INVESTIGATION OF THERMOGRAVITATIONAL-THERMOCAPILLARY STEADY-STATE CONVECTIVE FLOW STABILITY AT LOW PRANDTL NUMBERS

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The stability of gravitational-capillary flow in a square cavity with isothermal vertical and adiabatic horizontal boundaries is investigated. The region of stable regimes in the Grashof number—Marangoni number plane is determined for a fluid with a Prandtl number equal to 0.02.

In [1] the stability of steady-state thermogravitational convection regimes in a laterally heated square cavity was numerically investigated. The Galerkin method with a system of coordinate functions constructed as proposed in [2] was used to solve the system of equations of free convection in the Oberbeck—Boussinesq approximation. Below, the variant of the Galerkin method described in [2] is used to investigate the stability of steady-state regimes of free convection flow developing under the combined influence of thermogravitational and thermocapillary forces.

We will consider thermal convection in a square cavity whose vertical boundaries are maintained at constant and different temperatures. The horizontal boundaries are considered to be adiabatic. It is assumed that the upper boundary is a free surface, while the other boundaries are rigid. The flow is described by a system of convection equations in the Oberbeck—Boussinesq approximation with the corresponding boundary conditions

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\nabla p+\Delta \mathbf{v}+\mathrm{Gr}\,T\mathbf{e}_y,\quad \frac{\partial T}{\partial t}+(\mathbf{v}\nabla)T=\frac{\Delta T}{\mathrm{Pr}},\quad \mathrm{div}\,\mathbf{v}=0$$

$$\mathrm{Gr}=g\beta\frac{l^3}{\nu^2}\Delta T,\quad \mathrm{Pr}=\frac{\nu}{\chi},\quad \mathrm{Ma}=\frac{l}{\rho\nu^2}\left(\frac{\partial\sigma}{\partial T}\right)\Delta T \tag{1}$$

$$T|_{x=0}=1,\ T|_{x=1}=0,\ T_y'|_{y=0,1}=0 \tag{2}$$

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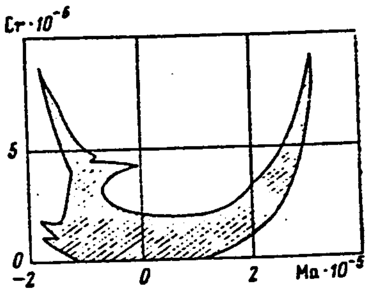


Fig. 1

$$\mathbf{v}|_{x=0,1}=\mathbf{v}|_{y=0}=0 \tag{3}$$

$$v_y|_{y=1}=0, \quad \left.\frac{\partial v_x}{\partial y}\right|_{y=1}=-\mathrm{Ma}\left.\frac{\partial T}{\partial x}\right|_{y=1} \tag{4}$$

Here, v is the fluid velocity, T is temperature, p pressure, g the free-fall acceleration, β the coefficient of volume expansion, ℓ the length of the cavity, ν the kinematic viscosity, χ thermal diffusivity, σ surface tension, ρ the density of the fluid, and $\mathbf{e}_y$ the unit vector in the direction of the y axis.

The solution of the problem (1)–(4) is sought in the form:

$$\mathbf{v}=\sum_{i=0}^{K_x}\sum_{j=0}^{N_y} c_{ij}(t)\varphi_{ij}(x,y)+\mathrm{Ma}\sum_{l=0}^{M} f_l(t)\varphi_l^*(x,y) \tag{5}$$

$$T=(1-x)+\sum_{i=0}^{K_x}\sum_{j=0}^{K_y} d_{ij}(t)q_{ij}(x,y) \tag{6}$$

The functions φ_{ij}, φ_l^*, and q_{ij} are constructed of linear combinations of Chebyshev polynomials of the first and second kinds so as to satisfy the continuity equations and all the boundary conditions except for that containing the Marangoni number. This condition is satisfied numerically, as a result of which the coefficients $f_\ell(t)$ are expressed in terms of the coefficients $d_{ij}(t)$. Following the application of the Galerkin method the problem reduces to a system of first-order ordinary differential equations of the form:

$$\dot{X}_i(t)=a_{ij}X_j(t)+b_{ijk}X_j(t)X_k(t)+F_i \tag{7}$$

where X_k is one of the coefficients c_{ij} or d_{ij}.

