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ASYMPTOTIC SOLUTIONS FOR CRACK CLOSURE IN AN ELASTIC PLATE UNDER COMBINED EXTENSION AND BENDING

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ABSTRACT

A coupled plane-bending problem is considered for an elastic Kirchhoff–Poisson plate containing a through-the-thickness or (part-through) surface crack under closure. The stress intensity factors at the ends of the crack are not zero. Asymptotic solutions are derived for cases in which the ratio of the crack length to its depth is large. As is shown, the width of the contact strip decreases as the crack length increases; the limiting contact force and moment distribution may be determined by considering an edge-cracked strip with zero stress intensity factor in the thickness direction. As is also shown, the crack surface interaction in-plane force and bending moment can be derived directly from the initial force and moment distribution acting in the intact plate on the prospective crack line. The same result is valid for a collinear system of cracks; this collinear system may include alternating open and closed crack segments. In addition, the closure stress distribution is determined. For the latter, the width of the contact strip asymptote is derived as a function of the crack length coordinate, and the asymptotic stress distribution is found as a product of the thickness averaged closure stress and a function of a normalized plate thickness coordinate. The latter stress distribution is unique and universal for any slowly curving crack or crack system under closure.

1. INTRODUCTION

One of the major difficulties in the analysis of either the load carrying capacity or the penetration of elastic plates undergoing brittle or semi-brittle deformations is the treatment of the cracking that occurs. Historically, the difficulties introduced by crack closure in cracked plates under bending have long been recognized (Smith and Smith, 1970; Jones and Swedlow, 1975; Heming, 1980; Alwar and Nambissan, 1983; Joseph and Erdogan, 1989; Young and Sun, 1992, 1993). The crack closure phenomena treated in this paper provide a framework for the application of fracture mechanics to cracked plates subjected to closure.

Consider a cracked infinite elastic plate (Fig. 1). The plate may be subjected to in-plane and transverse forces as well as bending moments of general distribution. The loading is assumed to induce internal in-plane force and moments distributions that vary slowly (compared to the crack depth) in the intact plate on the prospective crack path. Further, there are no shear stresses in the crack plane. The crack is assumed to be either through-the-thickness or part-through; the latter surface crack being formed by the contact interaction of the through-the-thickness crack surfaces. Closure causes

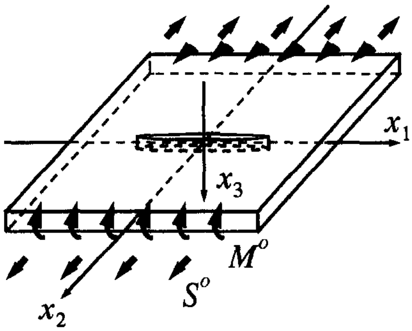

Fig. 1. The cracked plate configuration.

a zero stress intensity factor, except at the through-the-thickness crack tips. Nevertheless, almost throughout, the considerations are valid for a surface crack with a non-zero stress intensity factor.

Let the coordinates x_1, x_2 ($x_2 = 0$ on the crack) be placed in the plate middle plane, and x_3 be the transverse coordinate. The three-dimensional fields in the vicinity of the crack tips (in the x_1, x_2-plane) are neglected. The title problem is comprised of three sub-problems. The first problem is a plane problem (x_1, x_2-plane) for an elastic layer containing a through-the-thickness crack (Fig. 2) with a contact force, $S(x_1)$. This force $S(x_1)$ is assumed to be applied in the middle plane ($x_3 = 0$). The second problem is a bending problem for a Kirchhoff–Poisson plate containing the same crack (Fig. 3) with a contact-induced bending moment, $M(x_1)$. The third problem is a plane contact problem for an elastic layer containing a part-through surface crack (Fig. 4) with a normal stress distribution, $\sigma(x_1, x_3)$, acting on the continuation of the crack line, $-h \leqslant x_3 \leqslant h-a$. Such a formulation was given by Rice and Levy (1972); their problem was reduced to two coupled integral equations that were solved numerically for the case of a semi-elliptical part-through crack. Rice and Levy (1972) discussed several important aspects in their formulation of the surface crack problem; much of

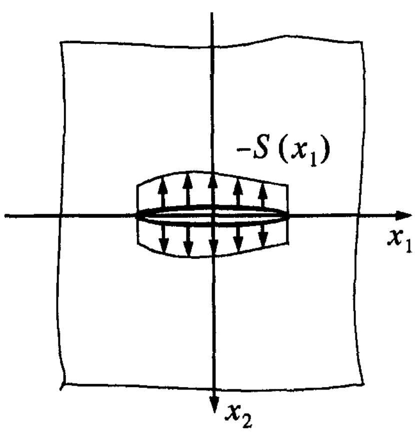

Fig. 2. The x_1, x_2-plane sub-problem.

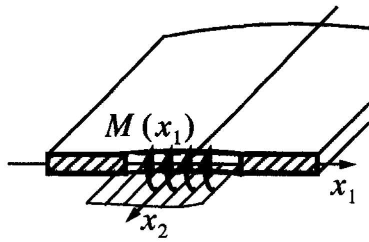

Fig. 3. Pure bending of the plate.

this discussion is relevant here. The coupled plane-bending problem for a through-the-thickness crack in an infinite elastic plate subjected to a uniform bending moment was recently considered by Young and Sun (1992, 1993) with the assumption of line contact at the topmost compressive edge of the crack face. Closed form solutions for the interaction force were presented therein for a uniform far-field moment distribution.

These papers (Rice and Levy, 1972; Young and Sun, 1992, 1993) are closely related to the present paper. In the present paper, the interconnections between the complete formulation by Rice and Levy (1972) and the simplified formulation by Young and Sun (1992) are established. The formulation and results obtained by Young and Sun (1992) are shown in this paper to be a particular case of the general asymptotic representation of the interaction force which follows from the formulation in Rice and Levy (1972) for long cracks. The three papers mentioned here also provide a representative survey of related papers.

In the present paper, the width of the contact strip asymptote is derived and the asymptotic stress distribution is found as well as the contact force and moment distribution for a collinear system of cracks.

In outline, the type of asymptotic solution considered depends on the lines of action of the resultant in-plane force. This resultant is expressed in terms of $S^0(x_1)$ and $M^0(x_1)$ via the coordinate $x_3 = e_f^0$, where $e_f^0 = M^0(x_1)/S^0(x_1)$; (only the absolute value is important, $|e_f^0|$). That is, this force and moment are statically equivalent to the force S^0 acting at $x_3 = e_f$. There are three possible types which correspond to the regions: (a) $0 \leqslant |e_f^0| \leqslant h/3$, (b) $h/3 < |e_f^0| < h$ and (c) $|e_f^0| \geqslant h$, where h is the half plate thickness. In (a), there is no crack opening displacement if $S^0 < 0$ (clearly, there is no closure if $S^0 > 0$), and uncoupled uncracked plane problem is at hand. In (b),

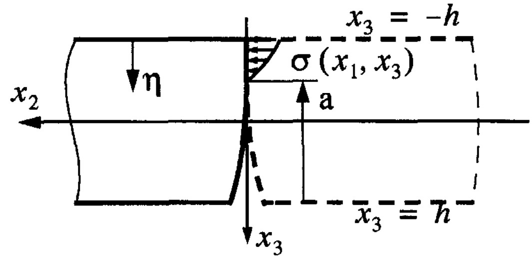

Fig. 4. Edge-cracked strip under closure.

the solution has a limit which corresponds to the x_2, x_3-plane problem; in this instance, the crack length influence on the crack opening displacement becomes negligible for long cracks. In fact, for long cracks, the problem reduces to an uncoupled edge-cracked x_2, x_3-plane problem, and the role of the x_1, x_2-plane and bending problems disappear. In other words, the closure force S and moment M are asymptotically the same as the initial values, S^0 and M^0, respectively.

