Source Decomposition as a Diakoptic Boundary Condition in FDTD with Reflecting External Regions

Sharon Malevsky, Ehud Heyman and Raphael Kastner

School of Electrical Engineering, Tel Aviv University, Tel Aviv 69987, Israel
kast@eng.tau.ac.il

Abstract—The diakoptic approach is utilized for the creation of boundary conditions for the truncation of the FDTD computational grid in the presence of reflecting external media. The interfaces between the regions can be of arbitrary shape and are in close proximity to the scatterers in both regions. Surface Green’s functions over these interfaces serve to characterize the external domain from a viewpoint of the computational domain. Previously used field definitions of such Green’s functions, that can be viewed as analogs of $Z_{ij}$ parameters in microwave networks, have involved effective hard boundaries at the interfaces. These boundaries, however, can cause artificial interactions with physical reflecting bodies in the external domain. To eliminate this phenomenon, the Green’s functions are now defined in terms of outgoing and incoming wave constituents, rather than fields. This distinction between the aforementioned constituents is made by viewing them as originating from sources within and outside the computational domain. The Green’s function - based boundary conditions are demonstrated to operate successfully in the two-dimensional case, resolving the aforementioned resonance issues efficiently and accurately.

I. INTRODUCTION

The finite difference time domain (FDTD) method [1], [2] for solving electromagnetic scattering problems is traditionally limited to closed region, or unbounded region in free space. For the latter case, a host of absorbing boundary conditions (ABCs) are available in the literature, including local ABCs that are based either on asymptotic representations of the one way wave equation [3], [4], or on perfectly matched layers (PMLs) [5]. These formulations usually require the boundary to be a convex surface, typically a box or sphere, which may not conform very well to the shape of the scattering obstacle. A homogeneous “white space” thus created between the scatterer and the ABC, whose size may compromise the inherent efficiency of the local formulations. More importantly, local ABCs are restricted to non-reflecting external domains such as free space or an infinite microstrip substrate.

In contrast to local ABCs, global boundary conditions (GBCs) involve integration over the entire boundary of the computational domain. Early formulations [6] and many subsequent versions of GBCs were based on Kirchhoff–like integration of the boundary field using the free–space Green’s function. GBCs provide the advantage of treating non–convex boundaries [7]–[11]; however, they are applicable to non reflecting external domains only. GBCs also suffer from efficiency issues due to the need to integrate over the entire boundary. Theses issues are addressed in [11] by a multilevel space–time integration scheme that manages to reduce the numerical complexity substantially, again in the context of non–reflecting external domains only.

The objective of this paper is to suggest boundary conditions that can handle reflective, as opposed to absorptive external domains. Reflecting external domains can be handled with the diakoptic approach. Diakoptics (Greek for “tearing apart”) is a domain decomposition approach whereby the physical space is divided into several distinct regions whose interfaces are referred to as “seams”. In our case, these regions would comprise the computational and external domains. This method was initially introduced into time domain problems in the form of the Johns matrix for the transmission–line modeling (TLM) formulation in [12]. Electromagnetic version has been introduced in a general multi–dimensional formulation of the Green’s function method (GFM) [13], [14]. In the GFM, adjacent regions are linked via impedance or admittance type Green’s functions defined over the seams, that can be of arbitrary shape and in close proximity to scatterers. A pre–processing stage is devoted to the construction of the Green’s functions transforming the magnetic (electric) field into the electric (magnetic) field over the seam. The Green’s functions serve to relate fields over the seam in the presence of a reflecting external domain as viewed from the computational domain. They are thus used as boundary conditions in the course of the main FDTD computational phase. The main advantage of this formulation is its capability of treating seams of general shapes, concave cases included, and thus economize of the size of the “white space”. The GFM also accommodates external regions that are reflective rather than absorptive. It is shown, however, (Section II) that the Green’s function used in the GFM introduces spurious late time multiple reflections between the external scatterers and the seam that are canceled in the overall summation yet may be a source of late time errors. This phenomenon is attributed to the field–type character of the GFM, reminiscent of $Z$ and $Y$ matrix representation in microwave circuit theory.