The steady-state solutions of system (7) ($\dot{X}_i = 0$), which with the aid of sums (5) and (6) determine the steady-state solutions of the problem (1)–(4), were calculated by Newton's method. For investigating the stability of the steady-state solutions we used the traditional procedure described in [3, 4].

In the calculations described below we used nine functions in the x direction, four functions in the y direction, and ten functions for satisfying the second of conditions (4) ($N_x = K_x = 9$, $N_y = K_y = 4$, $M = 10$). As shown in [5], the numerical method employed makes it possible to obtain results which are in good agreement with the various numerical and experimental data.

In all the calculations the Prandtl number was taken equal to 0.02, which is typical of molten metals and semiconductors. For a fixed Prandtl number the stability of the steady convective flows is determined by the Grashof and Marangoni numbers. The results of our numerical stability investigation are shown in the (Gr, Ma) plane (Fig. 1). The region of stability is shaded. The points of intersection of the region and the coordinate axes correspond to the critical numbers obtained in the absence of thermocapillary ($\mathrm{Ma} = 0$, $\mathrm{Gr} = 2.1\cdot10^6$) or thermogravitational ($\mathrm{Gr} = 0$, $\mathrm{Ma} = \pm1.14\cdot10^5$) forces.

The variation of the critical Grashof number with gradual increase in the Marangoni number is characterized by the upper boundary of the region of stability. Let us consider the case $\mathrm{Ma} > 0$. An increase in the Marangoni number from 0 to $8\cdot10^4$ causes only a slight decrease in the critical Grashof number Gr^*, from $2.1\cdot10^6$ to $1.9\cdot10^6$. A further increase in Ma is accompanied by the rapid growth of the critical value of the Grashof number, which when $Ma = 3.2 \cdot 10^5$ reaches a value of $9.1 \cdot 10^6$. Similar growth of the critical value of the Marangoni number, characterized by the lower boundary of the region of stability, is observed as the Grashof number gradually increases. The variation of Gr from 0 to $8 \cdot 10^6$ leads to an increase in the critical value of the Marangoni number from $1.1 \cdot 10^5$ to $3.2 \cdot 10^5$.

When $Ma > 0$ the liquid moves along the free surface under the influence of the thermocapillary force in the same direction as under the influence of the thermogravitational forces. Therefore an increase in either of the numbers Gr or Ma leads to an intensification of the convective heat transfer. The observed growth of the critical value of one of the parameters as the other increases shows that the intensification of the convective heat transfer by the combined action of the thermogravitational and thermocapillary forces may lead to stabilization of the motion. Obviously, when Gr or Ma exceeds a certain value, it will not be possible to stabilize the flow by increasing the other parameter. According to Fig. 1, these values of Gr and Ma are equal to $9.1 \cdot 10^6$ and $3.2 \cdot 10^5$, respectively.

When $Ma < 0$ the thermocapillary force acts along the free surface in a direction opposite to that of the thermogravitational flow. The different directions of action of the surface and volume forces may lead both to a weakening of the intensity of the convective flow and to the appearance of two convective eddies (see Fig. 5 below), one predominantly thermocapillary and the other predominantly thermogravitational. The stabilization of the flow with increase in the absolute value of one of the parameters Gr or Ma, which takes place in this case also (see Fig. 1), can be attributed to the decrease in the integral intensity of the flow as a result of the growth of Gr or Ma. On the other hand, the interaction of several convective eddies may create the conditions for destabilization of the flow. The presence of stabilizing and destabilizing factors explains the more complex (as compared with the case $Ma > 0$) shape of the neutral curve which, in Fig. 1, separates the regions of stable and unstable convection regimes. When $-1.8 \cdot 10^5 \leq Ma \leq -1.5 \cdot 10^5$, as the Grashof number increases repeated alternation of stable and unstable steady convective flow regimes is observed. When $Gr \approx 4.25 \cdot 10^6$ the region of stable regimes approaches the axis $Ma = 0$, without crossing it: in this case the flow is already stabilized at $Ma \approx -280$. When the Grashof number reaches the value $8.75 \cdot 10^6$, the convective flow will be unstable for any negative Marangoni number. The flows will also be unstable for any Grashof number if $Ma < -1.8 \cdot 10^5$.