Case (c) ($|e_f^0| \geqslant h$), however, identifies a fully coupled plane-bending problem in the sense that there is no plane problem limit, and the length of the crack remains an important variable. In this case the problem becomes asymptotically segmented in the following way. The coupled plane-bending problem can be considered separately under the equality $e_f = \pm h$ ($e_f = M(x_1)/S(x_1)$; the initial value is e_f^0), the same as was assumed by Young and Sun (1992, 1993). The latter solutions represent the particular case of $|e_f^0| = \infty$; that is, only a moment was applied ($S^0 = 0$). In fact, $e_f = \pm h$ is valid for $e_f^0 \geqslant h$ without any restrictions on the far-field loading (on the prospective crack line, the initial distribution is assumed to be slowly variable as compared to the crack depth). At the same time, the x_2, x_3-plane contact problem solution, which gives us the contact stress distribution, can be easily obtained using the solution to the coupled problem. Note that the above types of asymptotic solutions are restricted to the case of a zero stress intensity factor in the plane contact problem. A non-zero stress intensity factor would alter the ranges considered in (a)–(c) above.

Rice and Levy's coupled integral equation formulation is used below as well as an inverse formulation. The latter clarifies some additional issues and facilitates the consideration of a system of cracks. The fundamental solutions are given first for the inverse formulation in which the force and moment are expressed in terms of crack opening displacement and rotation (the first and the second problems accordingly).

2. FUNDAMENTAL GENERALIZED SOLUTIONS

In this section, the local rectangular (x, y) and polar coordinate (r, α) system shown in Fig. 5 are introduced. Neither system is connected with the global system shown in Fig. 1. Consider first the generalized forces (distributions) which must act on the intact plate to produce a δ-discontinuity in the u_y displacement on the x-axis but without any shear stresses on the same axis:

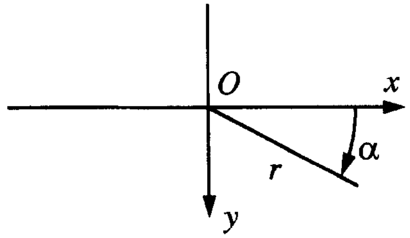

Fig. 5. Cartesian and polar coordinate systems.

$$v(x) = \lim_{y\to 0^+} u_y(x,y) = -\lim_{y\to 0^-} u_y(x,y) = \delta(x),$$

$$\sigma_{xy} = 0 \quad (-\infty < x < \infty, y = 0). \tag{1}$$

Using the complex variable $z = x+iy$ and assuming plane stress, the displacement and stress fields which correspond to the above conditions (Slepyan, 1981, Appendix I) are given by

$$u_x + iu_y = -\frac{1+v}{4\pi}\left(\frac{\kappa}{z} - \frac{2}{\bar{z}} + \frac{z}{\bar{z}^2}\right) \sim -\frac{1-v}{2\pi}x^{-1} \pm i\delta(x) \quad (y \to \pm 0),$$

$$\sigma_{xx} = \frac{E}{2\pi}\mathrm{Re}\frac{\bar{z}}{z^3} = \frac{E}{2\pi}\frac{\cos 4\alpha}{r^2} \to \frac{E}{2\pi}x^{-2} \quad (y \to 0),$$

$$\sigma_{yy} = \frac{E}{2\pi}\mathrm{Re}\left(\frac{2}{z^2} - \frac{\bar{z}}{z^3}\right) = \frac{E}{2\pi}\frac{2\cos 2\alpha - \cos 4\alpha}{r^2} \to \frac{E}{2\pi}x^{-2} \quad (y \to 0),$$

$$\sigma_{xy} = \frac{E}{2\pi}\mathrm{Im}\left(\frac{1}{z^2} - \frac{\bar{z}}{z^3}\right) = \frac{E}{2\pi}\frac{(\sin 4\alpha - \sin 2\alpha)}{r^2} \to 0 \quad (y \to 0). \tag{2}$$

in which E is Young's modulus, v is Poisson's ratio, and $\delta(x)$ is the Dirac δ-function.

Consider next the fundamental deflection function u_z that generates a δ-discontinuity in the angle of rotation $\partial u_z/\partial y$ of each side of the discontinuity, but without any shearing force R_y on the same axis:

$$\phi(x) = \lim_{y\to 0^+}\frac{\partial u_z}{\partial y} = -\lim_{y\to 0^-}\frac{\partial u_z}{\partial y} = \delta(x),$$

$$R_y = 0 \quad (-\infty < x < \infty, y = 0). \tag{3}$$

The required solution must satisfy the Kirchhoff–Poisson equation $\Delta^2 u_z = 0$ and is deduced to have the form

$$u_z = \frac{1}{4\pi}\left(\frac{\partial^2}{\partial y^2} + v\frac{\partial^2}{\partial x^2}\right)r^2\ln r, \quad r = \sqrt{x^2+y^2}. \tag{4}$$

This gives

$$u_z = \frac{1+v}{2\pi}\ln r + \frac{1}{4\pi}\left[3+v-2(1-v)\frac{x^2}{r^2}\right]. \tag{5}$$

This solution is anti-symmetrical with respect to y and behaves as required by (3):

$$\frac{\partial u_z}{\partial y} = \frac{1+v}{2\pi}\frac{y}{r^2} + \frac{1-v}{\pi}\frac{x^2 y}{r^4} \to \pm\delta(x) \quad (y \to \pm 0). \tag{6}$$

The jump in the rotation $\partial u_z/\partial y$, for a crack located on the x-axis undergoing symmetrical deformations can be expressed using the above fundamental solution and the superposition principle. This leads to an integral equation with respect to the slope; the same consideration applies in the plane problem where the main unknown is the jump in u_y-displacement. In and of itself, the problem of a through crack under bending is physically ill-posed because of the interpenetration of the crack surfaces. However, with suitable closure conditions, the coupled plane-bending problem is physically meaningful.

The bending and twisting moments corresponding to the expression in (5) are given by

$$M_x = -D\left(\frac{\partial^2 u_z}{\partial x^2}+\nu\frac{\partial^2 u_z}{\partial y^2}\right), \quad M_y = -D\left(\frac{\partial^2 u_z}{\partial y^2}+\nu\frac{\partial^2 u_z}{\partial x^2}\right),$$

$$M_{xy} = -(1-\nu)D\frac{\partial^2 u_z}{\partial x\,\partial y} = \frac{D}{2\pi r^2}((1-\nu^2)\sin 2\alpha+(1-\nu)^2\sin 4\alpha),$$

$$(M_x, M_x+M_y) = \frac{D}{2\pi r^2}((1-\nu)^2\cos 4\alpha, -2(1-\nu^2)\cos 2\alpha),$$

$$(M_x, M_y) \sim \frac{D}{2\pi}\left(\frac{(1-\nu)^2}{x^2}, -\frac{(3+\nu)(1-\nu)}{x^2}\right) \quad (y=0). \tag{7}$$

in which D represents the bending stiffness and is given by $D = E'I$; $\cos\alpha = x/r$ and $\sin\alpha = y/r$. In D, $E' = E/(1-\nu^2)$ and $I = 2h^3/3$, I and h being the moment of inertia and half the plate thickness, respectively.

