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On the Squire's transformation for stratified two-phase flows in inclined channels

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ABSTRACT

The applicability of the Squire's transformation for stability analysis of stratified two-phase flow in horizontal and inclined channels is examined. It is shown that for the considered flow such a transformation requires some additional constraints on the change of the inclination angle and flow rates of each of the phases. While the Squire's theorem (on the two-dimensionality of the critical disturbances) rigorously holds for the horizontal two-phase flow, for the inclined flow an exact mathematical theorem cannot be formulated. Nevertheless, it has been proven that 2D perturbations are the critical ones also for the case of inclined channel, since the transformation of a 3D stability problem to its 2D analog is associated with a stabilizing effect of reducing the system inclination, in addition to the reduction of the phases flow rates as in the case of horizontal flows.

1. Introduction

Stability of horizontal and inclined stratified two-phase plane-parallel flow is commonly studied in the framework of two-dimensional (2D) analysis, which considers only perturbations in the plane of the flow, while, to the best of our knowledge, oblique, i.e., three-dimensional (3D), perturbations have never been considered.

For a single-fluid plane Poiseuille flow, Squire (1933) was the first to show the equivalence between the linear stability of a 3D perturbation and that of a 2D perturbation but at a lower Reynolds number. Squire presented the relations between wavenumbers of 3D perturbations and their 2D equivalents and between the corresponding Reynolds numbers. Since then these relations are known as the Squire's transformation (Drazin and Reid, 2004). Using this transformation it can be easily proved that for each unstable 3D disturbance there is a more unstable 2D one. This allows one to conclude that consideration of the 2D stability problem is sufficient for a given plane Poiseuille flow. The latter is referred to in the literature as the Squire's theorem (Drazin and Reid, 2004).

Stability problems with several governing dimensionless parameters may also be subject to transformations of Squire's type. However, in contrast to those governed by the Reynolds number only, for some of the parameters the transformation may have a destabilizing effect. In such cases the Squire's theorem is not applicable for justifying the claim that the primary instability in the flow is associated with 2D perturbations (i.e., in the plane of the flow). For example, Gumerman and Homsy (1974) derived a Squire's transformation for thermally stratified horizontal two-phase flow and demonstrated that due to the competing effects of shear, surface tension and gravity the Squire's theorem (i.e., the criticality of 2D perturbations) cannot be claimed. Smyth and Peltier (1990) and later Gelfgat and Kit (2006) showed that in density stratified mixing layers instability can be triggered by 3D perturbations. On the other hand, Pearlstein (1985) proved the 2D nature of the critical disturbances for double-diffusive plane parallel (single phase) shear flows with varying temperature and concentration. For stratified two-phase flow, following the claim of Yih (1955), Hesla et al. (1986) established the sufficiency of 2D perturbations for the case of horizontal channel, provided that the density stratification is stabilizing (i.e., the upper phase is lighter than the lower phase). Yiantsios and Higgins (1988) and Tilley et al. (1994) discussed the possible inapplicability of the Squire's theorem for zero-gravity condition and flow in an inclined channel, respectively, drawing a conclusion that for such circumstances a three-dimensional analysis may be required. In a recent study, Allouche et al. (2015) compared stability with respect to 3D and 2D perturbations for a film flowing down an inclined surface. They argued that by combining the Squire's transformation and the results of 2D analysis, the 2D perturbations are found to be always more unstable. However, the validity of the Squire's theorem in the case of inclined stratified two-phase flow has never been established.

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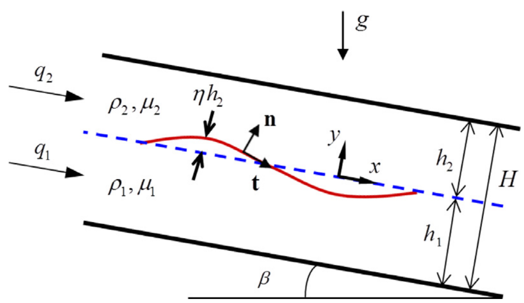

Fig. 1. Configuration of stratified two-layer flow in an inclined channel (z-axis comes out of the page).

In a recent study of Barmak et al. (2016b), the linear stability of stratified two-phase flows in inclined channels with respect to 2D perturbations of an arbitrary wavenumber was explored. The study was aimed at identifying the parameter regions for which a stable stratified configuration (with respect to all wavenumber perturbations) exists in gas-liquid and liquid-liquid concurrent and countercurrent flows. Attention was given to the operational conditions associated with the multiple-solution regions to reveal the feasibility of non-unique stable stratified configurations in inclined channels. Carrying out such a comprehensive analysis with respect to all possible 3D perturbations is practically unfeasible. Therefore, a question arises to what extent the stability of stratified two-phase flow can be determined by considering only 2D perturbations in the general case of inclined concurrent or countercurrent flows.

In this study, an appropriate transformation in the spirit of Squire (1933) is applied to convert the 3D stability problem to an equivalent 2D one for the case of inclined stratified two-phase flow. We show that for the considered flow such a transformation requires some additional constraints on the change of the inclination angle and flow rates of each of the phases. We show that the Squire's theorem rigorously holds for the horizontal two-phase flow. For the inclined flow, however, no exact mathematical theorem can be formulated. We argue, however, that in spite the absence of a rigorous mathematical formulation, the 2D perturbations are the most unstable one at least for most of practically important cases also in inclined two-phase flows.

2. Problem formulation

The flow configuration of a stratified two-layer flow of two immiscible incompressible fluids in an inclined channel ($0 \le \beta < \pi/2$) is sketched in Fig. 1. The flow, assumed to be isothermal, is driven by an imposed pressure gradient and a component of the gravity along the channel walls. The interface between fluids, labeled as $j=1, 2$ (1 – lower (heavy) phase, 2 – upper (light) phase), is assumed to be flat in the undisturbed base flow state. The flow in each of the fluids is described by the continuity and momentum equations that are rendered dimensionless in the standard manner (see Kushnir et al., 2014), scaling lengths by the height of the upper layer h_2, velocities by the interfacial velocity u_i, time by h_2/u_i, and pressures by $\rho_2 u_i^2$.

For the indicated three-dimensional coordinate system (where z comes out of the page), the dimensionless continuity and momentum equations governing the flow are

$$\frac{\partial u_j}{\partial x}+\frac{\partial v_j}{\partial y}+\frac{\partial w_j}{\partial z}=0,$$

$$\frac{\partial u_j}{\partial t}+u_j\frac{\partial u_j}{\partial x}+v_j\frac{\partial u_j}{\partial y}+w_j\frac{\partial u_j}{\partial z} = -\frac{\rho_1}{r\rho_j}\frac{\partial p_j}{\partial x}+\frac{1}{\mathrm{Re}_2}\frac{\rho_1}{r\rho_j}\frac{m\mu_j}{\mu_1}\left(\frac{\partial^2 u_j}{\partial x^2}+\frac{\partial^2 u_j}{\partial y^2}+\frac{\partial^2 u_j}{\partial z^2}\right)+\frac{\sin\beta}{\mathrm{Fr}_2},$$

$$\frac{\partial v_j}{\partial t}+u_j\frac{\partial v_j}{\partial x}+v_j\frac{\partial v_j}{\partial y}+w_j\frac{\partial v_j}{\partial z} = -\frac{\rho_1}{r\rho_j}\frac{\partial p_j}{\partial y}+\frac{1}{\mathrm{Re}_2}\frac{\rho_1}{r\rho_j}\frac{m\mu_j}{\mu_1}\left(\frac{\partial^2 v_j}{\partial x^2}+\frac{\partial^2 v_j}{\partial y^2}+\frac{\partial^2 v_j}{\partial z^2}\right)-\frac{\cos\beta}{\mathrm{Fr}_2},$$

$$\frac{\partial w_j}{\partial t}+u_j\frac{\partial w_j}{\partial x}+v_j\frac{\partial w_j}{\partial y}+w_j\frac{\partial w_j}{\partial z} = -\frac{\rho_1}{r\rho_j}\frac{\partial p_j}{\partial z}+\frac{1}{\mathrm{Re}_2}\frac{\rho_1}{r\rho_j}\frac{m\mu_j}{\mu_1}\left(\frac{\partial^2 w_j}{\partial x^2}+\frac{\partial^2 w_j}{\partial y^2}+\frac{\partial^2 w_j}{\partial z^2}\right), \quad (1)$$

where $\mathbf{u}_j=(u_j, v_j, w_j)$ and p_j are the velocity and pressure of the fluid j, ρ_j and μ_j are the corresponding density and dynamic viscosity. In the dimensionless formulation the lower and upper phases occupy the regions $-n \le y \le 0$, and $0 \le y \le 1$, respectively, where $n=h_1/h_2$. The other dimensionless parameters are the Reynolds numbers $\mathrm{Re}_{1,2}=\rho_{1,2}u_i h_2/\mu_{1,2}$ ($\mathrm{Re}_1=\mathrm{Re}_2 r/m$), the light fluid Froude number $\mathrm{Fr}_2=u_i^2/gh_2$, and the density and viscosity ratios $r=\rho_1/\rho_2$ and $m=\mu_1/\mu_2$, respectively.