Another possible formulation is wave, or spectral domain

1 a similar problem might occur with concave geometries.
decomposition, analogous to cascaded scattering matrices, where the field is resolved into outgoing and incoming harmonics across the seam. This approach can be considered as analogous the $S$-matrix representation in microwave circuit theory, as opposed to the field approach of the GFM. Definitions of the outgoing and incoming wave operators thus involve matched rather that short circuited “ports” at the seam. This eliminates the aforementioned spurious mechanism of reflections between the artificial seam and the external reflector. The main issue in this formulation is the restriction to a convex and separable seam that can compromise the efficiency of the procedure, having an excessively large white space.

$\begin{align*}
\frac{\partial }{ \partial z} \begin{pmatrix} E \cr H \end{pmatrix} = \frac{\partial }{ \partial t} \begin{pmatrix} \mu E \cr \epsilon H \end{pmatrix} + \begin{pmatrix} J_m \cr 0 \end{pmatrix}. 
\end{align*}$

Fig. 2. 1-D problem configuration. The area in the vicinity of the seam $z = z_0$ is homogeneous.

These equations apply to a wave of the form $E(z,t) = \tilde{x} E(z,t)$, $H(z,t) = \tilde{y} H(z,t)$ arising from the source terms $J = \tilde{x} J$, $J_m = \tilde{y} J_m$ and with $\epsilon, \mu$ that are, in general functions of $z$. This domain is sewn together with the “external domain” $z > z_0$ using the following BC at the seam $z = z_0^+$:

$H(z_0^+,t) = E(z_0^+,t) \rightarrow Y(z_0^+,t) \int \limits_{-\infty}^{t} E(z_0^+, \tau) Y(t-\tau) d\tau$

where the asterisk * denotes a temporal convolution and the first formulation leads to a field, or spatial domain procedure that can be considered analogous to cascaded impedance matrix representation in microwave circuit theory. This formulation generates a Green’s functions that relates the electric and magnetic fields over the surface enclosing the sub–problem (the “seam”), hence it is known as the Green’s function method (GFM). The main advantage of this formulation is its capability of treating seams of general shapes, concave cases included, and thus economize of the size of the “white space”. The issue with this formulation, however, is that the parameters are determined under short or open circuit conditions. Here, these conditions are defined by artificial perfect conductors on the seam, a situation that may lead to spurious multiple reflections outside the sub–problem domain. The second formulation is wave, or spectral domain decomposition, analogous to cascaded scattering matrices, where the field is resolved into outgoing and incoming spectral constituents across the seam. The main issue in this formulation is the restriction to a convex and separable seam that can compromise the efficiency of the procedure, having an excessively large white space.

The third formulation is the source decomposition method that mitigates the issues of both formulations mentioned above. It is a spatial domain method, however it is formulated under terminated conditions, similarly to a wave formulation, and yet accommodates seams of general shapes. The main characteristic of this approach, as demonstrated in is the distinction between sources within and outside the sub–problem region, as opposed to outgoing and incoming spectral constituents. To implement this approach, a two–step pre–processing stage is executed. This extra computational effort is relatively small, while enabling the generation of boundary conditions for finite regions, where the external regions are reflective rather than

II. THE FIELD–TYPE GREEN’S FUNCTION METHOD (GFM): ISSUES WITH REFLECTING EXTERNAL DOMAINS

For a brief review the GFM [13], [14], consider first the 1-D time dependent Maxwell’s equations within the “computa-
absorptive. $\overrightarrow{Y}(t)$ is the aforementioned “input admittance” into the external domain. Note that the external domain can be of arbitrary composition. The immediate vicinity of the seam on both sides, however, is considered homogeneous with $\epsilon = \epsilon_0, \mu = \mu_0$.

The discrete equivalent of Eq. (2), to be used in the context of the Yee grid, requires a pre-processing phase to evaluate the input admittance in a manner that is compatible with an FDTD computation of the entire space. Assuming a seam at the plane $z = z_0 = I\Delta z$, the discrete magnetic curl equation in its vicinity is

$$E_i^{n+1} = E_i^n + \gamma_0 H_{i+\frac{1}{2}}^{n+\frac{1}{2}} - \gamma_0 H_{i+\frac{1}{2}}^{n+\frac{1}{2}},$$