The Kirchhoff shear force, R_y, is expressed as

$$R_y = D\frac{\partial}{\partial y}\left(\frac{\partial^2 u_z}{\partial y^2}+(2-\nu)\frac{\partial^2 u_z}{\partial x^2}\right) = (1-\nu)^2\frac{3D}{\pi r^3}\sin\alpha\cos 4\alpha. \tag{8}$$

If the expression for u_z from (4) is substituted into (8) and use is made of the fact that $\Delta^2 u_z = 0$ for $y \neq 0$, it quickly follows that

$$R_y = -\frac{D(1-\nu)^2}{4\pi}\frac{\partial^4}{\partial x^4}(2y\ln r+y) \to 0 \quad (y \to 0). \tag{9}$$

The above fundamental solutions can be used for any crack systems to derive the integral equations for plane, bending or coupled problems (as mentioned above, the bending problem should not, in general, be considered by itself but only as a component of the coupled problem).

3. INTEGRAL EQUATION FORMULATION

Consider the crack on the x_1-axis: $-l < x_1 < l$, $x_2 = 0$ (Fig. 1). Using the above fundamental solutions for plane and bending problems one can express the force S_{x_2} and the moment M_{x_2} at $x_2 = 0^+$ by the superposition principle (in this case $\cos 2\alpha = \cos 4\alpha = 1$ and $S^0 \equiv S^0_{x_2}$ and $M^0 \equiv M^0_{x_2}$), giving

$$\frac{Eh}{\pi}\int_{-l}^{l}\frac{v(\xi)}{(\xi-x_1)^2}\,\mathrm{d}\xi = -S^0(x_1)+S(x_1),$$

$$-\frac{Eh}{\pi}\int_{-l}^{l}\frac{\phi(\xi)}{(\xi-x_1)^2}\,\mathrm{d}\xi = \hat{\nu}(-M^0(x_1)+M(x_1)), \tag{10}$$

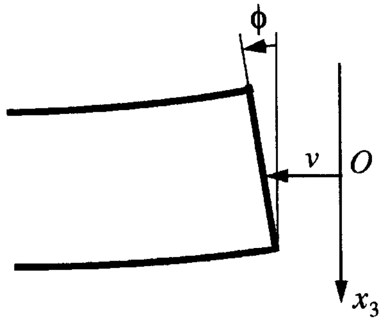

Fig. 6. Averaged crack opening displacement and rotation angle.

in which $\hat{v} = 3(1+v)/(3+v)h^2$, $-l < x_1 < l$ and S^0 and M^0 are the initial force and moment acting in the intact plate, and S and M are the crack surface interaction force and moment; v is the crack opening displacement, and ϕ is the crack face rotation (Fig. 6). Note that $S^0_{x_1}$ and $M^0_{x_1}$ exert no influence, as usual; in addition, it is assumed that the shear force $S^0_{x_1x_2}$ and the twisting moment $M^0_{x_1x_2}$ are both zero.

The inverse form of the integral equation formulation used by Rice and Levy (1972) can be derived from (10) using the following relations (Rice and Levy, 1972; Slepyan, 1981). These equations arise naturally in the consideration of a Griffith crack under arbitrary loading; in this context they link the crack opening displacement and the stress distribution on the plane $y = 0$. In this paper, (11) and (13) are useful in several contexts, as will become evident. Hence, for $-\infty < x = x_1 < \infty$ and $f_+(x) = 0$, given that the subscript "+" or "−" implies that $x^2 > l^2$ or $x^2 < l^2$, respectively,

$$\frac{1}{\pi}\int_{-l}^{l}\frac{f_-(\xi)}{(\xi-x)^2}\,\mathrm{d}\xi = \begin{cases}\alpha_-(x)\\ \alpha_+(x)\end{cases} \tag{11}$$

where

$$\alpha_+(x) = \frac{1}{\pi}\frac{\operatorname{sgn} x}{\sqrt{x^2-l^2}}\int_{-l}^{l}\frac{\alpha_-(\xi)\sqrt{l^2-\xi^2}}{\xi-x}\,\mathrm{d}\xi. \tag{12}$$

Then

$$f_-(x) = -\pi\int_{-l}^{l}\alpha_-(\xi)L(x,\xi)\,\mathrm{d}\xi \tag{13}$$

in which

$$L(x,\xi) = \frac{1}{\pi^2}\ln\left|\frac{\sqrt{l+\xi}\sqrt{l-x}+\sqrt{l-\xi}\sqrt{l+x}}{\sqrt{l+\xi}\sqrt{l-x}-\sqrt{l-\xi}\sqrt{l+x}}\right|. \tag{14}$$

Thus, the inverse of (10), for ($x_1^2 < l^2$), is

$$v(x_1) = \frac{\pi}{Eh} \int_{-l}^{l} [S^0(\xi) - S(\xi)] L(x_1, \xi) \, \mathrm{d}\xi, \tag{15}$$

$$\phi(x_1) = -\hat{v} \frac{\pi}{Eh} \int_{-l}^{l} [M^0(\xi) - M(\xi)] L(x_1, \xi) \, \mathrm{d}\xi.$$

Three different regions may be in existence along the crack line: crack closure, open non-contacting crack faces, and crack faces in full contact. In these regions, the solution has to satisfy conditions outlined in the next section.

4. EDGE-CRACKED STRIP

In the closure region, the global planar and bending deformations of the thin plate are well described by the usual assumptions (for example, planes remain plane and perpendicular to the neutral axis). However, "close" to the crack surface interaction area, the deformations can only be described by an "inner" or "local" elasticity solution. The stress distribution of the local problem differs from that for the global problem by a self-equilibrated stress field; the latter causes an additional crack opening displacement such that "far enough" from each crack the actual physical extent of contact can be deduced by prescribing kinematic compatibility. The plane contact problem (Fig. 4) describing the conditions in the closure region, similar to the paper by Rice and Levy (1972), relies heavily on the solution for an edge-cracked strip in plane strain subjected to an axial force S and moment M per unit thickness.

Following the procedure in Rice and Levy (1972), and given the configuration in Fig. 4, it quickly follows that the mode I stress intensity factor is given by

$$K(a) = \frac{1}{\sqrt{2h}} \left(g_s(\zeta) S(x_1) + \frac{3}{h} g_m(\zeta) M(x_1) \right) \tag{16}$$

in which $\zeta = a/2h$, while $a \equiv a(x_1)$. In terms of the weight function approach employed by Wu and Carlsson (1991) and the relevant expressions in Tada *et al.* (1985), it follows that

$$g_s(\zeta) = \sqrt{\pi\zeta} f_s(\zeta), \quad g_m(\zeta) = \sqrt{\pi\zeta} f_m(\zeta), \tag{17}$$

in which

$$f_s(\zeta) = \frac{F_s(\zeta)}{(1-\zeta)^{3/2}}, \quad f_m(\zeta) = \frac{F_m(\zeta)}{(1-\zeta)^{3/2}}. \tag{18}$$

In (18)

$$F_s(\zeta) = \sum_{i=0}^{7} \alpha_i^s \zeta^i, \quad F_m(\zeta) = \sum_{i=0}^{7} \alpha_i^m \zeta^i. \tag{19}$$

The results presented by Kaya and Erdogan (1987) (see their Table 3) apparently provide the most accurate normalized stress intensity factor and crack-mouth-opening displacement data for the tension and bending of an edge-cracked strip, especially for long crack lengths ($\zeta > 0.8$). A best fit of this SIF data leads directly to the following values for the coefficients α_i^s and α_i^m ($i = 0, 1, \ldots, 7$), consecutively, stated here as

$$\alpha_i^s : 1.1215, -1.6109, 6.9817, -17.044, 27.437, -27.441, 15.252, -3.5748. \quad (20)$$

$$\alpha_i^m : 1.1215, -2.9725, 8.8068, -21.257, 34.836, -35.100, 19.489, -4.5500. \quad (21)$$

At this juncture, it is important to note that the behavior of various quantities will be required for long crack lengths or ζ large ($\zeta \to 1$). The weight function approach employed in Wu and Carlsson (1991) is sound even for long crack lengths, but still loses accuracy for $\zeta > 0.85$. For this reason, the function g_m is obtained independently from Kaya and Erdogan (1987).

