity edge waves, described by Bretherton[6] as "counterpropagating Rossby waves" (CRWs), located on the two boundaries of the shear layer at $y=\pm b$ [Fig. 1(b)]. The vorticity perturbation then can be written as

$$q' = [q_1(k,t)\delta(y+b) + q_2(k,t)\delta(y-b)]e^{ikx}, \tag{1a}$$

$$q_1 = Q_1(k,t)e^{i\epsilon_1(k,t)}, \tag{1b}$$

$$q_2 = Q_2(k,t)e^{i\epsilon_2(k,t)}, \tag{1c}$$

where the CRW amplitudes and phases are $(Q_1,Q_2)$, and $(\epsilon_1,\epsilon_2)$, respectively. From the linearized vorticity equation they deduced the CRW dynamic equations

$$\dot{Q}_1 = \sigma Q_2 \sin\epsilon, \tag{2a}$$

$$\dot{Q}_2 = \sigma Q_1 \sin\epsilon, \tag{2b}$$

$$\dot{\epsilon}_1 = -kc_1^1 - \sigma\frac{Q_2}{Q_1}\cos\epsilon, \tag{2c}$$

$$\dot{\epsilon}_2 = -kc_2^2 + \sigma\frac{Q_1}{Q_2}\cos\epsilon, \tag{2d}$$

where $\epsilon=(\epsilon_2-\epsilon_1)$ is the CRW phase difference, and $\sigma=(\Lambda/2)\exp(-2kb)$ is the CRW interaction coefficient. The mean vorticity gradient is concentrated on the boundaries, i.e.,

$$\bar{q}_y = \Lambda[\delta(y+b) - \delta(y-b)], \tag{3}$$

where $\Lambda=\partial\bar{u}/\partial y=U/b$ is the mean shear. The CRW intrinsic phase speeds (i.e., without interaction) are

$$c_1^1 = \bar{u}(-b) + \frac{\Lambda}{2k} = -U\left(1-\frac{1}{2kb}\right), \tag{4a}$$

$$c_2^2 = \bar{u}(b) - \frac{\Lambda}{2k} = U\left(1-\frac{1}{2kb}\right), \tag{4b}$$

where each intrinsic phase speed comprises the Doppler shift term due to the basic state flow at the edge, and a propagation counter to this flow that is proportional to the wavelength, and to the shear (the basic state vorticity gradient) via the Rossby[7] wave propagation mechanism. Equation (2) can be written alternatively in the matrix form,

 

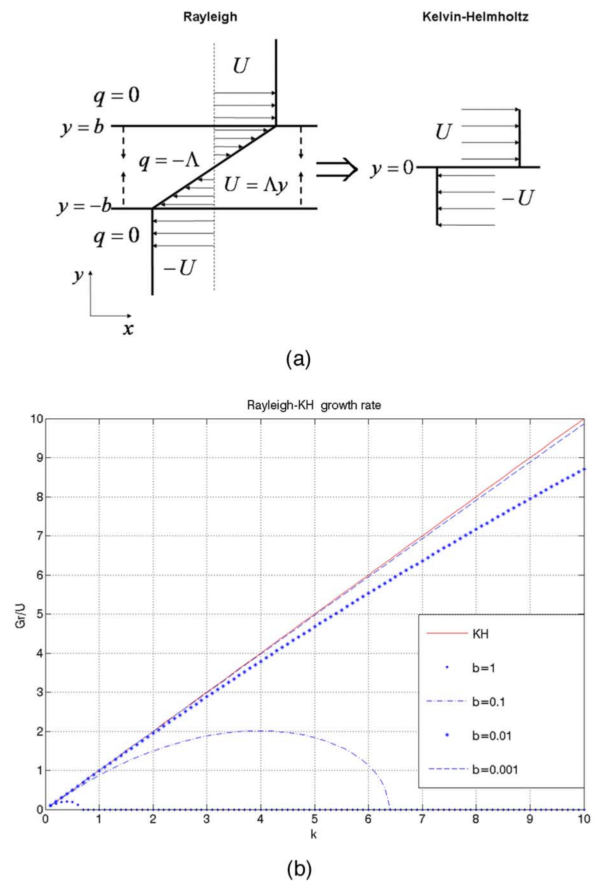


FIG. 1. (a) Schematic illustration of the transition of the Rayleigh problem to the Kelvin Helmholtz one. In the Rayleigh problem the mean velocity $U$ in the $x$ direction and the mean vorticity $q$ satisfy, respectively:

$$\bar{u} = \begin{cases} \Lambda b & \text{for } y \geqslant b \\ \Lambda y & \text{for } -b < y < b \\ -\Lambda b & \text{for } y \leqslant -b \end{cases} \quad \text{and} \quad \bar{q} = \begin{cases} 0 & \text{for } y \geqslant b \\ -\Lambda & \text{for } -b < y < b \\ 0 & \text{for } y \leqslant -b \end{cases} .$$

The Kelvin Helmholtz setup (right) is obtained as a limiting case of the Rayleigh problem (left) as the shear layer width $b \to 0$, the mean shear $\Lambda = \partial U / \partial y = U/b \to \infty$, while $U$ remains constant. (b) The growth rate of the Rayleigh problem, normalized by the mean velocity $U$ as a function of the wavenumber $k$ for different values of the half layer width $b$ (the length units of $k^{-1}$ and $(Gr/U)^{-1}$ are scaled by the length units of $b$), recall that

$$\frac{Gr}{U} = \frac{1}{2b}[e^{-4kb} - (2kb-1)^2]^{1/2}.$$

In the limit where $b$ goes to zero the Kelvin Helmholtz growth rate, $Gr/U = k$, is obtained.

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q}, \tag{5a}$$

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{5b}$$

$$\mathbf{A} = -i \begin{pmatrix} kc_1^1 & \sigma \\ -\sigma & kc_2^2 \end{pmatrix}. \tag{5c}$$

Generally $\mathbf{A}$ is a non-normal matrix (that is $\mathbf{A}\mathbf{A}^\dagger \neq \mathbf{A}^\dagger \mathbf{A}$, where $\mathbf{A}^\dagger$ is the Hermitian conjugate of $\mathbf{A}$) and therefore the CRWs can grow faster than the normal mode growth rate.[8] HM showed, in both enstrophy and energy norms, that optimal growth over a finite time is obtained by CRWs with equal amplitudes ($Q_1 = Q_2$) moving relative to each other so that their phase difference is $\pi/2$ at the midpoint of the interval.

In the limit where the shear layer goes to zero (i.e., $b \to 0$, but $U$ remains finite) all four terms in $\mathbf{A}$ become singular. However, carefully using the Taylor expansion it is straightforward to show that its eigenvalues converge to the zero stratification KH eigenvalues, $\lambda = \pm kU$. Also, it can be shown that (5) can be transformed into the energy norm and written as

$$\dot{\mathbf{e}} = \mathbf{D}\mathbf{e}, \tag{6a}$$

$$\mathbf{e} = \frac{\sqrt{k}}{2} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{6b}$$

$$\mathbf{D} = \begin{pmatrix} 0 & ikU \\ -ikU & 0 \end{pmatrix}, \tag{6c}$$

where $\mathbf{e}$ is the generalized coordinate energy vector (whose square magnitude is proportional to the integrated kinetic energy perturbation of a given wavenumber), and $(\psi_1, \psi_2)$ are the streamfunction amplitudes just below and just above the shear interface. Equation (6) can be obtained by using either the similarity transformation from enstrophy to energy norm, as suggested by HM, or by the direct standard KH analysis[9] which satisfies both the kinematic and dynamic conditions on the interface (i.e., that a fluid parcel remains on the interface while displaced and that the pressure is continuous across the interface). Equation (6) indicates that $\mathbf{D}$ is a normal matrix. Hence, we wish to understand why in the KH limit the Rayleigh problem becomes normal and why the dispersion relation alters.