The velocities satisfy the no-slip boundary conditions at the channel walls

$$\mathbf{u}_1(y=-n)=0, \quad \mathbf{u}_2(y=1)=0 \ . \quad (2)$$

The disturbed interface $y=\eta(x, z, t)$, which is a surface in the 3D problem, is defined by a unit-length normal vector $\mathbf{n}$, and $\mathbf{t_1}$, $\mathbf{t_2}$ denote two unit vectors tangential to the interface in xy- and yz-planes, respectively:

$$\mathbf{n}=\frac{(-\eta_x, 1, -\eta_z)}{\sqrt{1+\eta_x^2+\eta_z^2}},\ \mathbf{t_1}=\frac{(1, \eta_x, 0)}{\sqrt{1+\eta_x^2}},\ \mathbf{t_2}=\frac{(0, \eta_z, 1)}{\sqrt{1+\eta_z^2}}. \quad (3)$$

Boundary conditions at the interface $y=\eta(x, z, t)$ require continuity of the velocity components and the tangential stresses, and a jump of the normal stress due to the surface tension (the square brackets denote the jump of the expression value across the interface)

$$\mathbf{u}_1(y=0)=\mathbf{u}_2(y=0), \quad (4)$$

$$[\mathbf{t_1}\cdot\mathbf{T}\cdot\mathbf{n}]=\left[\frac{m\mu}{\mu_1}(-2\eta_x u_x+(u_y+v_x)-\eta_z(w_x+u_z)+2\eta_x v_y)\right] = 0. \quad (5)$$

$$[\mathbf{t_2}\cdot\mathbf{T}\cdot\mathbf{n}]=\left[\frac{m\mu}{\mu_1}(2\eta_z v_y-\eta_x(w_x+u_z)+(v_z+w_y)-2\eta_z w_z)\right] = 0. \quad (6)$$

$$[\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n}]=\Bigg[p-\frac{m\mu}{\mu_1}\frac{2\mathrm{Re}_2^{-1}}{1+\eta_x^2+\eta_z^2}\Big(\big(\eta_x^2-\eta_z^2\big)u_x+\big(1-\eta_z^2\big)v_y - \eta_x(u_y+v_x)-\eta_z(w_y+v_z)+\eta_x\eta_z(u_z+w_x)\Big)\Bigg] = \mathrm{We}_2^{-1}\frac{\eta_{xx}\big(1+\eta_z^2\big)+\eta_{zz}\big(1+\eta_x^2\big)-2\eta_x\eta_z\eta_{xz}}{\big(1+\eta_x^2+\eta_z^2\big)^{3/2}}. \quad (7)$$

where $We_2=\rho_2 h_2 u_i^2/\sigma$ is the Weber number, and σ is the surface tension coefficient.

The interface displacement and the normal velocity components are coupled by the kinematic boundary condition.

$$v_j=\frac{D\eta}{Dt}=\frac{\partial\eta}{\partial t}+u_j\frac{\partial\eta}{\partial x}+w_j\frac{\partial\eta}{\partial z}. \quad (8)$$

3. Base flow

The unperturbed base flow is assumed to be steady, laminar, and fully developed (the velocity $U(y)$ is parallel to the channel walls and varies only with the cross-section coordinate y). Details on the exact steady state solution can be found in the literature (e.g., Ullmann et al., 2003; Barmak et al., 2016b). The base flow solution is fully determined by three dimensionless parameters: the viscosity ratio m, the flow rate ratio $q=q_1/q_2$, and the inclination parameter $Y=\rho_2(r-1)g\sin\beta/(-dP/dx)_{2S}$. Here q_j is the feed flow rate of phase j (positive in the x direction), and $(-dP/dx)_{jS}=12\mu_j q_j/H^3$ is the corresponding superficial pressure drop for single phase flow in a channel of a height $H=h_1+h_2$. The Martinelli parameter, which represents the ratio of the superficial frictional pressure drop in the two phases, is $X^2=(-dP/dx)_{1S}/(-dP/dx)_{2S}=m\,q$. It can replace either q or m. Note that with $q>0$, the solution corresponds to concurrent upward flow in case of $Y<0$, and to concurrent downflow in case of $Y>0$. Countercurrent flow, $q<0$, is feasible only when the light phase flows upward, hence for $Y<0$.

Given the parameters (m,Y, q) the lower phase holdup, $h=h_1/H=n/(n+1)$, can be obtained by solving the algebraic Eq. (10) in Barmak et al. (2016b). Then, all the dimensionless characteristics of the base flow, including the interfacial velocity, $\tilde{u}_i=u_i/U_{2S}=\tilde{u}_i(m,q,h)$ (where $U_{jS}=q_j/H$ is the superficial velocity of fluid j), the dimensionless velocity profiles $\tilde{u}(y)=u(y)/u_i=\tilde{u}(m,q,h)$, and the driving forces, $\tilde{P}_{1,2}\equiv(dP/dx-\rho_{1,2}g\sin\beta)/(-dP/dx)_{2S}=\tilde{P}_{1,2}(m,q,h)$, can be determined. In horizontal flows ($Y=0$), where $h=h(m,\ q)$, and for a given two-fluid system (i.e., specified m) the holdup h and all the dimensionless flow characteristics are determined by the flow rate ratio, so that the solution is unique. In countercurrent flow ($Y<0$, $q<0$) two solutions are obtained for the holdup, which merge to a single solution at the flooding point. In concurrent upflow ($Y<0$, $q>0$) and concurrent downflow ($Y>0$, $q>0$) up to 3 different solutions for the holdup can be obtained in certain ranges of q and Y.

4. Linear stability for three-dimensional perturbations

In the framework of the linear analysis, the perturbed velocities and pressure fields are written as $u_j=U_j+\tilde{u}_j,\ v_j=\tilde{v}_j,\ w_j=\tilde{w}_j,\ p_j=P_j+\tilde{p}_j$, and $\eta=\tilde{\eta}$ for the dimensionless disturbance of the interface. The base flow is subject for infinitesimal 3D perturbations of the form:

$$\begin{pmatrix}\tilde{u}_j\\ \tilde{v}_j\\ \tilde{w}_j\\ \tilde{p}_j\\ \tilde{\eta}\end{pmatrix}=\begin{pmatrix}\bar{u}_j(y)\\ \bar{v}_j(y)\\ \bar{w}_j(y)\\ \bar{p}_j(y)\\ H_\eta\end{pmatrix}e^{(i(k_Xx+k_Zz)+\lambda t)} \tag{9}$$

where $\bar{u}_j,\ \bar{v}_j,\ \bar{w}_j,\ \bar{p}_j$ and H_η are the perturbation amplitudes, $k_X,\ k_Z$ are dimensionless real wavenumbers in the streamwise and spanwise directions respectively ($k_X=2\pi h_2/l_X$, $k_Z=2\pi h_2/l_Z$ with l_X and l_Z being the corresponding wavelengths) and λ is the complex time increment. In the following discussion the overbars in the notation of the perturbation amplitudes are omitted (e.g., v_j instead of $\bar{v}_j$) . Note also that since the collocation method based on the Chebyshev polynomials (defined in the interval [0, 1]) is applied to solve the stability problem (Barmak et al, 2016a), a new coordinate $y_1=(y+n)/n$ ($0\le y_1\le 1$) is introduced for the part of the channel occupied by the lower phase, while $y_2=y$ ($0\le y_2\le 1$) for the upper phase remains unchanged.

When (9) is substituted in the linearized governing equations and boundary conditions, the well-known Orr-Sommerfeld equations for the transverse component of velocity are obtained for each layer

$$\begin{array}{l}0\le y_1\le 1\\(-n\le y\le 0):\end{array}\quad \frac{\lambda}{k_X}D_1v_1=\left[i\left(-U_1D_1+\frac{U''_1}{n^2}\right)+\frac{1}{k_X\mathrm{Re}_1}D_1^2\right]v_1, \tag{10}$$

$$\begin{array}{l}0\le y_2\le 1\\(0\le y\le 1):\end{array}\quad \frac{\lambda}{k_X}D_2v_2=\left[i\left(-U_2D_2+U''_2\right)+\frac{1}{k_X\mathrm{Re}_2}D_2^2\right]v_2, \tag{11}$$

where $D_1v_1=\frac{v''_1}{n^2}-(k_X^2+k_Z^2)v_1$, $D_1^2v_1=\frac{v_1^{IV}}{n^4}-2(k_X^2+k_Z^2)\frac{v''_1}{n^2}+(k_X^2+k_Z^2)^2v_1$, $D_2v_2=v''_2-(k_X^2+k_Z^2)v_2$, $D_2^2v_2=v_2^{IV}-2(k_X^2+k_Z^2)v''_2+(k_X^2+k_Z^2)^2v_2$.