If v and ϕ denote the additional displacement and rotation of one end of the strip relative to the other due to the introduction of the crack,

$$v(x_1) = \frac{1}{E'}\left(\alpha_{ss}(\zeta)S(x_1) + \frac{3}{h}\alpha_{sm}(\zeta)M(x_1)\right),$$

$$\phi(x_1) = -\frac{3}{E'h}\left(\alpha_{ms}(\zeta)S(x_1) + \frac{3}{h}\alpha_{mm}(\zeta)M(x_1)\right). \quad (22)$$

By elastic reciprocity, $\alpha_{\lambda\mu} = \alpha_{\mu\lambda}$. The quantities v, ϕ, S and M are the same as in the bending-plane coupled problem (10) or (15). In terms of the closure contact problem,

$$v(x_1) = \frac{1}{2h}\int_{-h}^{h} u_2(x_1, 0^+, x_3)\,dx_3, \quad \phi(x_1) = -\frac{3}{2h^3}\int_{-h}^{h} x_3 u_2(x_1, 0^+, x_3)\,dx_3,$$

$$S(x_1) = \int_{-h}^{h} \sigma(x_1, x_3)\,dx_3, \quad M(x_1) = \int_{-h}^{h} x_3\sigma(x_1, x_3)\,dx_3, \quad (23)$$

where $u_2(x_1,\ 0^+,\ x_3)$ is the crack opening displacement, and $\sigma(x_1, x_3)$ is the stress distribution in the closure strip. The asymptotic stress distribution for a narrow contact strip as $a \to 2h$ is presented below. If S^0, M^0 and the crack length, $2l$, are given, the above equations together with the integral equations present the closed system of the equations for the problem under consideration.

The asymptotic solutions central to this paper are achieved by first providing a wide-ranging ($0 \leqslant \zeta = a/2h < 1$) description of the coefficients, $\alpha_{\lambda\mu}$, as functions of the ratio $e_f(x_1) = M(x_1)/S(x_1)$. From Dempsey *et al.* (1995), it is apparent that the most suitable form is

$$\alpha_{\lambda\mu}(\zeta) = \frac{\pi\zeta^2}{(1-\zeta)^2}\Lambda_{\lambda\mu}(\zeta), \quad \zeta = \frac{a(x_1)}{2h} \quad (24)$$

where a is the crack length in this plane problem (in the coupled plane-bending problem, $a = a(x_1)$ is the depth of the crack opening).

The $\alpha_{\lambda\mu}$ are dimensionless compliance coefficients and are defined by

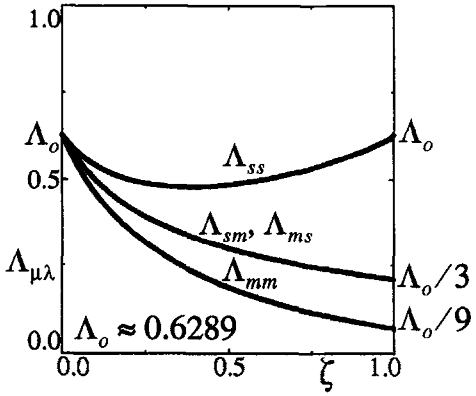

Fig. 7. Compliance coefficients $\Lambda_{\lambda\mu}(\zeta)$ versus $\zeta \equiv a(x_1)/2h$, a measure of the relative length of the crack in the thickness direction.

$$\alpha_{\lambda\mu}(\zeta) = \int_0^\zeta g_\lambda(\eta) g_\mu(\eta)\,\mathrm{d}\eta. \tag{25}$$

The expressions for $\Lambda_{\lambda\mu}$ in (24) are obtained first by the analytical integration of (25) and then a curve fit of the resulting long expression; in this manner, the key asymptotic behavior for small and large ζ is explicitly captured.

In (24)

$$\Lambda_{ss} = \sum_{i=0}^{7} \beta_i^{ss}\zeta^i, \quad \Lambda_{sm} = \sum_{i=0}^{7} \beta_i^{sm}\zeta^i, \quad \Lambda_{mm} = \sum_{i=0}^{7} \beta_i^{mm}\zeta^i \tag{26}$$

and the coefficients $\beta_i^{ss}, \beta_i^{sm}$ and β_i^{mm} $(i = 0, 1, \ldots, 7)$, consecutively, were found to be given by

$$\beta_i^{ss}: 0.6289, -1.1958, 3.9677, -7.5021, 8.4922, -4.1681, -0.3916, 0.7977. \tag{27}$$

$$\beta_i^{sm}: 0.6289, -1.6922, 4.7820, -9.9758, 13.3192, -10.4698, 4.3208, -0.70347. \tag{28}$$

$$\beta_i^{mm}: 0.6289, -2.1844, 5.9693, -12.5870, 17.6467, -15.0777, 7.0682, -1.39412. \tag{29}$$

The compliance functions $\Lambda_{\lambda\mu}(\zeta)$ are plotted in Fig. 7. Note that

$$F_s(0) = F_m(0) = F_s(1) = 3F_m(1) \approx 1.1215,$$

$$\Lambda_{ss}(1) = \Lambda_0 \approx 0.6289, \Lambda_{sm}(1) = \Lambda_{ms}(1) = \Lambda_0/3, \Lambda_{mm}(1) = \Lambda_0/9. \tag{30}$$

5. FRACTURE CRITERION

To complete the problem formulation a fracture criterion has to be specified. Griffith's energy criterion is suitable. In this case, the asymptotes of the force, S, and moment, M, as well as the crack opening displacement, v, and rotation, ϕ, are important on the crack line ahead of the crack tips. Based on the expressions

$$S \sim A_S/\sqrt{x_1 - l}, \quad M \sim A_M/\sqrt{x_1 - l} \quad (x_1 \to l+0), \tag{31}$$
$$v \sim A_v\sqrt{l - x_1}, \quad \phi \sim A_\phi\sqrt{l - x_1} \quad (x_1 \to l-0)$$

the energy release rate is given by

$$G = \frac{\pi}{2}(A_S A_v + A_M A_\phi). \tag{32}$$

In terms of the stress intensity factors, the same result can be expressed as

$$G = \frac{2h}{E}\left(K_v^2 + \frac{1+\nu}{3(3+\nu)}K_\phi\right) \tag{33}$$

where, for ($x_1 = \pm l$),

$$K_v = \frac{\sqrt{2\pi}}{2h}A_v = K_v^\pm, \qquad K_v^\pm = \frac{1}{2h\sqrt{\pi l}}\int_{-l}^{l}(S^0(\xi) - S(\xi))\sqrt{\frac{l \pm \xi}{l \mp \xi}}\,\mathrm{d}\xi, \tag{34}$$

$$K_\phi = \frac{3\sqrt{2\pi}}{2h^2}A_\phi = K_\phi^\pm, \quad K_\phi^\pm = \frac{3}{2h^2\sqrt{\pi l}}\int_{-l}^{l}(M^0(\xi) - M(\xi))\sqrt{\frac{l \pm \xi}{l \mp \xi}}\,\mathrm{d}\xi.$$

These energy release rate expressions are the same (in the present notation) as in the paper by Young and Sun (1992).