(3)

where $\gamma_0 = \sqrt{\mu_0/\epsilon_0}$ is defined in the homogeneous medium about the seam and $\gamma = \epsilon_0 \Delta z^2$, with $\epsilon_0 = 1/\sqrt{\mu_0\epsilon_0}$. The rightmost term in equation (3), which lies within the external domain field, and therefore is not available, is replaced by the boundary condition (BC). Similarly to its continuous domain counterpart (2), the BC is expressed as convolution of boundary field historical values with the discrete admittance-type Green’s function $Y_n$ that represents the external domain, as defined, in accordance with [13], by

$$H_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \sum_{n' = 0}^{\infty} Y_{n'} E_{i-\frac{1}{2}}^{n'-n'} \equiv \overrightarrow{Y}_n \ast E_i^n,$$

(4)

where $\ast$ denotes the discrete-time convolution. The definition in (4) includes the effect of the shift between the samples of $E_i$ and $H_{i+\frac{1}{2}}$. Note that in practice, the time series $\overrightarrow{Y}_n$ is truncated to its first $N$ terms, so that the convolution in (4) involves a history of $N$ terms of $E_i^n$ behind the wavefront. A dual impedance type Green’s function $\overrightarrow{Z}_n$ can also be defined by starting with the electric field curl equation instead of Eq. (3).

Usage of the BC (4) in an FDTD code is contingent upon the availability of $\overrightarrow{Y}_n$. Therefore, $\overrightarrow{Y}_n$ is evaluated in a pre-processing stage that precedes the main FDTD computation. This stage is performed over the external domain, whereby an impulsive (Kronecker delta) electric field $E_i^n = \delta_{n,0}$ at the boundary is used as the input. This type of input, with zero electric field at $I$ for all times except at $n = 0$ implies an impulsive magnetic source backed by a perfect electric conductor (PEC) at that boundary. Under these conditions, the solution of the external medium in the preprocessing stage produces the input admittance Green’s function via

$$H_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \overrightarrow{Y}_n.$$

(5)

(see also the left hand term in (4) with $E_0^n = \delta_{n,0}$).

When the external medium is homogeneous, we denote $\overrightarrow{Y}_n = \gamma_0 Y_n$, where $\gamma n$ is the “intrinsic” admittance that is the discrete counter part of the continuous free space admittance $\frac{1}{\eta_0}$ (see also explanation before Eq. (8)). For the dispersion-free case $\gamma \rightarrow 1$, $\overrightarrow{Y}_n \rightarrow \eta_0 \delta_{n,0}$ as one may expect, whereas for a general values of $\gamma$, $\overrightarrow{Y}_n$ is a rapidly decaying series.

Results for $\gamma n$, obtained via direct FDTD computations, $Z$-transform techniques and combinatorics are in the form of a converging series in $-\gamma^2$ that can be truncated after say $N$ terms (see also similar phenomena in [15], [16]).

The main advantage in the usage of the GFM equations (3),(4) as boundary conditions is their capability of treating seams of general shapes, concave cases included, and thus economize of the size of the “white space”. These boundary conditions also accommodate reflective external domains. In reflective cases, the conceptual PEC boundary, implied in (5), reflects the scattered waves generated in the external domain back into the same domain. Consequently, $\overrightarrow{Y}_n$ includes a series of spurious multiple reflections between the external domain and this conceptual PEC boundary, as seen in Fig. 3.²

In principle, the GFM has a built-in mechanism to offset the transmissions of the spurious reflections into the computational domain, at least over a limited time scale. Each of these reflections, arriving at a given time, comprises sub-reflections that have been excited earlier, and then propagated by different components of the admittance vector. The aggregated effect of these sub-reflections produces a zero field, as shown in Fig. 4. To facilitate the annulment over unlimited time scale, however, however,

²In analogy, consider the continuous-domain input admittance to an external medium with reflections. The frequency domain input admittance can then be defined in terms of the outward-looking reflection coefficient $\hat{\Gamma}(z_0, \omega)$ via (recall that the medium homogeneity in the immediate vicinity of the seam)

$$\overrightarrow{Y}(z_0, \omega) = \eta_0^{-1} \frac{1 - \hat{\Gamma}(z_0, \omega)}{1 + \hat{\Gamma}(z_0, \omega)} = \eta_0^{-1} \left[ 1 - \hat{\Gamma}(z_0, \omega) \right] \sum_{n=0}^{\infty} \left[ -\hat{\Gamma}(z_0, \omega) \right]^n.$$