The linearized boundary conditions are obtained by means of Taylor expansions of η in the vicinity of its unperturbed zero value

$$y_1=1,\ y_2=0\ :\quad \frac{\lambda}{k_X}\Pi=-i(v_2(0)+\Pi),$$
$$\text{with }\Pi=\frac{v'_2(0)-v'_1(1)/n}{U'_1(1)/n-U'_2(0)},\quad \text{where }\Pi=-ik_XH_\eta, \tag{12}$$

$$\begin{aligned}y_1=1,\ y_2=0\ :\ &\frac{\lambda}{k_X}\left(v'_2(0)-r\frac{v'_1(1)}{n}\right)\\&=i\Bigg[\left(\left(v_2(0)U'_2-v'_2(0)U_2\right)-r\left(v_1(1)\frac{U'_1}{n}-\frac{v'_1(1)}{n}U_1\right)\right)\\&\quad+\frac{(k_X^2+k_Z^2)}{k_X^2}\left(\frac{\cos\beta}{\mathrm{Fr}_2}(r-1)+\frac{(k_X^2+k_Z^2)}{\mathrm{We}_2}\right)\cdot\Pi\Bigg]\\&\quad+\frac{1}{k_X\mathrm{Re}_2}\Bigg(\left(v'''_2(0)-3\left(k_X^2+k_Z^2\right)v'_2(0)\right)\\&\quad-m\left(\frac{v'''_1(1)}{n^3}-3\left(k_X^2+k_Z^2\right)\frac{v'_1(1)}{n}\right)\Bigg),\end{aligned} \tag{13}$$

$$y_1=0:\quad v_1=v'_1=0, \tag{14}$$

$$y_2=1:\quad v_2=v'_2=0, \tag{15}$$

$$y_1=1,\ y_2=0\ :\quad v_1(1)=v_2(0), \tag{16}$$

$$\begin{aligned}y_1=1,\ y_2=0\ :\quad &m\left(\frac{v''_1(1)}{n^2}+\left(k_X^2+k_Z^2\right)v_1(1)\right)\\&-\left(v''_2(0)+\left(k_X^2+k_Z^2\right)v_2(0)\right)\\&+\left(m\frac{U''_1}{n^2}-U''_2\right)\cdot\Pi=0.\end{aligned} \tag{17}$$

The temporal linear stability for 3D perturbations (Eqs. (10)–(17)) can be solved in the same manner as for 2D perturbations (see Barmak et al., 2016a,b). However, in addition to a wavenumber in the flow direction k_X, all wavenumbers k_Z should be considered (since the channel is infinite in the z-direction), for the specified two-phase system and operational conditions (i.e., the flow rate of each of the phases). Neutral stability of the flow corresponds to $\max(\lambda_R)=0$ for $u_i>0$ ($\min(\lambda_R)=0$ for $u_i<0$), where λ_R is the perturbation growth rate. The flow is considered to be stable if the real parts of all eigenvalues are negative for $u_i>0$, or if the real parts of all eigenvalues are positive while $u_i<0$. The dimensionless phase speed of the perturbation is defined as a quantity $c_R=-\lambda_I/k$, where λ_I is the wave angular frequency with $k=\sqrt{k_X^2+k_Z^2}$. The details on the numerical method can be found in Barmak et al. (2016a). The numerical solution was verified by comparison with the two-dimensional analysis ($k_Z=0$).

5. Squire's transformation

The linear stability with respect to 2D perturbations was studied for stratified two-phase flows in inclined channel in Barmak et al. (2016b). The governing equations and boundary conditions for the 2D case can be easily deduced for v-component of the velocity from Eqs. (10)–(17), taking $w=0$ and $k_Z=0$. The resulting 2D formulation would be identical to that presented in Barmak et al. (2016b), where the transverse velocity component has been expressed in terms of the stream function.

Comparing the formulations, one can easily reduce the three-dimensional problem to an equivalent two-dimensional problem by using the following transformation (the parameters for this analog are denoted with a superscript "2D"):

$$k_X^{2D} = \sqrt{k_X^2 + k_Z^2},$$
$$\mathrm{Re}_{1,2}^{2D} k_X^{2D} = \mathrm{Re}_{1,2}^{\ 2D}\sqrt{k_X^2 + k_Z^2} = \mathrm{Re}_{1,2} k_X,$$
$$\mathrm{Fr}_2^{2D}\left(k_X^{2D}\right)^2 = \mathrm{Fr}_2^{2D}\left(k_X^2 + k_Z^2\right) = \mathrm{Fr}_2 k_X^2,$$
$$We_2^{2D}\left(k_X^{2D}\right)^2 = \mathrm{We}_2^{2D}\left(k_X^2 + k_Z^2\right) = \mathrm{We}_2 k_X^2. \tag{18}$$

The velocity components and pressure perturbations are transformed in the following way $k_X^{2D} u^{2D} = k_X u + k_Z w$, $p^{2D}/k_X^{2D} = p/k_X$. The relation for the obtained eigenvalues (time increments) reads $\lambda^{2D} = \lambda\sqrt{k_X^2 + k_Z^2}/k_X$.

The fluids considered in both cases (2D and 3D) are identical, hence m and r are invariant. In terms of dimensional parameters, the only component that is changing in the Reynolds, Froude, and Weber numbers is the interfacial velocity:

$$u_i^{2D}\sqrt{k_X^2 + k_Z^2} = u_i k_X. \tag{19}$$

The analogy holds only when the base state velocity profile, included in the governing equations, remains unchanged. To meet this requirement the holdup should stay the same, $h^{2D}=h$. In horizontal flow, where $h=h(m,\ q)$, this requirement corresponds to $q^{2D}=q$, or in terms of the superficial velocities the following relations should be satisfied

$$\frac{U_{1S}^{2D}}{U_{1S}} = \frac{U_{2S}^{2D}}{U_{2S}} = \frac{u_i^{2D}}{u_i} = \frac{k_X}{\sqrt{k_X^2 + k_Z^2}}. \tag{20}$$

Thus, for horizontal flow the transformation from a 3D linear stability problem into an equivalent 2D linear stability problem (Squire's transformation) is exact. As the equivalent 2D problem is associated with lower flow rates of both phases, the addition of a spanwise wave component always has a stabilizing effect provided that stable density stratification is considered ($r > 1$). Therefore, in such cases the critical instability is two-dimensional in the plane of the flow and the Squire's transformation fully applies.

In inclined systems, however, the angle of inclination is an additional parameter and $h=h(m,\ q,\ Y)$. Hence, constant holdup can be maintained if in addition to $q^{2D}=q$ the inclination parameter is invariant, whereby $Y^{2D}=Y$. Using the definition of Y, the latter condition can be satisfied for fixed fluid properties and channel size if

$$\sin\beta^{2D} = \frac{U_{2S}^{2D}}{U_{2S}}\cdot\sin\beta = \frac{k_X}{\left(k_X^2 + k_Z^2\right)^{1/2}}\cdot\sin\beta. \tag{21}$$

This means that the channel inclination considered in the equivalent 2D stability problem is changed when the Squire's transformation is applied, whereby:

$$\beta^{2D} = \arcsin\left(\frac{k_X}{\left(k_X^2 + k_Z^2\right)^{1/2}}\cdot\sin\beta\right). \tag{22}$$

Eqs. (20) and (22) suggest that the Squire's transformation for inclined system is associated with reduction of the flow rates of each of the phases and decrease of the channel inclination. Generally, both modifications have stabilizing effects. Therefore, one may expect the validity of the Squire's theorem also in inclined systems. However, due to the presence of the term with $\cos\beta \neq 1$ in the boundary condition on the normal stress (Eq. (13)), the resulting 2D problem is not a complete equivalent of the 3D problem. Nevertheless, the effect of $\cos\beta$ is relatively small for shallow inclinations (e.g., it varies from 1 to 0.9 while β is from 0° to 26°). The main effect of gravity in inclined channels is due to the stream wise component ($g\sin\beta$) via its impact on the holdup and velocity profiles. For steeper channel inclinations, the 2D analysis indicates that the stratified flow is unstable in the entire range of practical flow rates (Barmak et al., 2016b). Thus, the 3D stability analysis can reveal only additional unstable disturbance modes. In inclined channels of shallow inclinations, at which stratified flow with a smooth interface can be stable, an approximate transformation can be utilized with sufficient accuracy to convert the 3D into its equivalent 2D stability problem in order to determine the stability of the flow. This will be shown and discussed in the following section.