Note that, in this paper, the contact force and moment (S and M) will not be accurately determined in the vicinity of the crack tips. However, this shortcoming is of no concern as far as the determination of the energy release rate is concerned; the latter is defined, as usual by global considerations. Further, this shortcoming is of decreasing significance as the crack length versus plate thickness increases.

6. LIMITING CASES AND ASYMPTOTES

The nature of the problem under consideration depends, to a large extent, on the ratio $e_f^0 = M^0/S^0$. Different scenarios are outlined here that serve to identify the influence of e_f^0.

Consider the case of a long through-the-thickness crack subjected to a compressive in-plane force initially located such that $0 \leqslant |e_f^0| \leqslant h/3$. In this case, $S^0 \leqslant 0$ and there is complete closure of the through crack: $a \equiv 0$, $v = \phi = 0$, and $\sigma(x_1, x_3) \leqslant 0$ is negative for $-h \leqslant x_3 \leqslant h$. The equations in (10) and (22) are satisfied by $S = S^0$, $M = M^0$ and by the fact that the parameters $\alpha_{\lambda\mu} = 0$ (since $\zeta = 0$). For this range of e_f, an intact infinite layer would experience only compressive stresses through the depth. Alternatively, consider the plane problem of two semi-infinite layers (of thickness $2h$) laid end to end and subjected to smooth compressive contact by far-field forces S^0 whose line of action is restricted to vary between $-h/3 \leqslant x_3 \leqslant h/3$. In this case, the same complete closure would result and the above parameter values apply.

Consider next the case of either a long surface crack with $K = 0$ or a long through-the-thickness-crack under closure. The in-plane force and moment combination is such that while crack opening takes place $(a > 0)$, $h/3 < |e_f^0| < h$. As $l/a \to \infty$, the solution is simply that for the edge-cracked strip described above in (22) for $S = S^0$, $M = M^0$. In other words, while the quantities S, M, v and ϕ can be obtained and are functions of crack length $2l$, the limit as $l/a \to \infty$ exists. For long cracks and for $h/3 < |e_f^0| < h$, the solution is obtained by simply considering a plane problem (Fig. 4). This solution is valid everywhere on the crack line except small (as compared the crack length) vicinities of the crack tips, $x_1 = \pm l$.

Now consider the case $|e_f^0| \geqslant h$ which has no plane problem limit. This fact is readily apparent because the value of $|e_f| = |M/S|$ cannot be greater than unity. Clearly, given a zero stress intensity factor at $x_3 = a(x_1)$ and $e_f \leqslant -h$, the edge-cracked strip would not remain in equilibrium as $l/a \to \infty$. This leads directly to the conclusion that, in this coupled case, the interaction force S, and moment M cannot tend to the applied force S^0 and moment M^0, respectively, as the crack length $2l$ increases. However, from (15), it is evident that in-plane (averaged) crack opening displacement v and rotation ϕ increase under a constant force, S and moment M, as the crack length increases. Under this increase, (22) can only be satisfied by an associated increase in the coefficients, $\alpha_{\lambda\mu}$. The latter is possible only if the ratio, $|e_f/h|$ (generally less then unity) tends to unity; such behavior is crucial and is evident in (24) and Fig. 7.

Thus, as the crack length increases the ratio $|e_f/h|$ tends to unity; the larger the value of l, the less the difference $h-|e_f|$. This is true everywhere on the crack line $-l < x_1 < l$ except in the vicinity of the crack tips $(x_1 \sim \pm l)$—on a scale relative to the plate thickness.

Knowledge concerning the limiting behavior of e_f allows one to separate the plane contact problem (22) from the coupled plane-bending problem (10) (or (15)). First, note that the averaged crack opening displacement is zero at $x_3 = e_c$; that is,

$$\bar{v}(x_1, x_3) = v(x_1) - x_3\phi(x_1) = 0 \quad \text{for} \quad x_3 = e_c. \tag{35}$$

Clearly, $e_c = v/\phi$. To determine the limiting closure behavior for long cracks, the procedure is to first let $e_f = \pm h$, and hence, $e_c = \pm h$ in (10) [(15)], then substitute the crack opening displacement and rotation so obtained into (22) to find the difference, $h-|e_f|$. The closure behavior and associated stress distribution are determined in this manner below.

7. CLOSURE CONTACT PROBLEM

The through-the-thickness or surface crack in Fig. 4 seeks its own natural extent of contact; smooth crack-tip closure requires that

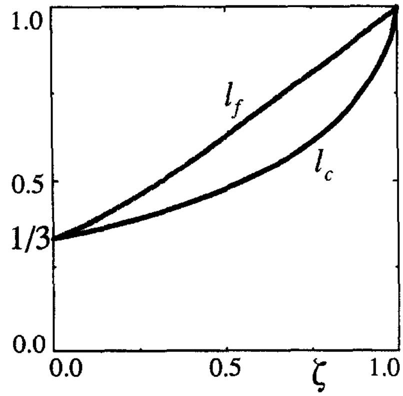

Fig. 8. Ratios l_f and l_c versus ζ.

$$K(a) = 0 \quad \text{or} \quad l_f = -\frac{e_f}{h} = \frac{1}{3}\frac{F_s(\zeta)}{F_m(\zeta)}. \tag{36}$$

In addition, given that $e_c = v/\phi$, it is evident from (22) and (24) that

$$l_c = -\frac{e_c}{h} = \frac{1}{3}\frac{\Lambda_{ss}(\zeta)-3l_f\Lambda_{sm}(\zeta)}{\Lambda_{ms}(\zeta)-3l_f\Lambda_{mm}(\zeta)}. \tag{37}$$

The value of ζ appearing in (37) is tied to the value of l_f appearing in the same equation through (36).

The parameters l_f and l_c are plotted versus ζ in Fig. 8.

8. CLOSURE FORCE AND MOMENT DISTRIBUTION

Consider now the case in which $e_c = \text{const}$; that is, $v(\xi)-e_c\phi(\xi) = 0$ and from (10)

$$S(x_1) = \frac{S^0(x_1)+\hat{v}e_cM^0(x_1)}{1+\hat{v}e_ce_f}, \quad M(x_1) = e_fS(x_1). \tag{38}$$

From this, neglecting any shortcomings in the crack tip regions, and assuming the crack segment under consideration to be subjected to closure, one can express the interaction force and moment as follows, given that $M = e_fS$, $e_f = -h$, $e_c = -h$,

$$S(x_1) = \frac{S^0(x_1)-\hat{v}hM^0(x_1)}{1+\hat{v}h^2} \tag{39}$$

By the result in (39), the formulation by Young and Sun (1992), (viz. $e_fe_c = h^2$), is justified for long cracks. At the same time, note that such a formulation is valid not only for the case $M^0 = \text{const}$, $S^0 = 0$, but for a general distribution of the in-plane force, S^0, and bending moment, M^0, along the crack line. However, the crack does have to be long, and the variation has to be slow on the scale of the plate thickness.