To each continuous interval of the boundary of the region of stability (see Fig. 1) there corresponds a particular physical mechanism causing instability of the steady convective flow. An idea of the physical instability mechanism can be obtained by analyzing the spatial structure of the most dangerous infinitesimal perturbation. The latter is determined by the eigenvector, calculated for the critical values of the parameters, of system (7) linearized in the neighborhood of the steady-state solution. Together with the sums (5) and (6), the eigenvector of the linearized system (7), which corresponds to an eigenvalue with non-negative real part, determines (correct to multiplication by a constant) the most dangerous infinitesimal perturbation whose exponential growth destroys the steady convective flow. As the calculations show, to each smooth interval of the boundary of the region of stability there corresponds a particular spatial structure of the most dangerous perturbation.

Figures 2–6 show the steady convective flows and the corresponding most dangerous infinitesimal perturbations for values of Gr and Ma lying on different smooth intervals of the boundary of the region of stability. The continuous curves in Figs. 2–6 represent the streamlines (a) and isotherms (b) of the steady flows, the broken curves the isolines of the most dangerous perturbations of the stream function (a) and temperature (b). The Grashof and Marangoni numbers determining the flows and perturbations in Figs. 2–6 are equal to: 0, $1.87 \cdot 10^6$, $3.5 \cdot 10^6$, $4 \cdot 10^6$, $2.1 \cdot 10^6$ and $1.14 \cdot 10^5$, 10^5, $-7 \cdot 10^4$, $-1.1 \cdot 10^5$, 0, respectively.

The existence of qualitatively different structures of the most dangerous perturbation indicates that instability of the convective flow may be caused by different physical mechanisms which, as the calculations show, abruptly replace each other on transition from one smooth interval of the stability limit to another (see Fig. 1). At the same time, on each smooth interval the spatial structure of the most dangerous perturbation varies continuously.

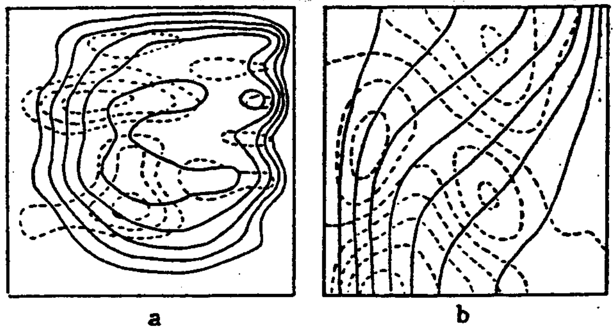


Fig. 2

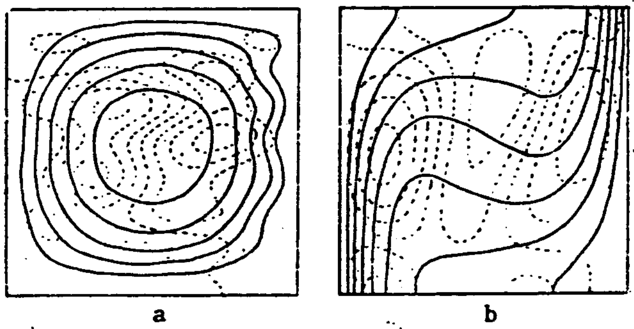


Fig. 3

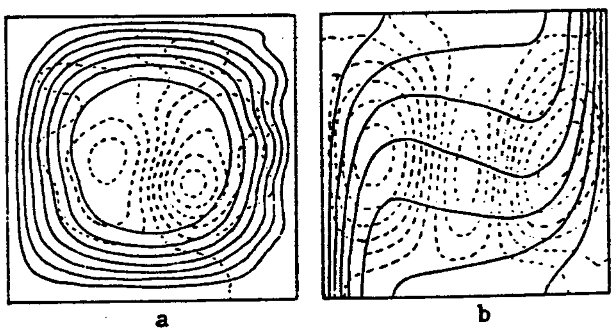
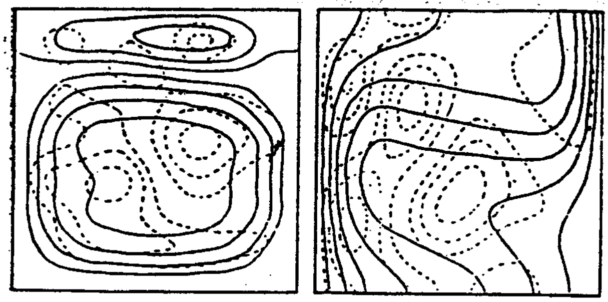


Fig. 4

In the absence of thermogravitational forces when Gr = 0 and Ma = $1.14 \cdot 10^5$ (Fig. 2) the maxima of the most dangerous perturbation of the stream function are distributed around the perimeter of the convective eddy. The maximum values of the most dangerous temperature perturbation are located along the horizontal walls of the cavity, which may be associated with the intense motion of the liquid along the free boundary.