6. Results and discussion

The parameter regions, in which stratified flow with a smooth interface is stable with respect to two-dimensional perturbations in the plane of the flow, are known for horizontal (Barmak et al., 2016a) and inclined (Barmak et al., 2016b) flows. The results were presented in the form of stability boundaries on the flow pattern map for each steady state solution (configuration). In the following we provide justification of the completeness of such a 2D analysis in order to determine the linear stability boundaries with respect to all possible perturbations (three-dimensional in general). In all cases the upper layer is considered to be lighter than the lower layer ($r > 1$), so that the Rayleigh-Taylor instability is not encountered. It will be shown that in all practical situations 2D perturbations in the flow plane are the most unstable and thereby determine the critical conditions for the onset of instability.

6.1. Horizontal flow

For horizontal stratified two-phase flows, the Squire's transformation is rigorously valid. Nevertheless, as the studied stability problem involves several dimensionless parameters, the mathematical statement on the 2D nature of the critical perturbations (Squire's theorem) is not straightforward. Therefore, the consequences of applying the Squire's transformation are demonstrated in this subsection.

Given a particular two-phase system, for which the channel height, the viscosity ratio (m) and density ratios (r) are specified, the stability problem for an oblique wave defined by the two wavenumbers (k_X, k_Z) is equivalent to the stability problem for plane wave, where k_X is replaced by $\sqrt{k_X^2 + k_Z^2}$, the flow rates are reduced by the factor $k_X/\sqrt{k_X^2 + k_Z^2}$, while the flow rates ratio q is kept constant. Maintaining q and m ensures that the holdup and all the other dimensionless characteristics of the base flow remain invariant.

The above procedure is illustrated in Fig. 2 for the case of stratified horizontal air-water flow, where air forms the upper layer ($m=55$ and $r=1000$). The solid (blue) curve confines the region which was found to be (linearly) stable with respect to all 2D perturbations ($k_Z=0$, see Barmak et al. 2016a). Each point along the stability boundary corresponds to water and air flow rates (i.e., superficial velocities U_{1S} and U_{2S}), for which the flow is neutrally stable for a particular 2D perturbation, k_X (denoted the critical 2D perturbation), and is stable with respect to all other 2D perturbations. For higher flow rates, beyond the 2D neutral stability bound-

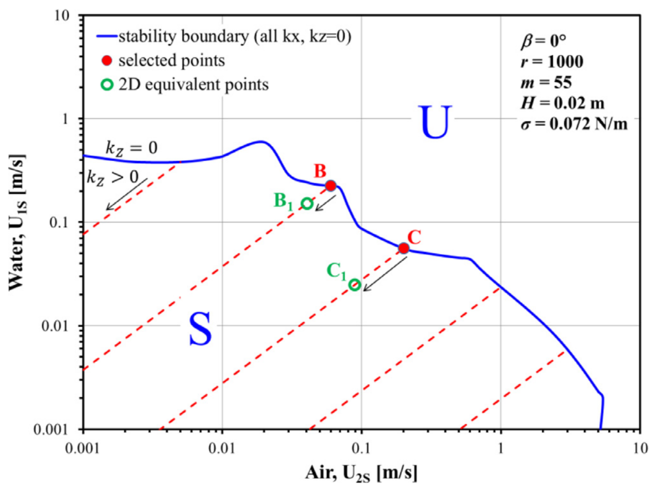

Fig. 2. Stability boundary with respect to 2D perturbations for horizontal air-water flow ($k_Z = 0$, blue solid line). According to the Squire's transformation, stability of any 3D disturbance can be reduced to an equivalent 2D stability problem (e.g., B_1, C_1 instead of B, C, respectively), which corresponds to lower superficial velocities along the line of constant flow rate ratio (red dashed line). For such conditions the flow is stable.

ary, there is a range of 2D perturbations that are amplified. The dashed (red) straight lines correspond to the locus of constant flow rate ratio, q, and hence of constant holdup as well.

Adding an arbitrary oblique perturbation (3D, $k_Z > 0$) at any operational conditions along the 2D neutral stability boundary (e.g., at point **B**) will have identical effect on the flow stability as that of the corresponding plane (2D) perturbation imposed on a flow of lower U_{1S} and U_{2S}. Such a point (e.g., $\mathbf{B_1}$) is situated along the line of the same flow rate ratio as that of the original (3D) problem. The higher is k_Z, the smaller are the equivalent superficial velocities of each of the phases. As all such points are within the 2D stable region, each of the points along the (2D) neutral stability boundary (solid (blue) curve) must be stable with respect to any 3D perturbation. Note that this actually holds in all cases where the structure of stable region is such that the constant flow rate ratio line drawn from a point on the neutral stability curve towards lower flow rates is within the 2D stable region. This is the case in horizontal stratified two-phase flows. Hence, the Squire's transformation allows us to reduce the 3D stability problem to the 2D one for horizontal flows.

Additional validation of the theorem can be provided by examining the growth rate of perturbations with arbitrary wavenumbers k_X and k_Z for operational conditions along the stability boundary. The growth rate of perturbations for points **B** and **C** are illustrated in Fig. 3(a) and (b), respectively. It is seen that except the two-dimensional perturbation ($k_Z = 0$) corresponding to the critical (neutrally stable) wavenumber k_X (for which the growth rate, $\lambda_R = 0$), all other perturbations are stable and decay in time (negative growth rate).

Short-wave mode of instability is the critical at point **B** ($k_X = 0.46$), while long waves ($k_X \to 0$) determine stability for the conditions of point **C**. As demonstrated by Fig. 3(a) and (b), the effect of the spanwise component k_Z is especially weak in case of short streamwise waves (k_X). This can be deduced by comparison of the vertical axis coordinates in Fig. 3(a) and (b) that correspond to about the same (negative) growth rate. Fig. 4 illustrates the growth rate dependence on k_Z for a fixed wavenumber in x-direction (e.g., the critical one). As shown, the damping of the perturbation becomes stronger as k_Z is larger. In Fig. 5, two different patterns of the perturbation amplitudes are presented for point **B**: the critical (neutrally stable) 2D perturbation with a maximum in the bulk of the upper (air) layer in proximity of the interface, and a 3D interfacial mode (which is equivalent to the 2D perturbation at point $\mathbf{B_1}$), that eventually decays (stable).

The case of particular interest is a system under zero-gravity conditions (or fluids of equal densities, $r = 1$). Yiantsios and Higgins (1988) argued that under such conditions and the absence of surface tension the Squire's theorem may not be valid. However, according to the 2D analysis (Barmak et al., 2016a), the flow is always unstable in the absence of surface tension (We $\to \infty$). Therefore, there is no need to consider also 3D perturbations. With the inclusion of surface tension, there is a stable region on the flow pattern map, which is located in the region where the more viscous phase is faster (above the critical flow rate line for $m > 1$ (e.g., Fig. 6), or below it for $m < 1$). The structure of stable region for systems under zero-gravity conditions with $m > 1$ and $m < 1$ is such that the constant flow rate ratio line (e. g., red lines in Fig. 6) drawn from the point on the neutral stability curve towards lower flow rates is within the stable region. This means that Squire's theorem is valid, and 2D perturbations are the most un-

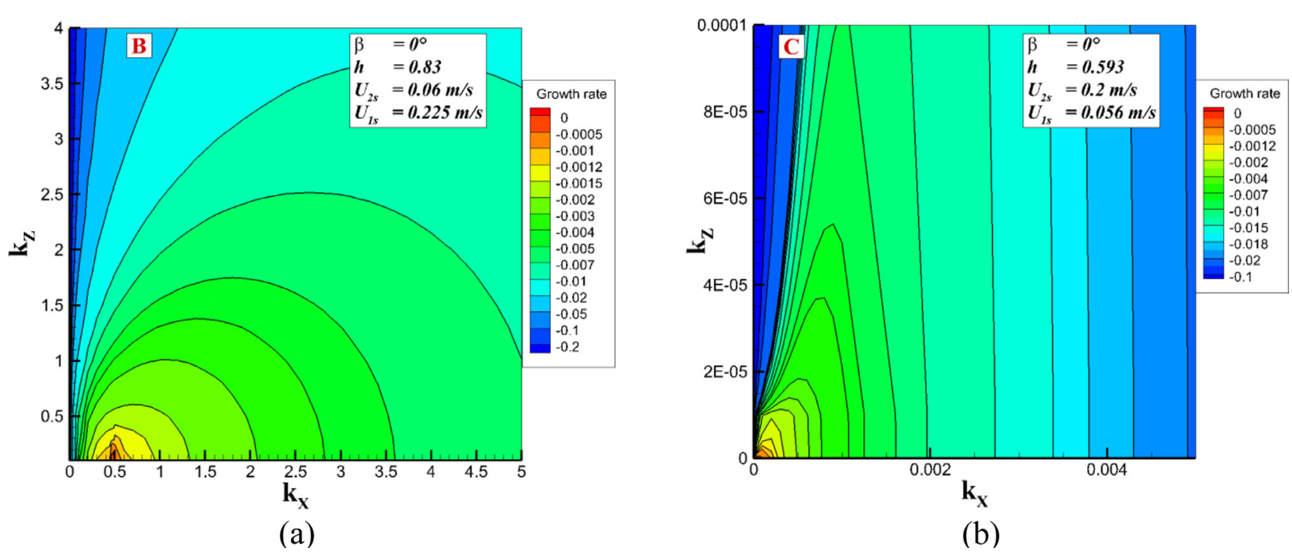

Fig. 3. Contours of the growth rate of 3D perturbations (a) at point B and (b) at point C. All perturbations are stable (growth rate ≤ 0). The growth rate attains a zero value (neutral stability) only for $k_Z = 0$ and the k_X value corresponding to the critical 2D wavenumber: $k_X = 0.46$ for point B and $k_X \to 0$ for point C.