Now consider a more general problem in which a (multi-segment) portion is subjected to closure. In this case, the difference $v(\xi)-e_c\phi(\xi)$ with $e_c = \pm h$ is non-zero only in the open crack segments (called hereafter open crack area or OCA) because $v(\xi)-e_c\phi(\xi) = 0$ in the closure region. It is assumed that $e_c = \text{const}$ in the closure region; that is, either $e_c = +h$ or $e_c = -h$. This more general case may be described by

$$\frac{Eh}{\pi}\int_{\text{OCA}} \frac{(v-e_c\phi)}{(\xi-x_1)^2}\,\mathrm{d}\xi = \begin{cases} -S^0-\hat{v}e_cM^0, & x_1 \in \text{OCA} \\ -S^0-\hat{v}e_cM^0+(1+\hat{v}e_ce_f)S, & x_1 \notin \text{OCA} \end{cases} \tag{40}$$

where $e_c = e_f = \pm h$. In (40), the difference $v-e_c\phi$ plays the same role as the function $f(\xi)$ in (11) or the role of crack opening displacement as in plane crack theory. In this sense, the open segments may be viewed as cracks. Note that the difference $v-e_c\phi$ is zero in the closure segments and intact regions.

Now let there be a single open segment only; in this case, (40) can be solved using (13) assuming S^0 and M^0 to be known. The interaction force outside of the "crack" is thus given, for ($x_1 \notin$ OCA), by

$$S(x_1) = \frac{S^0(x)+\hat{v}e_cM^0}{1+\hat{v}e_ce_f} - \frac{\operatorname{sgn} x}{\pi\sqrt{x_1^2-l^2}}\int_{\text{OCA}} \frac{S^0+\hat{v}e_cM^0}{1+\hat{v}e_ce_f}\,\frac{\sqrt{l^2-\xi^2}}{\xi-x_1}\,\mathrm{d}\xi. \tag{41}$$

To solve (40) for several open crack segments, the equivalent of (12) for several collinear cracks must be used. Note that the transition points between the open and closed segments, excluding the "real" crack tips at $x_1 = \pm l$, must be associated with zero stress intensity factors at these points. This requirement serves to locate these points.

9. SYSTEM OF CRACKS

The main difference between the integral equation formulations summarized by (10) and by (15), is that the former can be used to model more general crack system due to the validity of the superposition principle with respect to the averaged crack opening displacement, v, and rotation, ϕ. The general representation for the integral equation kernels (2b), (2c) and (7) must be used. However, in the present paper, anti-symmetrical crack opening displacements caused by non-zero lateral forces (8) are not considered. Any system of cracks considered must, at this stage, be subjected to zero lateral force on the crack lines.

The solution in (38) and (39) for the closure force and moment distribution is independent of the crack tip coordinates and the evolution of the contact (closure) region. Therefore, this solution is valid for any collinear system of cracks with $e_f = h$ (or $e_f = -h$) over each crack (that is, each crack has to be long). Clearly, the limits $-l$, l in the integrals in (10) need only cover the extent of closure, similar to the considerations in (40) with respect to the OCA. Then, if e_c is constant on the closure segments, the closure force and moment are expressed by (39) because $v-e_c\phi = 0$. Thus, in this case of a collinear system of cracks under closure, and the unilateral contact conditions on these long crack surfaces, the interaction is *independent* of the

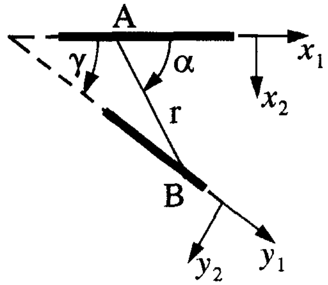

Fig. 9. A non-collinear crack system.

details of the actual crack lengths and spacing on this line. At the same time, the crack opening displacement and rotation cannot be expressed in terms of the right hand sides of (10) by the formula (12) as for a single crack. The integral representation of the crack opening displacement (and hence, for the rotation) can be formed for any number of cracks and such solutions are available, albeit considerably more complicated than (12). Thus, the crack opening displacement and rotation, in contrast to the closure force and moment, depend on the specifics of the collinear crack distribution.

The ability of the method put forward in this paper to treat non-collinear crack systems is now briefly examined. Consider two cracks located asymmetrically on different crack lines with an angle γ between the crack lines (Fig. 9). The crack opening displacement, $\delta(x_1)$ at point A on the first crack line with coordinate x_1, in accordance with the relations in (2) induces the force, S_{y_2}, at point B on the second crack line coordinate y_1 (Fig. 9) as follows

$$S_{y_2} = 2h\sigma_{y_2y_2} = \frac{Eh}{\pi r^2}\{\cos 2\alpha + \mathrm{Re}\,[(e^{2i\alpha} - e^{4i\alpha})e^{-2i\gamma}]\}. \tag{42}$$

In the same way, the moment M_{y_2} at point B caused by a δ-crack rotation at point A (see formulas (7)) is given by

$$M_{y_2} = -\frac{(1-\nu^2)D}{2\pi r^2}\left\{\cos 2\alpha + \mathrm{Re}\left[\left(e^{2i\alpha} + \frac{1-\nu}{1+\nu}e^{4i\alpha}\right)e^{-2i\gamma}\right]\right\}. \tag{43}$$

By superposition, these expressions lead to integral equations similar to those in (10) but with more complicated kernels.

10. CONTACT STRESS DISTRIBUTION

Consider the closure contact problem (Fig. 4) under the force, S, applied along the line of action $x_3 = e_f < 0$. The stress distribution can be expressed as

$$\sigma(x_1, x_3) = f\left(\frac{2h}{2h-a}, \eta\right)\frac{S}{2h-a}, \quad \eta = \frac{h+x_3}{2h-a}. \tag{44}$$

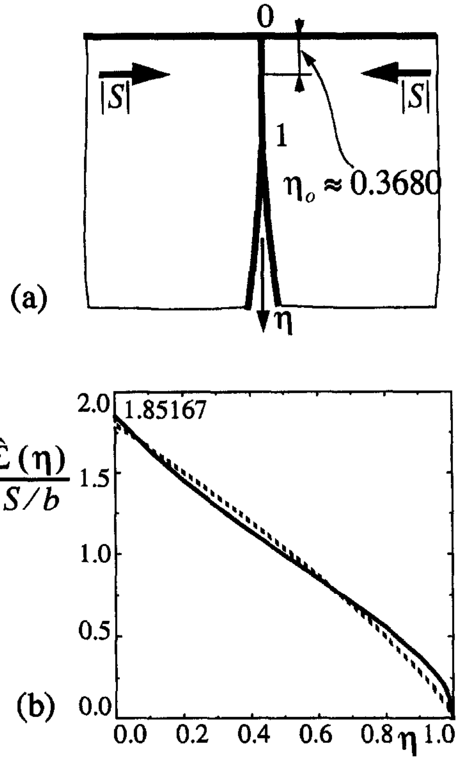

Fig. 10. (a) Two elastic quarter planes under closure. (b) Closure stress distribution for long cracks: solid curve—(46); dashed curve—(49).

In this qualitative expression, the crack depth a is defined by e_f. In terms of the η-coordinate, the extent of contact is invariable and remains equal to unity as $(x_3+h)/(2h-a)$ tends to infinity under the condition $a \to 2h$. This limit corresponds to the unilateral contact of two elastic quarter-planes contacting along a zone of contact of unit length $0 \leqslant \eta < 1$ (Fig. 10a) under the force $S/(2h-a)$ applied at infinity along the line of action $x_3 = e_f$. The asymptotic closure stress distribution is a function solely of η and directly proportional to S/b, where $b = 2h-a$. Such a contact problem is the same as the corresponding problem for an elastic half-plane with a semi-infinite crack perpendicular to the surface (with a zero stress intensity factor). Significantly, the desired closure contact stress is *unique* and valid in general under the condition $b \ll 2h$.

