Nonlinear dynamics and control of an inverted spherical pendulum maneuvering on a horizontal plane

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INTRODUCTION

In this work, a dynamic model for a maneuvering spherical pendulum on an uneven 3D surface was developed using several coordinate systems. The motion of the system is controlled by a force vector, which is tangent to the 3D surface. The magnitude and direction of the force are the control parameters of the system. Several control laws for the dynamic system were tested. In order to gain insight into the complex dynamics, a simplified planar model was also investigated.
2. DYNAMICAL SYSTEM

The generalized coordinates $x_1$ and $y_1$ describes the distance of the cart system from the inertial coordinate frame $\hat{e}_i$. The generalized coordinates and $x_2$ and $y_2$ describe the distance of the center of gravity of the relative to the base of the cart.

2.1 Energies of the system

The kinetic energy of the system is:

$$T = \frac{m_1 + m_2}{2} (\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2} (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}^2 + 2(\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2))$$  \hspace{1cm} (1)

$z$ is the height of the center of gravity of the pendulum above the cart:

$$z = \sqrt{1 - x_2^2 - y_2^2}$$

$$\dot{z} = \frac{(x_2\dot{x}_2 + y_2\dot{y}_2)}{\sqrt{1 - x_2^2 - y_2^2}}$$  \hspace{1cm} (2)
The potential of the gravity field has the following form:

\[ V = m g z \quad (3) \]

2.2 Generalized forces

The generalized forces can be separated to dissipation force and control force:

2.2.1 Dissipation force

The damping of the dynamical system will be described as viscose damping by the dissipation function of Rayleigh as follows:

\[ Q^D_j = - \frac{\partial D}{\partial \dot{w}_j} \quad (4) \]

Where: \( D = \frac{d_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{d_2}{2}(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}^2) \) - is the dissipative energy.

\[ \dot{w}_j = \{ \dot{x}_j; \dot{y}_j \} \]

\( d_1, d_2 \) - are viscous damping coefficients.

2.2.2 Control force

The dynamical system is controlled by two forces, \( F_x \) and \( F_y \), that are applied on the cart in the directions \( \hat{e}_1 \) and \( \hat{e}_2 \) respectively. The generalized forces caused by the control forces has the following form:
\[
\begin{bmatrix}
Q^c_x \\
Q^c_y
\end{bmatrix} =
\begin{bmatrix}
F_x \\
F_y
\end{bmatrix}
\]  \hspace{1cm} (5)

Where: \( F_x \) - is the control force in \( \hat{e}_1 \) direction.

\( F_y \) - is the control force in \( \hat{e}_2 \) direction.

2.3 EQUATIONS OF MOTION

The equations of motion are normalized by the natural frequency of a simple pendulum, \( \omega^2 = \frac{g}{l} \), and the distance of the center of mass of the pendulum from a cart, \( l \).

\[
\begin{align*}
(1 + \mu)\ddot{q}_1 &= -\dot{q}_1 + F_1 - \delta_1 \dot{q}_1 - \delta_2 (\dot{q}_1 + \dot{q}_3) \\
(1 + \mu)\ddot{q}_2 &= -\dot{q}_2 + F_2 - \delta_1 \dot{q}_2 - \delta_2 (\dot{q}_2 + \dot{q}_4)
\end{align*}
\]  \hspace{1cm} (6)

\[
\begin{bmatrix}
\dot{q}_3 \\
\dot{q}_4
\end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix}
\mu (1 - q_3^2) + q_4^2 & -q_3 q_4 \\
-(1 - q_3^2 - q_4^2)(\mu + 1) & \mu (1 - q_3^2) + q_4^2
\end{bmatrix} \begin{bmatrix}
F_3 \\
F_4
\end{bmatrix}
\]

Where:

\[
F_{3/4} = \left(-\frac{(q_3 \dot{q}_3 + q_4 \dot{q}_4)}{1 - q_3^2 - q_4^2}\right)^2 - \frac{\dot{q}_3^2 + \dot{q}_4^2}{1 - q_3^2 - q_4^2} + \frac{1}{\sqrt{1 - q_3^2 - q_4^2}} q_{3/4} + \frac{\delta_1 \dot{q}_{1/2}}{1 + \mu}
\]

\[
+ \delta_2 \left(-\frac{\mu}{1 + \mu} (\dot{q}_{1/2} + \dot{q}_{3/4}) - \frac{(q_3 \dot{q}_3 + q_4 \dot{q}_4) q_{3/4}}{(1 - q_3^2 - q_4^2)}\right) - \frac{F_{1/2}}{1 + \mu}
\]

\[
\Delta = \frac{(\mu + q_3^2 + q_4^2)\mu}{(1 - q_3^2 - q_4^2)(1 + \mu)^2} \] - is the determinant of the inertial matrix of the last two equations of motion.
\( q = \frac{1}{t} \begin{bmatrix} x_1 & y_1 & x_2 & y_2 \end{bmatrix}^T \) - are the normalized generalized coordinates.

\[ F_{1/2} = \frac{F_{x/y}}{lm_2\omega^2} \] - are the normalized control forces.

\[ \delta_{1/2} = \frac{d_{1/2}}{m_2\omega} \] - are normalized viscous damping coefficients.