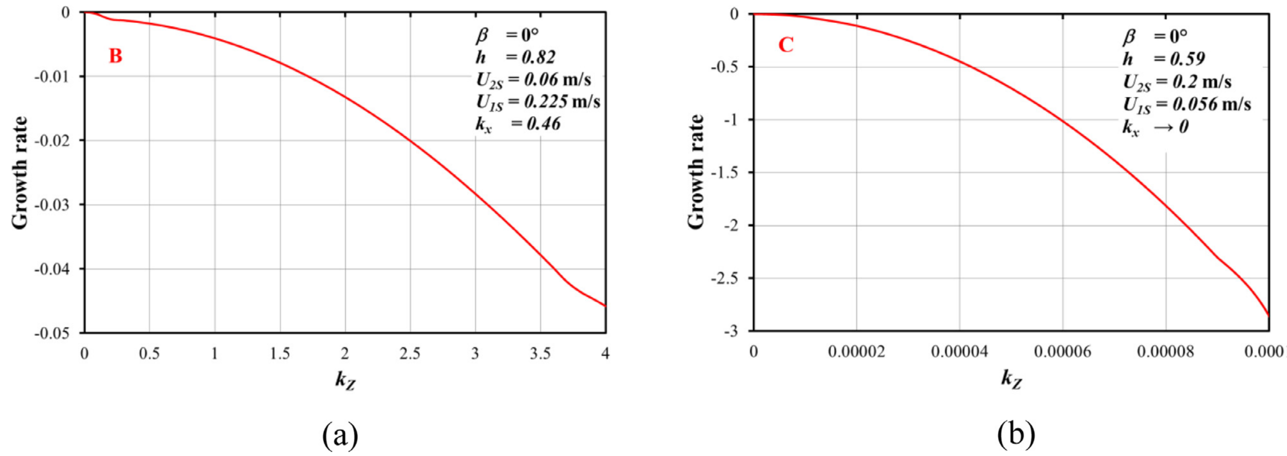

Fig. 4. Growth rate of the perturbation with k_X set at the critical (2D) wavenumber vs. the wavenumber in spanwise direction k_Z (a) at point B and (b) at point C.

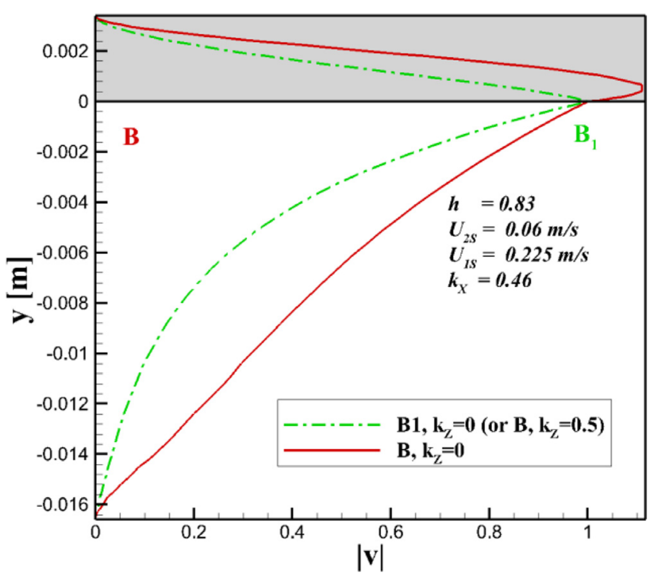

Fig. 5. Amplitude of the transverse component of velocity (normalized by the value at the interface) corresponding to the critical perturbation ($k_X = 0.46$, red solid line) and that of a decaying 3D perturbation ($k_X = 0.46$, $k_Z = 0.5$) at point B (green dash-dot line). The latter is equivalent to a 2D disturbance $k_X^{2D} = 0.68$at point B_1.

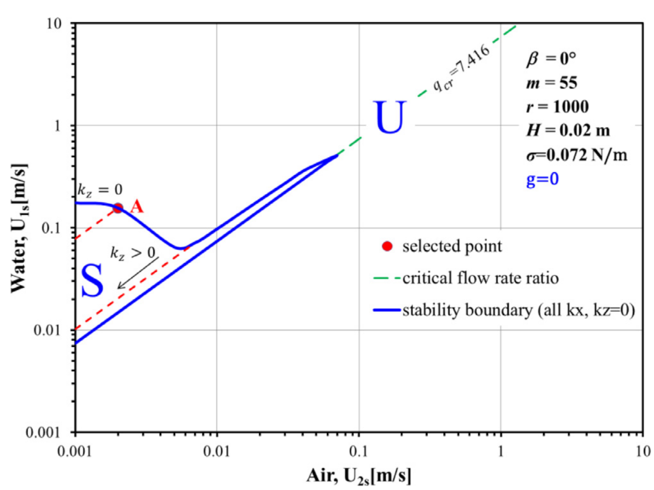

Fig. 6. Stability boundary for horizontal air-water flow with surface tension under zero-gravity condition for 2D perturbations ($k_Z = 0$, blue solid line), which are the most unstable ones. According to the Squire's transformation, stability of the 3D disturbances can be reduced to an equivalent 2D problem shifted to lower superficial velocities along the line of constant flow rate ratio (red dashed line).

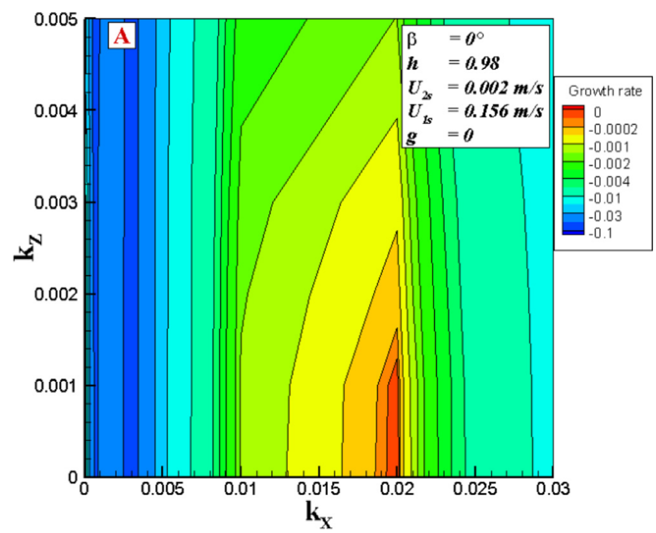

Fig. 7. Contours of the growth rate of 3D perturbations at point A (see Fig. 6). All perturbations are stable (growth rate ≤ 0). The growth rate attains a zero value (neutral stability) only for the critical 2D wavenumber $k_X = 0.02$, $k_Z = 0$.

stable ones also for this type of systems. This is also confirmed by the non-positive growth rate of any perturbation of an arbitrary wavenumber at point **A** (marked in Fig. 6), which is shown in Fig. 7. A zero value is observed only for the 2D perturbation of the critical wavenumber at point **A**, $k_X = 0.02$.

6.2. Inclined flow

In order to implement the Squire's transformation to an inclined stratified flow to study its stability with respect to 3D perturbations, two conditions should be satisfied. In addition to maintaining a constant flow rate ratio (Eq. (20)), the inclination parameter should be kept at a constant value. This means that, according to Eq. (22), the inclination angle of the equivalent 2D problem should be reduced compared to that of the original 3D problem. Due to the possible non-uniqueness of the base two-phase stratified flow in an inclined channel (e.g., Ullmann et al., 2003; Barmak et al., 2016b), the transformation should be accomplished separately for each branch of the base flow solutions. In the following we demonstrate application of this transformation for concurrent and countercurrent inclined flows, and its implication on the criticality of 2D perturbations.