As $a \to 2h$, and as h becomes very large (as for two quarter-planes), the parameter $\zeta \to 1$. The "outer" problem's closure parameters l_f and l_c both tend to unity (Fig. 8). For the case of two contacting quarter-planes, the "inner" parameter, the difference between a and $2h$ has decreased to a finite constant value, here called the closure width, $l_{cl} = 2h-a$. The line of action of the far-field force S, denoted now by $\eta_0 l_{cl}$ (Fig. 10a), has a unique value if $K(a) = 0$. This value is readily obtained from the paper by Kipnis (1979) as

$$\eta_0 = \frac{\pi/\sqrt{\pi^2-4}}{2\sqrt{2}G^-(1)} = 0.3680338, \tag{45}$$

in which $G^-(1) = 1.245698$. Referring again to the situation depicted in Fig. 10a (the unilateral contact of two quarter-planes), note that η_0 defines the line of action of the compressive forces $|S|$, with zero applied moment, such that $K(a) = 0$.

A closed form solution to the closure stress distribution is available from the paper by Kipnis (1979), who studied the problem of a semi-infinite crack lying on the bisector of an elastic wedge of angle 2α subjected to a far-field load and moment. For the case of $\alpha = \pi/2$ the eigenvalue equation in (1.4) of Kipnis gives $\lambda = n$, $(n = 1, 2, \ldots)$. Setting $K = 0$ to satisfy the closure condition, the accurate stress distribution $\hat{\Sigma}$ is found to be given by

$$\hat{\Sigma}(\eta) = \frac{S}{b}\sqrt{\frac{2\pi}{\pi^2-4}}\left(\frac{8/\pi\sqrt{\pi}}{G^+(-1)} - \sum_{n=2}^{\infty} \frac{2n^2-1+\cos(n\pi)}{n\pi(n-1)\cos(n\pi)}\,\frac{\Gamma(1+n)}{\Gamma(\frac{1}{2}+n)}\,\frac{\eta^{n-1}}{G^+(-n)}\right), \tag{46}$$

in which

$$G^{\pm}(x) = \exp\left\{\pm\frac{|x|}{\pi}\int_0^{\infty}\frac{\ln g(\xi)}{\xi^2+x^2}\,\mathrm{d}\xi\right\}. \tag{47}$$

In (47),

$$g(x) = \frac{\cosh(2\pi x) - 2(1+2x^2)\cosh(\pi x) + 1}{\cosh(2\pi x) - 1}. \tag{48}$$

The function $g(0) = (\pi^2-4)/2\pi^2$, while $g(x)$ rapidly tends to unity for large x.

This closure stress is well approximated by the simple expression $\Sigma = \Sigma_0(1-\eta)^p$; for this distribution, the average value $(\Sigma_0/(p+1))$ acts at $\eta_0 = 1/(p+2)$. From (45), $p = 0.7171$ and then

$$\Sigma(\eta) \approx 1.7171(S/b)(1-\eta)^{0.7171}, \quad \sigma_{\max} \approx 1.85S/b. \tag{49}$$

The maximum contact stress, which is especially important with regard to crushing considerations, is defined as well.

A general simple approximation for the closure stress distribution, applicable for arbitrary values of b, is now chosen here:

$$\Sigma(\eta) = 2(S/b)(1-\eta). \tag{50}$$

Such a distribution is exact for $b = 2h-0$. This linear approximation for the closure stress distribution corresponds with the linear approximation of the function l_f: $l_f = -e_f/h \sim (1+2\zeta)/3$.

The accurate closure stress distribution in (46) and the approximate expression in

Table 1. *Closure stress distribution* (*Fig.* 10*b*)

η	$b\hat{\Sigma}(\eta)/S$	η	$b\hat{\Sigma}(\eta)/S$	η	$b\hat{\Sigma}(\eta)/S$
0.00	1.85167	0.10	1.64177	0.20	1.45709
0.30	1.29061	0.40	1.13697	0.50	0.99169
0.60	0.85049	0.70	0.70837	0.80	0.55774
0.90	0.38118	0.95	0.26520	1.00	0.00000

(49) are plotted in Fig. 10b. Values for the accurate curve are provided in Table 1. Note that the accurate value for the maximum stress is only slightly modified from the approximate value in $(49)_2$: $\sigma_{\max} \approx 1.85S/b$.

An alternative deduction of the value of η_0 is as follows. First, the behavior of e_f as $a \to 2h$ needs to be determined. Remembering that $l_f = -e_f/h$, note that for $\zeta = 1-\varepsilon$ and $\varepsilon \to 0$

$$l_f \sim 1-k_f\varepsilon, \quad k_f = \frac{\mathrm{d}l_f}{\mathrm{d}\zeta}(1). \tag{51}$$

Recalling that $l_f(1) = 1$, k_f can be defined as $l_f'(1)/l_f(1)$, or, equivalently, as

$$k_f = \hat{F}_s(1) - \hat{F}_m(1), \quad \hat{F}_\mu = \frac{1}{F_\mu(1)}\frac{\mathrm{d}F_\mu}{\mathrm{d}\zeta}(1). \tag{52}$$

From (51), $h+e_f \sim (1/2)k_f b$. Given that $\eta_0 \equiv (h+e_f)/b$, it quickly follows that $\eta_0 = k_f/2$. However, the curve fit expressions in (19), along with the data in (20) and (21), give an inaccurate value of 0.3824 for η_0. A more precise curve fitting of the original data from Kaya and Erdogan (1987) using Tablecurve (an automated non-linear curve fitting program that uses a 64-bit Levenburg–Marquardt algorithm) was attempted. The best-fit was obtained using a rational function of two fourth order polynomials and gave a value of 0.3687. The interesting point here is that an extra constraint on the functions $F_s(\zeta)$ and $F_m(\zeta)$ is now available via (52); that is,

$$k_f = \hat{F}_s(1) - \hat{F}_m(1) = 2\eta_0 = 0.7360675. \tag{53}$$

The values listed in (20) and (21) are quite adequate unless rather special asymptotic considerations are involved, as above.

11. ASYMPTOTIC CLOSURE WIDTHS

The asymptotic equalities, $e_f = e_c = \pm h$, developed to describe the coupled plane-bending cracked-plate problem, are suitable for the determination of the closure force and moment, but are not suitable for the determination of the stress distribution in the contact area. With this goal in mind, it is first necessary to find the asymptote for the contact strip width $2h-a$, where a is the crack depth portrayed in Fig. 4.