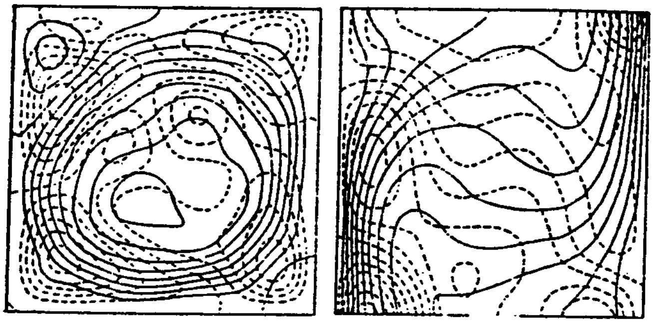
An example of a stability-losing steady convective flow with Gr = $1.87 \cdot 10^6$, Ma = 10^5, associated with the upper boundary of the region of stability, is given in Fig. 3. This case corresponds to the same branch of the neutral curve as that on which the point Ma = 0, Gr = $2.1 \cdot 10^6$ is located, thus making it possible to compare Figs. 3 and 6. The comparison shows how the shape of the thermogravitational flow and the structure of the disrupting perturbation are affected by the thermocapillary force. The presence of a thermocapillary convection mechanism leads to the disappearance of the small back eddies which when Ma = 0 (see Fig. 6) are located in the upper left and lower right corners of the cavity. The maxima of the stream function perturbation are displaced from the boundaries into the center of the flow region. When Ma = 10^5 the greatest values of the temperature perturbation (Fig. 2b) are located in the middle of the vertical boundaries. The relative values of the maxima of the temperature perturbations near the lower left and upper right corners of the cavity decrease with increase in the Marangoni number.

Figures 4 and 5 show examples of the steady flows and corresponding perturbations for Ma < 0. Obviously, as a result of the symmetry of the problem the case Gr = 0, Ma = $-1.14 \cdot 10^5$ will correspond to Fig. 2a reflected about the straight line x = 0.



a b

Fig. 5



a b

Fig. 6

Figure 4 shows the flow and perturbations for Gr = $3.5\cdot10^6$ and Ma = $-7\cdot10^4$. This case corresponds to the same branch of the neutral curve as the stability-losing flows shown in Figs. 3 and 6. A comparison of Figs. 3 and 4 indicates that in these two cases the flow and perturbation structures are similar. In the latter case the velocity perturbations have a simpler structure, and in the temperature perturbation diagram the maxima located near the corners of the cavity in Figs. 3 and 6 are missing.

As already noted, when Ma < 0 the interaction of the thermogravitational and thermocapillary forces may lead to a decrease in the intensity of the convective flow. Therefore on the branch of the neutral curve in question the values of the critical Grashof number in the neighborhood of Ma = 0 increase when Ma < 0 and decrease when Ma > 0.

When Gr = $4\cdot10^5$, Ma = $-1.10\cdot10^5$ (Fig. 5) the flow consists of two eddies, one due to the predominance of the thermocapillary force at the free surface and the other due to the action of the thermogravitational force in the part of the flow remote from the boundary y = 1. The distribution of the velocity perturbation isolines shows (see Figs. 3 and 5) that after loss of stability the upper (thermocapillary) eddy retains its shape, while the lower (thermogravitational) eddy is heavily deformed. The convective transport of hotter fluid from the heated to the cooled boundary due to the action of the top of the lower eddy and the bottom of the upper eddy (Fig. 5) leads to a sharp deflection of the isotherms and the displacement of the temperature perturbation maxima towards the cooled wall.

It should be noted that the most interesting result of investigating the stability of steady thermogravitational-thermocapillary convection regimes is the flow stabilization effect associated with an increase in the absolute value of the Grashof or Marangoni number. However, in practice it is difficult to make use of this effect because of the comparative narrowness of the region of stable flow (Fig. 1). Further investigation is required to determine the dependence of the shape and size of the stability zone on the Prandtl number or the geometric parameters of the flow region.

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