In the time domain, the second equality in (6) describes an infinite series of reflections between the external medium and the PEC at $z_0$. 
of this impedance is of each constituent are then related by the intrinsic impedance \( \epsilon \) of external sources, respectively. The electric and magnetic fields \( \epsilon \) and \( \mu \) that are also the contributions of the sources in the computational and external domains, respectively, in this case. constituents via (recall that the immediate vicinity of the seam is trivially planar. To implement this approach, one first resolves the electric field into the outgoing and incoming wave components, that are also the contributions of the sources in wave-type approach discussed in the Introduction, since the distance of the farthest discontinuity \( \delta_n \), that may be slow, is compatible with the distance of the farthest discontinuity \( \delta_n \), whose physical meaning is revealed, for example, in the form of an excessively large white space. The length of the filter is also available beforehand. This mandates a pre-processing stage that contains only outgoing constituents of the magnetic field:

\[
\begin{align*}
E^n_i &= E^n_i + E^n_0 = \bar{Z}^n \left( H^n_{i-\frac{1}{2}} - R^n_{i+\frac{1}{2}} H^n_{i-\frac{1}{2}} \right),
\end{align*}
\]

Equation (13) serves as the BC for the internal domain at the seam, applied concurrently with the FDTD procedure. The quantities \( \bar{H}^{n-\frac{1}{2}}_{i-\frac{1}{2}} \) and \( \bar{H}^{n-\frac{1}{2}}_{i+\frac{1}{2}} \) used there are also evaluated recursively in the course of the FDTD computation, using the following combination of (10) and (12):

\[
\bar{H}^{n-\frac{1}{2}}_{i+\frac{1}{2}} = \bar{H}^{n-\frac{1}{2}}_{i+\frac{1}{2}} - T^n \ast R^n_{i+\frac{1}{2}} H^n_{i-\frac{1}{2}} \ast \bar{H}^{n-\frac{1}{2}}_{i+\frac{1}{2}}.
\]

The reflection coefficient \( R^n_{i+\frac{1}{2}} \) that characterizes the external domain, is independent of the excitation and needs to be available beforehand. This mandates a pre-processing stage for the calculation of \( R^n_{i+\frac{1}{2}} \). In line with the comment after (4), note that in practice the temporal series \( R^n \) is truncated to its first \( N \) terms, such that the BC in (13) involves an \( N \)-term history behind the wavefront.

III. SOURCE DECOMPOSITION IN ONE DIMENSION

In the one dimensional case, the SDM coincides with the wave-type approach discussed in the Introduction, since the seam is trivially planar. To implement this approach, one first resolves the electric field into the outgoing and incoming wave components, that are also the contributions of the sources in the computational and external domains, respectively, in this case. constituents via (recall that the immediate vicinity of the seam is homogeneous with \( \epsilon = \epsilon_0, \mu = \mu_0 \):

\[
\begin{align*}
\bar{E} &= \frac{1}{2} \left( E(t, z^n_{i+\frac{1}{2}}) + \eta_0 H(t, z^n_{i+\frac{1}{2}}) \right), \\
\bar{H} &= \frac{1}{2} \left( E(t, z^n_{i-\frac{1}{2}}) - \eta_0 H(t, z^n_{i-\frac{1}{2}}) \right),
\end{align*}
\]

with analogous expressions for \( \bar{H} \) and \( \bar{H} \). As noted above, the outgoing and incoming constituents originate from internal and external sources, respectively. The electric and magnetic fields of each constituent are then related by the intrinsic impedance \( \eta_0 \) that plays a role analogous to the characteristic impedance in transmission line representation. The discrete counterpart of this impedance is \( \bar{Z}^n \), with \( \bar{Z}^n \) being the corresponding intrinsic admittance (see discussion after (5)). These operators are used in the context of the Yee grid as follows:

\[
\begin{align*}
\bar{H}^{n+\frac{1}{2}}_{i+\frac{1}{2}} &= Y^n_{i+\frac{1}{2}} \ast E^n_{i+\frac{1}{2}}, \\
\bar{H}^{n+\frac{1}{2}}_{i-\frac{1}{2}} &= -Z^n_{i} \ast \bar{H}^{n-\frac{1}{2}}_{i+\frac{1}{2}}. 
\end{align*}
\]
The pre-processing stage for finding \( R_{n, \frac{1}{2}, \frac{1}{2}} \) proceeds in two stages as follows. In the first stage, the entire space is replaced with a homogeneous medium, creating an analog of a matched load in transmission line theory. The field \( H_{n, \frac{1}{2}, \frac{1}{2}} \) that is due to constrained test sources in the internal domain is then calculated. The 0th order designation is used for this first stage. Specifically, we identify the \( i = 1 \) term \( H_{n, \frac{1}{2}, \frac{1}{2}} \) as the desired outgoing field \( H_{n, \frac{1}{2}, \frac{1}{2}} \) near the seam. In the second stage, the test problem includes an external domain that contains the true inhomogeneities whereas the internal domain is replaced with a homogeneous medium. Using the same internal sources, the total field \( H_{n, \frac{1}{2}, \frac{1}{2}} \) is calculated. In this stage, the 1st order designation is used. We can now identify the backscattered field as

\[
H_{n, \frac{1}{2}, \frac{1}{2}}^{\text{out}} = H_{n, \frac{1}{2}, \frac{1}{2}}^{\text{in}} - H_{n, \frac{1}{2}, \frac{1}{2}}^{0, \text{in}}
\]

from which the reflection coefficient can be extracted by de-convolving Eq. (12), now written in the form

\[
H_{n, \frac{1}{2}, \frac{1}{2}}^{\text{out}} = H_{n, \frac{1}{2}, \frac{1}{2}}^{\text{in}} - H_{n, \frac{1}{2}, \frac{1}{2}}^{0, \text{in}} = R_{n, \frac{1}{2}, \frac{1}{2}, \text{in}} \cdot H_{n, \frac{1}{2}, \frac{1}{2}}^{\text{in}},
\]

(16)

In order to perform the de-convolution operation, the convolution (16) is re-cast in matrix representation, as follows:

\[
\begin{align*}
\tilde{H}_{n, \frac{1}{2}, \frac{1}{2}} &= \bar{R}_{n, \frac{1}{2}, \frac{1}{2}, \text{in}} \cdot \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{\text{in}} \\
\tilde{H}_{n, \frac{1}{2}, \frac{1}{2}} &= \left( \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{1}, \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{2}, \ldots, \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{N} \right)^{T},
\end{align*}
\]

(17)

where

\[
\begin{align*}
\tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{1} &= \left( \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{1}, \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{2}, \ldots, \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{N} \right)^{T}, \\
\tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{2} &= \left( \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{2}, \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{2}, \ldots, \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{N} \right)^{T}, \\
\tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{N} &= \left( \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{N}, \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{N}, \ldots, \tilde{H}_{n, \frac{1}{2}, \frac{1}{2}}^{N} \right)^{T},
\end{align*}
\]

(18)

and \( R_{n, \frac{1}{2}, \frac{1}{2}, \text{in}} \) is the causal–circulant matrix

\[
R_{n, \frac{1}{2}, \frac{1}{2}, \text{in}} = \begin{pmatrix}
R_{1, \frac{1}{2}, \frac{1}{2}}^{1} & 0 & 0 & \cdots & 0 \\
R_{2, \frac{1}{2}, \frac{1}{2}}^{1} & R_{2, \frac{1}{2}, \frac{1}{2}}^{1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_{N, \frac{1}{2}, \frac{1}{2}}^{1} & \cdots & \cdots & \cdots & R_{N, \frac{1}{2}, \frac{1}{2}}^{1}
\end{pmatrix}.
\]

(19)

Then, the elements of \( R_{n, \frac{1}{2}, \frac{1}{2}, \text{in}} \) can be extracted recursively, starting from the first row and adding one equation at a time for each element.

IV. SOURCE DECOMPOSITION IN TWO DIMENSIONS

The one dimensional formulation Eq. (12) of Section III can branch out into either a wave or source decomposition two dimensional representations. The wave representation is less useful in two or three dimensions, where it is related only to the spectral constituents. For this wave representation to hold in multidimensional problems, the enclosing surface has to be convex. The SDM is still valid, though, for concave as well as convex surfaces. In this representation, we redefine the outgoing and incoming field constituents as the outcomes of sources in the computational and external domain, respectively. Then, the generalization to more dimensions can be carried out in the spatial (field) rather than the spectral (wave) domain in the form of the source decomposition method detailed below. In the one–dimensional case treated above, the two definitions coincide.