In countercurrent flow, there are two possible stratified-smooth configurations, which are stable in closed regions of low flow rates

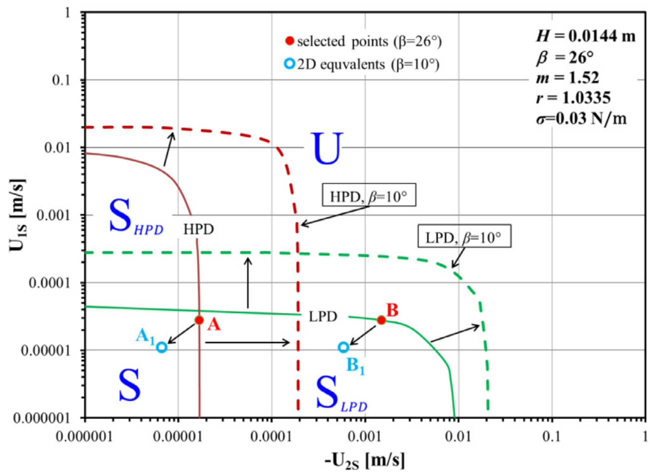

Fig. 8. Stability boundaries for countercurrent liquid-liquid flow for 2D perturbations ($k_Z = 0$, blue solid line). According to the proposed transformation, stability of a 3D disturbance can be reduced to an equivalent 2D problem which corresponds to lower superficial velocities (along constant q lines) in a channel of lower inclination (e.g., from red dots A and B to blue hollow dots A_1 and B_1, respectively).

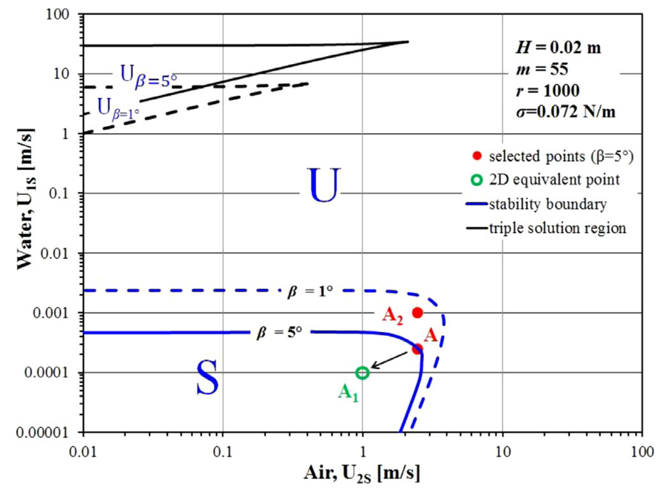

Fig. 10. Stability boundaries for downward inclined air-water flow with respect to 2D perturbations ($k_Z = 0$, blue solid line). According to the suggested transformation, stability of a 3D disturbance can be reduced to an equivalent 2D problem at lower superficial velocities (along s constant q line) and lower channel inclination (e.g., from red dot A of $\beta = 5°$ to green hollow dot A_1 of $\beta = 1°$).

of each of the phases. The 2D stability results are shown in Fig. 8, where the upper solution for the heavy phase holdup is denoted as the Heavy Phase Dominated (HPD) configuration, while the lower holdup solution is denoted as the Light Phase Dominated (LPD) configuration. It was shown in Barmak et al. (2016b) that the stable region for each of the holdup solutions widens with decreasing the inclination. This is demonstrated in Fig. 8 by comparing the stability boundaries of the HPD and LPD configurations for $\beta = 26°$ (solid curves) and $\beta = 10°$ (dashed curves). Any 3D perturbation at a point located on the (2D) neutral stability curve of one of the holdup solutions obtained for a particular inclination (e.g., points **A** and **B** for $\beta = 26°$ in Fig. 8, for the HPD and LPD solutions, respectively) can be converted to a 2D perturbation in a channel of lower inclination and for lower flow rates (e.g., points $\mathbf{A_1}$ and $\mathbf{B_1}$ for $\beta = 10°$). The flow is stable with respect to 2D disturbances for these conditions, indicating the stability of the original 3D problem. Examination of the growth rate of arbitrary 3D perturbations at points **A** and **B** (Fig. 9) gives an additional evidence for the correctness of the suggested transformation and the associated conclusions for countercurrent flows.

Air-water flows in downward inclined channels are stable only in the region of single base flow solution of low water flow rates (see Fig. 10). In this system the multiple (triple) solution region is located at high water flow rates, and it was found that none of those solutions are stable with respect to short-wave perturbations in the flow plane (Barmak et al., 2016b). Increase of the inclination angle has a destabilizing effect and results in shrinkage of the stable region to lower liquid flow rates. Thereby, similarly to the previous example of countercurrent flow, the 3D problem for a point located on the neutral stability curve can be converted to an equivalent 2D problem corresponding to stable conditions. For example, a 3D perturbation at point **A** on the stability boundary of $\beta = 5°$ (obtained with respect to 2D perturbations), is transformed to a 2D equivalent for lower flow rates and shallower channel inclination (point $\mathbf{A_1}$, $\beta = 1°$). As shown in Fig. 10, the latter conditions are stable. Hence, the 2D disturbances remain the most unstable also for downward inclined concurrent flows.

Comparison of the growth rate of arbitrary 3D perturbations for neutrally stable conditions (point **A**) and for unstable conditions

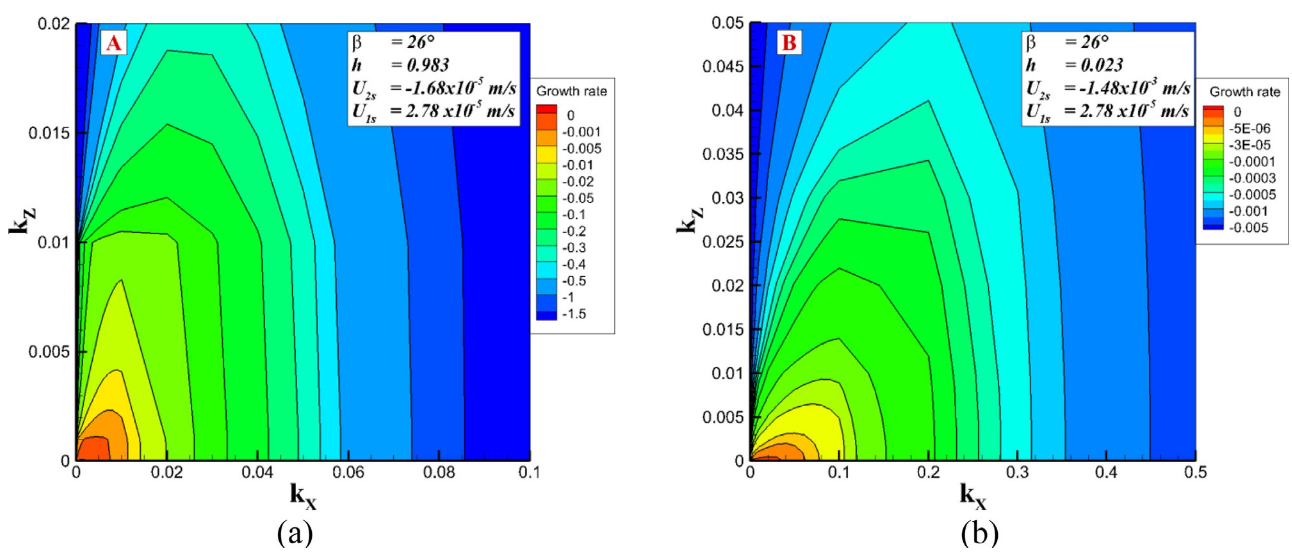

Fig. 9. Contours of the growth rate of three-dimensional perturbations (a) for the HPD solution at point A and (b) for the LPD solution at point B (see Fig. 8). All perturbations are stable (growth rate ≤ 0). The growth rate attains a zero value (neutral stability) only for $k_Z = 0$ and very long in the streamwise direction waves $k_X \to 0$.