For any finite shear $\Lambda$, (4) indicates that without interaction, short wavelengths will be moving in the direction of the mean velocity on their boundaries (advection dominates counterpropagation), while long waves (when $k < 1/2b$) will be moving in the direction opposite to the flow. In both cases the two CRWs can phase-lock to form a normal mode if the CRWs interact to help (in the case of short waves) or to hinder (in the case of long waves) the Rossby counterpropagation rate (cf. Fig. 3 in HM). Supermodal transient growth is possible due to relative motion between the CRWs. If the CRWs are initially not in a phase-locked configuration then during the optimal evolution their relative phase, $\epsilon$, decreases with time for short waves, and increases with time for long waves. If growing normal modes exist, then in both cases the two CRWs converge eventually to the growing modal phase-locking configuration. Supermodal optimal growth is obtained because the phase difference for fastest growth ($\epsilon = \pi/2$) is always passed through en route to the modal phase-locked configuration (for any optimization time interval).

For infinite shear, when $b \to 0$, all wavelengths become "long" in the sense that the counterpropagation rate becomes infinite. Since in the Rayleigh problem optimal growth is obtained when the two CRW amplitudes are equal (in both enstrophy and energy norms), i.e., $Q_1 = Q_2$, the change in their relative phase (2c) and (2d) becomes

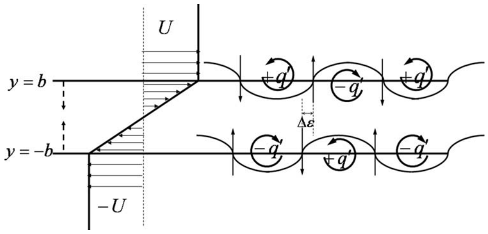


FIG. 2. Schematic illustration of the modal phase locking configuration of the two CRWs while approaching the KH limit, as $b \to 0$. The CRWs are of equal amplitudes and approach the phase locking $\epsilon = \pi (\Delta\epsilon = \pi - \epsilon \to 0)$. The undulating curves represent the cross-stream displacement $\eta$ of the CRWs and the arrows indicate the velocity field they induce by their vorticity anomaly $q$.

$$\dot{\epsilon} = -k(c_2^2 - c_1^1) + 2\sigma \cos \epsilon = U\left( \frac{1 + e^{-2kb} \cos \epsilon}{b} - 2k \right). \tag{7}$$

Thus, in the limit of infinite shear, $\dot{\epsilon} \to U(1/b - 2k) \times (1 + \cos \epsilon)$ goes to infinity unless $\epsilon = \pi$. Hence, for any finite wavelength, and for any initial phase difference, the CRWs will approach $\pi$ infinitely fast and will converge in zero time to this only possible phase-locking configuration where the CRW vorticities are in antiphase. Since no relative motion between the CRWs is possible in finite time, no transient growth is allowed in this limit, which is consistent with the normality of $\mathbf{D}$.

The conclusion that the CRWs must be $\pi$ out of phase is puzzling because at first glance (2a) and (2b) appear to indicate that the growth rate should be zero because $\sin \pi = 0$. However, the interaction coefficient tends to infinity with the shear, so the growth rate is nonzero in the limit as we now show. Normal mode growth corresponds to a synchronously growing [$Q_1 = Q_2$ required from (2a) and (2b)] phase-locked ($\dot{\epsilon} = 0$) state, so using (7) in (2a) and (2b) we find

$$\frac{\dot{Q}}{Q} = \sigma \sin \epsilon = \frac{U}{2b}[e^{-4kb} - (2kb - 1)^2]^{1/2}. \tag{8}$$

As $b \to 0$ the right-hand side of (8) converges to the KH growth rate $kU$. Hence, the modal KH limit can be regarded, for all wavelengths, as the consequence of phase-locking between two "very long" CRWs (compared to the shear layer width $b$), whose self-propagation speeds (4a) and (4b) approach infinity with opposite signs $\{[e^{-4kb} - (2kb-1)^2]^{1/2} \to 2kb\}$, but interaction $(U/2b)$ hinders their propagation to the extent that they are both stationary. The modal growth rate remains finite despite the infinite interaction coefficient because, as the layer width decreases, the vorticity anomalies of the two CRWs approach antiphase. Therefore, the cross-stream velocity induced by CRW-2 at the edge where CRW-1 exists (and vice versa) becomes almost in quadrature with the vorticity anomalies there (Fig. 2). It is only the fact that the CRWs are just off antiphase, such that $\sigma \sin \epsilon \neq 0$ as $b \to 0$, that enables growth from this perspective.