![Fig. 5. 2-D example: Parallel plate waveguide with $\frac{\partial E}{\partial z} = 0$ and a reflective external domain.](image-url)

In order to develop the source decomposition method (SDM) for the two dimensional case, consider a typical geometry with $\frac{\partial E}{\partial z} = 0$, that can be defined in the form of a parallel plate waveguide of width $a$ in the $y$ direction (see Fig. 5), supporting a $TE_z$ propagating mode. The discrete index notations in this case are $F_{n, j} = F(i\Delta z, j\Delta y, n\Delta t)$ where $F$ is any field component, and the corresponding FDTD equations are (note that $\epsilon$ and $\mu$ are now functions of $(z, y)$)

\[
\begin{align*}
H_{n, i, \frac{1}{2}, j}^{\text{out}} &= H_{n, i, \frac{1}{2}, j}^{\text{in}} - \frac{\gamma_y}{\eta_{i, j}}(E_{n, i, j} - E_{n, i, j+1}) \quad (20a) \\
H_{n, i, j, \frac{1}{2}}^{\text{out}} &= H_{n, i, j, \frac{1}{2}}^{\text{in}} + \frac{\gamma_y}{\eta_{i, j}}(E_{n, i, j+1} - E_{n, i, j}) \quad (20b) \\
E_{n, i, j} &= E_{n, i, j} + \gamma_y \eta_{i, j}(H_{n, i, j}^{\text{out}} - H_{n, i, j+1}^{\text{out}}) \quad (20c)
\end{align*}
\]

where $\gamma_y = \frac{c}{\Delta y}$ and $\gamma_z = \frac{c}{\Delta z}$. In the sequel we use the shorthand notation $E = E_x$ and $H = H_y$. The field at each point $j$ over the seam at $z = l\Delta z$ in now a summation of contributions from all nearby points $j'$ along the seam. This summation is expressed, for the case of an $E$-type boundary, in the following generalization of equation (11):

\[
E_{n, i, j} = \sum_{j'=0}^{J} \left( Z_{n, j'}^{\text{in}} * H_{n, i, j'}^{\text{out}} + Z_{n, j'}^{\text{out}} * H_{n, i, j'}^{\text{in}} \right)
\]

(21)

where $J = \frac{a}{\Delta y}$, and the medium is assumed to be homogeneous ($\epsilon = \epsilon_0$, $\mu = \mu_0$) in the immediate surrounding of the seam.\(^3\)

\(^3\)In general, the spatial summation is not a convolution because the spatial matrices are not circulant.
The generalization of the wave reflection coefficient \( R_{\ell+\frac{1}{2},j}^{n+\frac{1}{2},j'} \) of (12) would apply to a wave representation, however it does not apply the SDM. Instead, it is replaced by the “external return operator” \( R_{\ell+\frac{1}{2},j}^{n+\frac{1}{2},j'} \), defined as follows:
\[
    H_{\ell+\frac{1}{2},j}^{n+\frac{1}{2},j'} = \sum_{j''=0}^{\infty} R_{\ell+\frac{1}{2},j-j'}^{n+\frac{1}{2},j''} H_{\ell+\frac{1}{2},j'}^{n+\frac{1}{2},j''} ,
\]
the two dimensional generalization of Eq. (14) is
\[
    H_{\ell+\frac{1}{2},j}^{n+\frac{1}{2},j'} = H_{\ell+\frac{1}{2},j}^{n+\frac{1}{2},j'} - \sum_{j''=0}^{\infty} T_{j,j''}^{n+\frac{1}{2},j'} \sum_{j'=0}^{\infty} R_{\ell+\frac{1}{2},j-j''}^{n+\frac{1}{2},j''} H_{\ell+\frac{1}{2},j'}^{n+\frac{1}{2},j''} ,
\]
where \( T_{j,j''}^{n+\frac{1}{2},j'} \) is defined in analogy with (10). As with Eq. (14), usage of the boundary condition (23) is contingent upon the availability of \( R_{\ell+\frac{1}{2},j}^{n+\frac{1}{2},j'} \), that has been computed in a pre-processing stage. The procedure for this stage is again analogous to the one dimensional case. One sets the medium within the internal domain to be homogeneous. Then, the outgoing fields on both sides of the seam, \( H_{\ell+\frac{1}{2},j}^{\pm\frac{1}{2},j'} \), and the total field \( H_{\ell+\frac{1}{2},j}^{\pm\frac{1}{2},j} \) are computed for the same trial sources in the presence of a homogeneous and the actual external domains, respectively. These quantities are used to construct the two dimensional equivalent of (16), from which \( R_{\ell+\frac{1}{2},j}^{n+\frac{1}{2},j'} \) is then extracted.