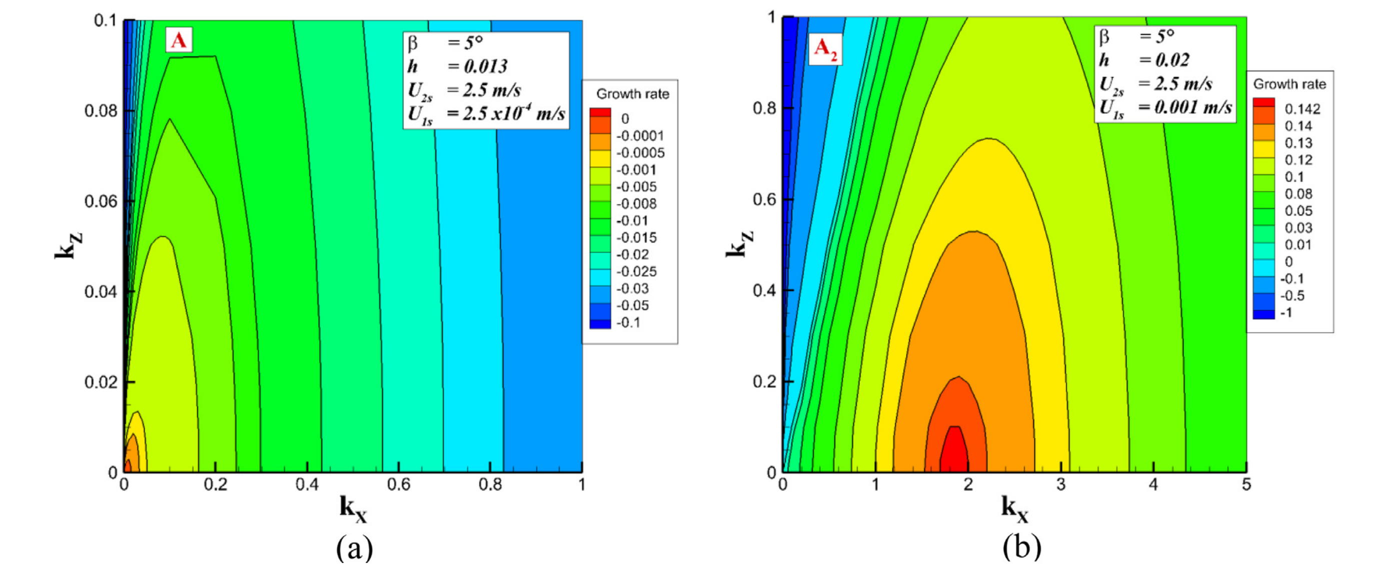

Fig. 11. Contours of the growth rate of 3D perturbations in downward inclined air-water flow, $\beta = 5°$ (a) for point A on the stability boundary and (b) for unstable conditions of point A_2. All perturbations decay for point A (growth rate ≤ 0). The growth rate at point A attains a zero value (neutral stability) only for $k_Z = 0$ and very long streamwise waves $k_X \to 0$.

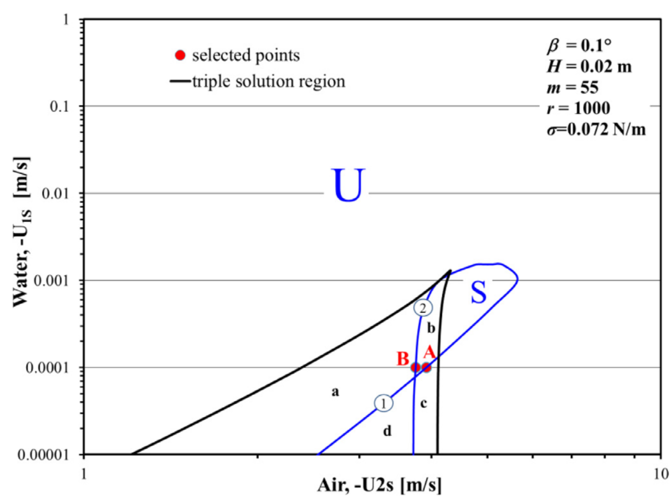

Fig. 12. Stability boundaries in the triple solution region (black solid triangle) of 0.1° upward inclined air-water flow: **a** – lower and middle solutions are stable; **b** – only the lower solution is stable; **c** – all three solutions are unstable; **d** – only the middle solution is stable.

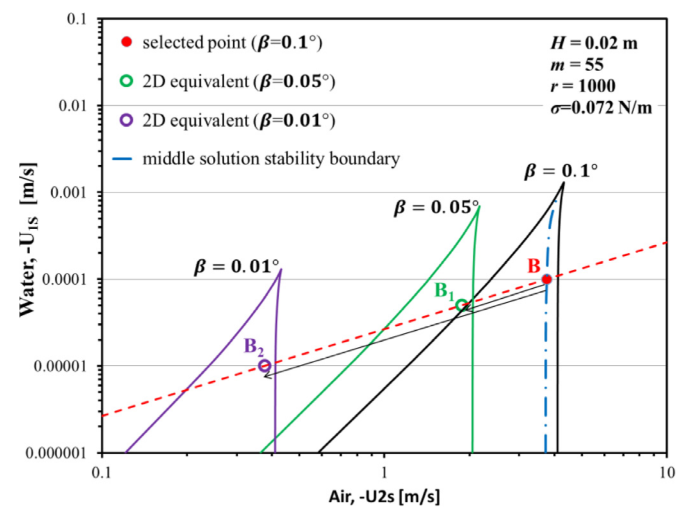

Fig. 13. Triple solution regions for air-water flows in upward inclined channels of different inclinations (solid lines). Blue dash-dot line depicts a stability boundary for the middle solution with respect to 2D perturbations ($k_Z = 0$) for $\beta = 0.1°$. According to the suggested transformation, a 3D stability problem can be reduced to an equivalent 2D problem at lower inclination and lower flow rates while maintain the same q (red dashed line). For example, from red dot B (neutrally stable for $k_X = 3.9$, $k_Z = 0$), to the middle holdup solution at the green hollow dot B_1, $\beta = 0.05°$ for ($k_X = 3.9$, $k_Z = 6.74$), or to the middle holdup solution at purple hollow dot B_2, $\beta = 0.01°$ for ($k_X = 3.9$, $k_Z = 38.7$) applied at B.

(**A_2**) is demonstrated in Fig. 11. In contrast to point **A**, at which all wavelength perturbations are stable and only very long 2D waves have zero-value growth rate, in the unstable region, a range of various wavenumber perturbations, including 3D ones, are amplified. The wavenumber k_X of the most amplified perturbation is also shifted (from $k_X \to 0$ to $k_X \approx 1.8$).

For slightly upward inclined air-water flows $\beta = 0.1°$, there exists either a single or triple solution base state region. The stability map for the latter is shown in Fig. 12. In part of the triple solution region, the lower and middle holdup solutions were found to be stable (Barmak et al., 2016b). The two-dimensionality of the critical disturbances in the single solution region (at low air flow rates) can be proved in the same manner as for downward flow, since the closed stable region expands with decreasing the channel inclination. However, the implementation of the Squire's theorem in the triple-solution region requires more careful examination.

The upper solution is always unstable and therefore is not of interest. The lower holdup solution is stable in subdomains **a** and **b** of the triple solution region. In shallow channel inclinations the flow configuration of this water layers (low holdup) is insensitive to the inclination and is similar to the flow configuration obtained in a horizontal channel. In fact, the neutral stability curve for the lower holdup solution (line 1 in Fig. 12) coincides with that obtained for horizontal flow and thus remains invariant in this range of shallow inclinations. The location of the triple solution (3-s) region is however sensitive to the channel inclination (see Fig. 13). As channel inclination is reduced ($\beta < 0.1°$), the triple solution is shifted to lower air and water superficial velocities. Consequently, the lower (and middle) solutions are found to be stable with respect to 2D perturbations in larger parts of the triple solution region as the channel inclination is reduced. The middle solution be-

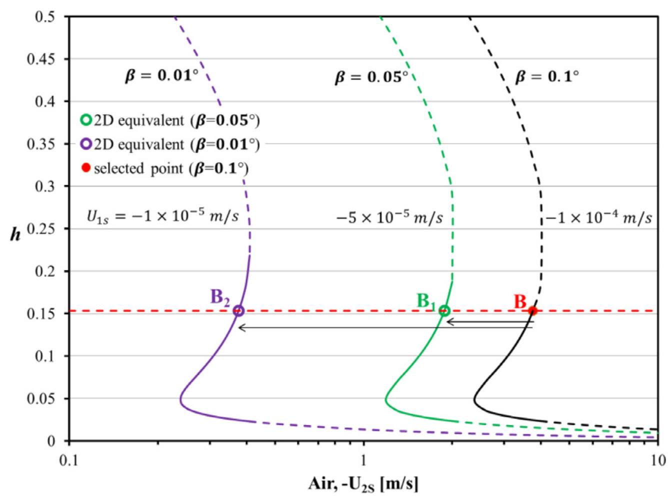

Fig. 14. Holdup curves for constant water flow rate in slightly upward inclined flow. Dashed lines represent unstable conditions.

comes stable in the whole triple solution region for inclination angle of $\beta = 0.05°$ and lower.

Point **A** is located on the neutral stability boundary of the lower holdup solution. Considering any arbitrary 3D perturbation at point **A**, and applying the proposed transformation, the resulting conditions of reduced air and water flow rates along the same constant q line as point **A** are always in the stable region (stability boundary is independent of the inclination for the lower holdup solution).

The consequences of applying the proposed transformation for converting a 3D perturbation to its equivalent 2D perturbation is demonstrated in Fig. 13 with respect to point **B** (marked in Fig. 12). Point **B** is located on the stability boundary of the middle solution ($\beta = 0.1°$) and corresponds to the critical 2D short wave ($k_X = 3.9$, $k_Z = 0$). The location of point **B** on the holdup curve (water holdup vs. the air superficial velocity at a fixed water superficial velocity corresponding to point **B**) is shown in Fig. 14. The stability of point **B** with respect to a 3D perturbation ($k_X = 3.9$, $k_Z = 6.74$) is equivalent to perturbing the middle solution of point $\mathbf{B_1}$ in a channel inclined at $\beta = 0.05°$ with a 2D perturbation ($k_X = 7.79$, $k_Z = 0$). Similarly, the 2D stability of the middle solution of point $\mathbf{B_2}$ in a $\beta = 0.01°$ inclined channel is the 2D equivalent to a stability of a 3D the perturbation ($k_X = 3.9$, $k_Z = 38.7$) at point **B**. Since for the shallower inclinations ($\beta \le 0.05°$) the middle holdup solution is stable over the entire triple solution region, both $\mathbf{B_1}$ and $\mathbf{B_2}$ are stable with respect to the equivalent 2D perturbation. Hence, point **B** is stable with respect to the considered 3D perturbations.