The cross-stream displacement $\eta$ is related to the vorticity anomalies through the linearized vorticity equation

$$\frac{D}{Dt} q' = -v \bar{q}_y = \frac{D}{Dt}(-\eta \bar{q}_y), \tag{9}$$

where $D/Dt = \partial/\partial t + \bar{u}\partial/\partial x$ is the linearized Lagrangian time derivative and $v = D\eta/Dt$ is the cross-stream velocity. If the vorticity perturbation $q'$ results solely from advection of the mean vorticity then $\eta = -q'/\bar{q}_y$. Hence, the Rayleigh mean vorticity gradient (3), implies that a northward displacement $(\eta > 0)$ yields a negative vorticity anomaly on the northern boundary and a positive anomaly on the southern one (see also Fig. 2). Therefore, in the modal KH limit, as the vorticity anomalies of the CRWs approach antiphase their cross-stream displacements become identical.

Understanding the velocity structure of the KH normal modes from the CRW perspective is also puzzling at first sight. As the width of the shear layer goes to zero, the vorticity anomalies of the two CRWs lay next to each other almost in antiphase. However, the superposition of the velocities induced by the CRWs does not vanish. Like a dipole, the velocity field does not vanish because the intensity of the vortices tends to infinity as they approach each other. In order to see that, consider the streamfunction induced by the two CRWs (as introduced by HM),

$$\begin{aligned} \psi' &= [\psi_1(k,y,t) + \psi_2(k,y,t)]e^{ikx} \\ &= \left[ -\frac{q_1(k,t)}{2k} e^{-k|y+b|} - \frac{q_2(k,t)}{2k} e^{-k|y-b|} \right] e^{ikx}. \end{aligned} \tag{10}$$

Therefore, using (1), the cross-stream velocity $v = \partial\psi/\partial x = (v_1 + v_2)e^{ikx}$, is composed of

$$v_1 = -\frac{i}{2} Q_1 e^{i\epsilon_1} e^{-k|y+b|}, \tag{11a}$$

$$v_2 = -\frac{i}{2} Q_2 e^{i\epsilon_2} e^{-k|y-b|}. \tag{11b}$$

Hence, in the region $y > b$ (when $Q_1 = Q_2$)

$$v = -\frac{i}{2} q_1 e^{-ky}(e^{-kb} + e^{i\epsilon} e^{kb}) e^{ikx}. \tag{12}$$

When the CRWs are phase-locked, $\dot{\epsilon} = 0$, (7) indicates that $\cos \epsilon = e^{2kb}(2kb - 1)$, and thus

$$\begin{aligned} v = -\frac{i}{2} q_1 e^{-ky}(e^{-kb} + e^{kb}\{e^{2kb}(2kb - 1) \\ + i[1 - e^{4kb}(2kb - 1)^2]^{1/2}\}) e^{ikx}. \end{aligned} \tag{13}$$

In the limit $kb \to 0$, (13) becomes $v = (1+i) q_1 e^{-ky} kb e^{ikx}$. However, since (1) and (9) evaluated on $y = -b$ suggest that $q_1 \delta(y+b) = -\eta \bar{q}_y = \eta U / b \, \delta(y+b)$, we obtain for $y > 0$

$$v = (1+i) kU e^{-ky} \eta. \tag{14}$$

Similarly, it can be shown that the complete velocity field perturbation becomes

$$u = kU e^{-k|y|} \eta \begin{cases} (1-i) & \text{for } y > 0 \\ (1+i) & \text{for } y < 0 \end{cases} \tag{15a}$$

$$\text{and } v = kUe^{-k|y|}\eta\begin{cases}(1+i) & \text{for } y>0\\ (1-i) & \text{for } y<0\end{cases}. \qquad (15b)$$