Following the comment after (4), note again that in practice the temporal series are truncated to their first \( N \) terms, so that the BC in (13) involves an \( N \)-terms history behind the wavefront.

V. NUMERICAL EXAMPLES

The performance of the SDM is evaluated in comparison with a FDTD reference simulation in a pseudo–infinite computational domain that extends over both the original computational and external domains. Scatterers within the original external domain can be included. The figure of merit that quantifies the BC performance is the normalized RMS error over the entire original computational domain, evaluated at the maximal temporal point:
\[
    err = \sqrt{\frac{\sum_{n=1}^{N} (E_{\text{sdm}}^{n} - E_{\text{ref}}^{n})^2}{\sum_{n=1}^{N} (E_{\text{ref}}^{n})^2}}
\]
where \( E_{\text{sdm}}^{n} \) and \( E_{\text{ref}}^{n} \) are the results of the SDM and the reference problem, respectively.

The SDM is first tested for a 1-D case as an ABC with a homogeneous external domain for several values of \( \gamma \). The excitation is defined by the electric field at the aperture plane \( z = 0 \) (see Fig. 2) that can be replaced by its equivalent magnetic current backed by a perfect electric conductor at the same plane. The exciting pulse for this case is a Blackman-Harris window
\[
    E(t) = \begin{cases} 
    a_0 + a_1 \cos \frac{\pi(t-t_c)}{t_h} + a_2 \cos \frac{2\pi(t-t_c)}{t_h} + a_3 \cos \frac{3\pi(t-t_c)}{t_h}, \\ 0,
    \end{cases} \quad \frac{|t-t_c|}{t_h} < 1 \text{ otherwise}
\]
with \( t_h = 12\Delta t \) and \( a_0 = 0.35875 \), \( a_1 = 0.48829 \), \( a_2 = 0.14128 \), and \( a_3 = 0.01168 \). The temporal size of the vector \( \mathbf{r}_{\text{in}} \) in equations (12)–(13) is referred to henceforth as “filter length”. The multiple reflection issue is not present in this case, and indeed the results track those found for the impedance/admittance approach in [13], see Fig. 6. Ideally, for an infinite filter length, the error would be negligible. For filter lengths of \( N \geq 60 \), the error is still under \( \sim 60 \text{ dB} \). Next, an external domain that contains a step discontinuity in the electrical permittivity from \( \epsilon_r = 1 \) at \( i < I + 10 \) to \( \epsilon_r = 9 \) at \( i \geq I + 10 \). Results for the filter function \( R_{\ell+\frac{1}{2},j}^{n+\frac{1}{2},j'} \) now account for the reflections in the external domain, as can be seen in Fig. 7 for several values of \( \gamma \). The continuous value of \( R = 0.58(t-20\Delta t) \) has more dispersion errors as \( \gamma \) is decreased.

![Fig. 6. RMS normalized error vs. the BC filter length (defined as the temporal size the vector \( \mathbf{r}_{\text{in}} \) in equations (12)–(13)) for a 1-D problem as an ABC for homogeneous external domain (no multiple reflections issue). Size of computational domain: \( I = 100 \), Simulation was carried for \( N = 1000 \) times steps. Results track those in [13].](image-url)

The 2D SDM of Sec. IV is tested in the parallel plate waveguide configuration, with the excitation again defined by the electric field at the aperture plane \( z = 0 \) (see Fig. 5). The excitation is designed as a TE\(_0\) doubly differentiated Gaussian pulse whose spectrum fits below the FDTD upper frequency limit. Also, to show a propagating wave in the context of this example, the spectrum fits above the waveguide cutoff frequency, a condition necessary in general (see Fig. 9). The total normalized RMS error compared with the FDTD solution
is shown in Fig 8. The error for finite filter lengths of $N \geq 60$ the error is typically under $-30$ dB.