7. Conclusions

In this work the sufficiency of 2D analysis to explore the stability of stratified horizontal and inclined gas-liquid and liquid-liquid flows was established based on the Squire's transformation that allows us to derive either rigorous or rationale-based conclusion saying that two-dimensional perturbations remain most unstable also in stratified two-phase plane-parallel flows.

For flows in horizontal channels the analogy between the 2D and 3D problem formulations is rigorously followed from the Squire's transformation. As a result it has been proven that 2D perturbations are the most unstable and should be considered for identifying the stability limits. However, in the case of inclined channels, additional arguments should be invoked in order to draw a similar conclusion. These arguments were elaborated and their consequences were demonstrated for several gas-liquid and liquid-liquid inclined concurrent and countercurrent flows. It is shown that in inclined stratified flows the transformation of a 3D stability problem to its 2D analog is associated with a stabilizing effect of reducing the system inclination, in addition to the reduction of the phases flow rates as in the case of horizontal flows. The slower growth of 3D perturbations compared to 2D perturbations is elaborated by considering the corresponding stability maps and contour plots of the growth rate levels.

Appendix: 3D boundary conditions on the stress components

To formulate the B.C. on the stress components, a unit-length normal to the disturbed interface, and two unit vectors tangential to the interface in xy- and yz-planes, $\mathbf{t_1}$, $\mathbf{t_2}$, respectively, are used:

$$\mathbf{n} = \frac{(-\eta_x, 1, -\eta_z)}{\sqrt{1+\eta_x^2+\eta_z^2}}, \quad \mathbf{t_1} = \frac{(1, \eta_x, 0)}{\sqrt{1+\eta_x^2}}, \quad \mathbf{t_2} = \frac{(0, \eta_z, 1)}{\sqrt{1+\eta_z^2}}. \tag{A23}$$

The stress tensor is

$$\mathbf{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}, \tag{A24}$$

where stress tensor components are

$$T_{xx} = 2\mu\frac{\partial u}{\partial x}, \quad T_{yy} = 2\mu\frac{\partial v}{\partial y}, \quad T_{zz} = 2\mu\frac{\partial w}{\partial z},$$
$$T_{xy} = T_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad T_{yz} = T_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),$$
$$T_{xz} = T_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right).$$

The stress vector acting across the interface is given by:

$$\mathbf{T^{(n)}} = \mathbf{T}\cdot\mathbf{n} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \cdot \begin{pmatrix} \frac{-\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}} \\ \frac{1}{\sqrt{1+\eta_x^2+\eta_z^2}} \\ \frac{-\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}} \end{pmatrix}$$
$$= \begin{pmatrix} -\eta_x T_{xx} + T_{xy} - \eta_z T_{xz} \\ -\eta_x T_{xy} + T_{yy} - \eta_z T_{yz} \\ -\eta_x T_{xz} + T_{yz} - \eta_z T_{zz} \end{pmatrix} \cdot \frac{1}{\sqrt{1+\eta_x^2+\eta_z^2}}. \tag{A25}$$

Whereby the normal stress is:

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n} = \frac{(-\eta_x, 1, -\eta_z)}{\sqrt{1+\eta_x^2+\eta_z^2}} \cdot \begin{pmatrix} -\eta_x T_{xx} + T_{xy} - \eta_z T_{xz} \\ -\eta_x T_{xy} + T_{yy} - \eta_z T_{yz} \\ -\eta_x T_{xz} + T_{yz} - \eta_z T_{zz} \end{pmatrix} \cdot \frac{1}{\sqrt{1+\eta_x^2+\eta_z^2}}$$
$$= \frac{\eta_x^2 T_{xx} + T_{yy} + \eta_z^2 T_{zz} - 2\eta_x T_{xy} - 2\eta_z T_{yz} + 2\eta_x\eta_z T_{xz}}{1+\eta_x^2+\eta_z^2}. \tag{A26}$$

According to the Young-Laplace equation, the jump in the normal stress across the interface is balanced by surface tension, hence:

$$[\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n}] = -\sigma\nabla\cdot\mathbf{n} = -\sigma\left(\frac{\partial}{\partial x}\left(\frac{-\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{-\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right)\right), \tag{A27}$$

where the jump of the quantity across the interface (i.e.,$[f] = f_2 - f_1$) is denoted by [•].

Eq. (A27) can be rewritten in dimensionless form with the stress tensor component expressed in terms of the velocity components:

$$[\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n}] = \Bigg[p - \frac{m\mu}{\mu_1}\frac{2\mathrm{Re}_2^{-1}}{1+\eta_x^2+\eta_z^2}\Big(\big(\eta_x^2-\eta_z^2\big)u_x + \big(1-\eta_z^2\big)v_y - \eta_x(u_y+v_x) - \eta_z(w_y+v_z) + \eta_x\eta_z(u_z+w_x)\Big)\Bigg] = \mathrm{We}_2^{-1}\frac{\eta_{xx}\big(1+\eta_z^2\big)+\eta_{zz}\big(1+\eta_x^2\big)-2\eta_x\eta_z\eta_{xz}}{\big(1+\eta_x^2+\eta_z^2\big)^{3/2}}. \quad (A28)$$

The interface deformations (η) are assumed to be small, whereby Eq. (A28) can be reduced (neglecting quadratic terms of the order η^2) to:

$$[\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n}] = \Big[p - \frac{m\mu}{\mu_1}2\mathrm{Re}_2^{-1}(v_y - \eta_x(u_y+v_x) - \eta_z(w_y+v_z))\Big] = \mathrm{We}_2^{-1}(\eta_{xx}+\eta_{zz}). \quad (A29)$$

The shear stress components are:

$$\mathbf{t_1}\cdot\mathbf{T}\cdot\mathbf{n} = \frac{(1,\eta_x,0)}{\sqrt{1+\eta_x^2}}\cdot\begin{pmatrix}-\eta_x T_{xx}+T_{xy}-\eta_z T_{xz}\\ -\eta_x T_{xy}+T_{yy}-\eta_z T_{yz}\\ -\eta_x T_{xz}+T_{yz}-\eta_z T_{zz}\end{pmatrix}\cdot\frac{1}{\sqrt{1+\eta_x^2+\eta_z^2}} = \frac{-\eta_x T_{xx}+T_{xy}-\eta_z T_{xz}-\eta_x^2 T_{xy}+\eta_x T_{yy}-\eta_x\eta_z T_{yz}}{\sqrt{1+\eta_x^2}\cdot\sqrt{1+\eta_x^2+\eta_z^2}}, \quad (A30)$$

$$\mathbf{t}_2\cdot\mathbf{T}\cdot\mathbf{n} = \frac{(0,\eta_z,1)}{\sqrt{1+\eta_z^2}}\cdot\begin{pmatrix}-\eta_x T_{xx}+T_{xy}-\eta_z T_{xz}\\ -\eta_x T_{xy}+T_{yy}-\eta_z T_{yz}\\ -\eta_x T_{xz}+T_{yz}-\eta_z T_{zz}\end{pmatrix}\cdot\frac{1}{\sqrt{1+\eta_x^2+\eta_z^2}} = \frac{-\eta_x\eta_z T_{xy}+\eta_z T_{yy}-\eta_z^2 T_{yz}-\eta_x T_{xz}+T_{yz}-\eta_z T_{zz}}{\sqrt{1+\eta_z^2}\cdot\sqrt{1+\eta_x^2+\eta_z^2}}. \quad (A31)$$

Since the interface deformations are small, the shear stress balance on xy-plane can be written as:

$$[\mathbf{t_1}\cdot\mathbf{T}\cdot\mathbf{n}] = [-\eta_x T_{xx}+T_{xy}-\eta_z T_{xz}+\eta_x T_{yy}] = \Big[\frac{m\mu}{\mu_1}(-2\eta_x u_x+(u_y+v_x) - \eta_z(w_x+u_z)+2\eta_x v_y)\Big] = 0. \quad (A32)$$

The shear stress balance on yz-plane is:

$$[\mathbf{t}_2\cdot\mathbf{T}\cdot\mathbf{n}] = [\eta_z T_{yy}-\eta_x T_{xz}+T_{yz}-\eta_z T_{zz}] = \Big[\frac{m\mu}{\mu_1}(2\eta_z v_y - \eta_x(w_x+u_z) + (v_z+w_y)-2\eta_z w_z)\Big] = 0. \quad (A33)$$

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