This velocity field is identical to the classical KH modal analysis (cf. Batchelor,[9] noting that in our analysis the shear is in the opposite direction with twice the magnitude). In the expression for $u$ [(15a)] the component in phase with cross-stream displacements is continuous across the vortex sheet, and arises from the cos $\epsilon$ term in the superposition of CRWs. It is in the same sense as advection by the mean flow either side of $y=0$ which Batchelor describes as "sweeping vorticity" towards the positive vorticity anomalies and away from the negative anomalies, resulting in growth. Taking the curl of the velocity field of (15), the perturbation vorticity field resulting from superposition of CRWs in the KH limit, $q'=2kUi\eta\delta(y)$, is located $\pi/2$ out of phase with the displacement $\eta$, as shown in Batchelor's Fig. 7.1.3. In the expression for $v$ [(15b)] the component in phase with cross-stream displacements is also continuous across the vortex sheet. However, this component arises from sin $\epsilon$ in (12) and is responsible for the growth of the cross-stream displacement through advection. Only the velocity components in phase with displacements and associated with growth are depicted in Batchelor's Fig. 7.1.3. The situation is not so simple and Batchelor's figure does not show the other components of velocity that are in quadrature with displacements of the vortex sheet and change sign across it.

It is simple to see that the cross-stream velocity $v$ cannot be continuous across the interface since under linearization $v=[\partial/\partial t+\bar{u}(y)\partial/\partial x]\eta$. Substituting the growing normal mode solution of the form of $\eta\sim e^{ik(x-ct)}=e^{ikx}e^{kUt}$ (using $c_r=0$, $c_i=U$) either side of the vortex sheet we again find (15b) (with $y=0$ at the sheet). The continuous component of $v$ (represented in Fig. 7.1.3 in Batchelor) is in phase with $\eta$ and is responsible for the amplification mechanism, while the discontinuous components of $v$ and $u$ act to resist the shear in order to maintain the coherent structure of the mode.

To conclude, we have shown that the CRW perspective can explain the transition from the Rayleigh model to the KH one. Shear instability in the limiting case of a vortex sheet can be viewed in terms of two CRWs on the boundaries of the sheet that grow and phase-lock through mutual interaction, just as for a shear layer of finite width. This growth mechanism takes account of the full velocity perturbation, not just the component in phase with cross-stream displacements. However, since the cross-stream displacements of the boundaries of a shear layer must become equal for continuity as its thickness tends to zero, the limit is subtle; the vorticity anomalies associated with displacements on each edge tend towards infinity but also tend towards antiphase so that they largely cancel. For a thin shear layer ($kb \ll 1$), the slight phase difference between the displacements, $\pi-\epsilon$, enables mutual growth at a rate proportional to the shear times this phase difference [(8)]. As layer width tends to zero, although the phase difference also tends to zero, the shear tends to infinity so that their product is nonzero and the KH growth rate $kU$ is obtained.

The differences in the complex dispersion relation of the two models are well known; however, the vanishing of the non-normality property of the Rayleigh model in the KH limit has not been noted previously to the best of our knowledge. Shear problems are generally highly non-normal, and therefore provide the efficient supermodal transient growth mechanism. In this sense the KH setup stands as an exception; therefore, if a thin shear layer is approximated by a vortex sheet, the supermodal growth mechanism is erroneously excluded.

This work is supported by the Israel Science Foundation (Grant 240/01). E.H. wishes to acknowledge Orkan Mehmet Umurhan and Hylke DeVries for illuminating discussions.

[1]E. Heifetz and J. Methven, "Relating optimal growth to counter-propagating Rossby waves in shear instability," Phys. Fluids **17**, 064107 (2005).
[2]Lord Rayleigh, "On the stability, or instability, of certain fluid motions," Proc. London Math. Soc. **9**, 57 (1880).
[3]E. Heifetz, C. H. Bishop, and P. Alpert, "Counter-propagating Rossby waves in the barotropic Rayleigh model of shear instability," Q. J. R. Meteorol. Soc. **125**, 2835 (1999).
[4]H. C. Davies and C. H. Bishop, "Eady edge waves and rapid development," J. Atmos. Sci. **51**, 1930 (1994).
[5]E. T. Eady, "Long waves and cyclone waves," Tellus **1**, 33 (1949).
[6]F. P. Bretherton, "Baroclinic instability and the short wave cut-off in terms of potential vorticity," Q. J. R. Meteorol. Soc. **92**, 335 (1966).
[7]C. G. Rossby, "Planetary flow patterns in the atmosphere," Q. J. R. Meteorol. Soc. **66**, 68 (1940).
[8]B. F. Farrell and P. J. Ioannou, "Generalized stability theory. Part I: Autonomous operators," J. Atmos. Sci. **53**, 2025 (1996).
[9]G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967).