Results are shown in Fig. 10 for both homogeneous and step discontinuity external domains, the latter being similar to the 1-D case of Figures 7 and 8. In order to achieve a normalized RMS error of $-30$ dB or lower for both cases, a filter length of $N \geq 120$ is needed. This order of accuracy is the same as the one obtained in the GFM, however the SDM requires a shorter computation time (not including the pre-processing stage) as seen in Fig. 11.

This calculation is performed once for all possible configurations within the computational domain and all excitations, therefore it cannot be uniquely quantized. Regarding memory consumption, it is roughly proportional to the total number of samples, similarly to the computational time behavior. In this simulation, a comparison is made between the two methods. The normalized simulation time is measured for a fixed normalized RMS error of $-30$ dB while varying the distance of the external scatterer from boundary. In the GFM, the length of the filter has to be increased in accordance with the distance of the external scatterer from the boundary, in order to include the scatterer within the computational domain and thus avoid the multiple reflection phenomenon described in Section II. Therefore, run times increase roughly in proportion to this distance as seen in Fig. 11. The SDM run times, in comparison, remain about stationary. Consequently, although the GFM is more efficient for a nearby scatterer, the SDM becomes much more efficient in more general cases.

If one keeps the filter length fixed, then the GFM computation is bound to show higher errors at later times. This phenomenon is seen in Fig. 12 that shows the electric field near the boundary as a function of time for both methods.
Fig. 11. Run time of the SD compared with the GFM for a normalized RMS error of −30 dB vs. the scatterer distance. The ensure this level of error, the computational domain for the GFM case is chosen to include the external discontinuity, while the GFM filter length remains constant, describing the free space boundary conditions only. For the SD case, both size of the computational domain and the filter length remain stationary.

A discontinuity in the dielectric constant is introduced at 20 samples beyond the boundary. For this case, the SDM reconstructs an incident pulse plus one physical reflected pulse, while the gfm (dashed line) generates spurious reflections. \( \gamma = 0.7 \).

Fig. 12. Simulation of the total electric field near the boundary as a function of time for a step discontinuity in the dielectric constant at 20 samples beyond the boundary. The sd (solid line) method reconstructs an incident pulse plus one physical reflected pulse, while the gfm (dashed line) generates spurious reflections.

VI. Conclusions

The time domain diakoptic/domain decomposition method approach is capable of stitching together regions of arbitrary shape and composition, that may contain reflective objects. The source decomposition method (SDM), presented above, has been shown to be a viable variant of this approach that eliminates spurious reflections between physical and artificial boundaries. Since the entire computation is done within the framework of the second order accurate, explicit Yee scheme, no issues of instability are encountered as long as the CFL condition is adhered to. Although implementation of this method requires a pre-processing stage, this added burden is small relative to the main computational stage. The method is particularly suitable for multiple scatterers scenarios, where the scatterers are distinct. It is most useful in a design process, where the computational domain undergoes several modifications with many excitations while the external boundary remains unchanged. Another computational advantage is seen in the reduction of the size of the white space by letting the computational domain boundary track the boundary of a scatterer. This advantage is especially important for a multi-body problem with many excitations, where an outright FDTD computation would involve an expanded computational domain that includes all scatterers, internal and external, with a substantial white space between them. Clearly, for tightly packed scatterers with a small number of excitations, the conventional FDTD method suffices.

The formulation in Section IV, although written in two-dimensional terms for the sake of relative simplicity, is the same for three dimensions, except for extra variables necessary to model the three dimensional dyad involved. Although outside the scope of the present work, three dimensional problems are routinely implemented in FDTD simulations. Once combined with the SDM, problems with highly complex structures (e.g., the human body) could be torn into smaller regions, that can be dealt with efficiently. This is the topic of future work.

Although the pre-processing stage of calculation of \( R \) is done only once per a given configurations for all excitations, acceleration of this stage may be in order. For the homogeneous case, methods for by-passing the direct computation of FDTD-compatible Green’s functions can be found in [15], [16]. More complex cases would probably involve hybridization with integral equations techniques.

References