This asymptotic result can be found from an asymptotic representation of the coefficients $\alpha_{\lambda\mu}$ in (22). If $\zeta = 1-\varepsilon$ and $\varepsilon \to 0$

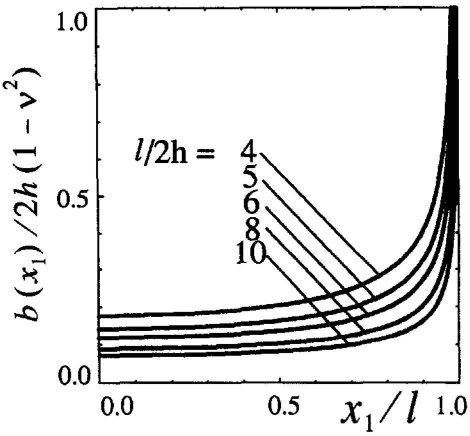

Fig. 11. Normalized closure width $b(x_1)$ versus location along the crack length.

$$\alpha_{\lambda\mu} \sim \frac{\pi}{(1-\zeta)^2}\Lambda_{\lambda\mu}(1)(1-2\varepsilon-k_{\lambda\mu}\varepsilon), \quad k_{\lambda\mu} = \frac{1}{\Lambda_{\lambda\mu}(1)}\frac{\mathrm{d}\Lambda_{\lambda\mu}}{\mathrm{d}\zeta}(1). \tag{54}$$

Further, note (52) and the following identities:

$$\Lambda_{\lambda\mu}(1) = F_\mu(1)F_\lambda(1)/2, \quad k_{\lambda\mu} = 2\hat{F}_\mu+2\hat{F}_\lambda. \tag{55}$$

Directly from (55),

$$2k_f = k_{ss}-k_{sm} = k_{ms}-k_{mm} \quad \text{and} \quad k_{ss}+k_{mm} = 2k_{sm}. \tag{56}$$

Using these asymptotic equalities, and (22)

$$v(x_1) \sim -\frac{\pi\Lambda_0 k_f}{E'(1-\zeta)}S(x_1), \quad \phi(x_1) \sim \frac{\pi\Lambda_0 k_f}{E'h(1-\zeta)}S(x_1). \tag{57}$$

Thus, the required contact width, $b(x_1)$, can be asymptotically expressed as

$$b(x_1) = 2h-a = 2h(1-\zeta) = -\frac{\pi\Lambda_0 k_f}{E'}\frac{2hS(x_1)}{v(x_1)}. \tag{58}$$

In this formula, the force S is defined by (39), and the displacement v can be found by $(15)_1$ for a single crack and a collinear crack system. Note that this asymptotic formula is valid in contact region(s) where $b/2h \ll 1$, and that it does not hold in the vicinity of the crack tips ($x_1 \sim \pm l$), where $v(x_1) \to 0$.

The contact width function b in (58) is shown in Fig. 11 for various values of the half-crack length to plate thickness ratio $l/2h$ ($v = 0.3$). The latter ratio is chosen to facilitate comparison with Fig. 11 in the paper by Joseph and Erdogan (1989). Figure 11 is for the case of zero initial force and a uniform moment distribution: $S^0 \equiv 0$, $M^0 = M^0_*$, where $M^0_* \neq 0$ is a constant. From (38), it is apparent in this case that $S = S_*$ is also a constant. Since S_* also appears in the expression for $v = v_*$, in this particular instance, the closure force does not actually appear in the contact width expression $b = b_*$. Noting that v_* is given by $(15)_1$ and that b_* is given by (58), it is found that

$$v_*(x_1) = -\frac{S_*}{Eh}\sqrt{l^2 - x_1^2}, \quad \frac{b_*}{2h} = \frac{\pi}{2}\frac{(1-v^2)\Lambda_0 k_f}{\sqrt{1-(x_1/l)^2}}\frac{2h}{l}. \tag{59}$$

As has already been discussed, the asymptotic expressions in this paper are expected to lose validity in the vicinity of the crack tips $x_1 \sim \pm l$. This remark notwithstanding, the agreement between the analytical result in (58) that is plotted in Fig. 11 above and the numerical calculations plotted in Fig. 11 in the paper by Joseph and Erdogan (1989) is very good, especially for larger values of $l/2h$. Preliminary numerical calculations have revealed that the agreement is remarkable for $l/2h \geqslant 8$.

Using the same asymptotic representations, (54) and (56), the second formula in (22) reveals that the crack rotation asymptote behaves as

$$\phi \sim -\frac{v}{h}. \tag{60}$$

This result corresponds to straight crack surfaces which are normal to the plate neutral axis and plate surfaces over almost the entire plate thickness. That is, in the asymptotic sense, the elastic layer considered in Fig. 4 behaves as a Kirchhoff–Poisson plate.

12. FUTURE CONSIDERATIONS

Crack closure is a major influence in terms of the compliance of cracked plates under bending, the energy release rate during crack propagation, and the different types of (complex) crack systems formed under different loading conditions. As a result, crack closure is a key ingredient in applications involving the quasi-static and dynamic forcing of cracked plates. The asymptotic distributions of the closure contact width and the closure stresses are applicable to even rather short cracks of rather general shape as well as to general systems of cracks. In the case of non-collinear systems of cracks, only the solution to the integral equations with respect to the contact forces and moments poses a problem. The closure contact width and the closure stress distribution, however, are defined in closed form in terms of these forces, moments, and crack opening displacements. In the dynamics of a plate subjected to crack closure, in particular, the mechanics of wave propagation and reflection have yet to be considered. In this topic, a combined formulation may be suitable: (i) quasi-static considerations that apply in the context of the closure contact (elasticity) problem given that the time to establish the self-equilibrated closure stresses is found to be short enough, plus (ii) a dynamic formulation for the global cracked-plate problem; the contact width and the contact stress distribution would depend on time as a parameter. For supported cracked plates, such as an ice sheet or pavements, the global, plate–fluid or plate–soil problem can be separated, in the above sense, from the closure contact problem the same as in the dynamic problem discussed. The closure contact width and closure stress distributions remain valid for this topic as well. Finally, the fracture mechanics presented in this paper can be used to analyze the stability of compression loaded symmetrically cracked configurations undergoing crack closure.

13. CONCLUSIONS

Four main conclusions follow from the analysis in this paper (valid both for any through-the-thickness crack, including a curvilinear crack and for more general crack systems):

1. Consideration of the crack closure problem as being one of line contact along the topmost edge is suitable for the determination of the closure force and moment distribution but not for the determination of the contact width and the contact stresses.
2. The asymptotic distribution of the closure forces and moments has been determined in terms of the initial force and moment. This result is applicable to not only a single crack but also a collinear system of cracks.
3. The asymptotic distribution of the extent of contact under closure has been determined in terms of the closure force and moment and the (far-field) Poisson–Kirchhoff plate averaged crack opening displacements.
4. The asymptotic closure stress distribution is unique and universal; the generality of the local plane contact conditions forms the basis for this universality.

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APPENDIX: FUNDAMENTAL SOLUTIONS

In this appendix, the fundamental solutions stated in (2) and (4) which reproduce the conditions stated in (1) and (3), respectively, are derived. In the following, the referenced x, y rectangular coordinate system is shown in Fig. 5; u_x and u_y denote the displacements in the x- and y-directions, respectively.

To derive the plane fundamental solution for the conditions stated in (1), consider an infinite elastic plane subjected to the concentrated force, $P = X^0 + iY^0$, which is applied at the origin of the coordinate system. The Kolosov–Muskhelishvili complex representation (Muskhelishvili, 1977) is used; the displacements and stresses are expressed in terms of two analytical functions, $\phi(z)$ and $\psi(z)$ of the complex variable, $z = x + iy$

$$\begin{aligned} 2\mu(u_x + iu_y) &= \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2[\bar{z}\phi''(z) + \psi'(z)], \\ \sigma_{xx} + \sigma_{yy} &= 4\,\mathrm{Re}\,\phi'(z), \end{aligned} \tag{61}$$

in which μ denotes the shear modulus, $\kappa = (3-\nu)/(1+\nu)$, while ν denotes the Poisson's ratio. The solution to this problem is well known (Muskhelishvili, 1977; §56a.4°):

$$\phi = -\frac{X+iY}{2\pi(1+\kappa)}\ln z, \quad \psi = \frac{\kappa(X-iY)}{2\pi(1+\kappa)}\ln z, \tag{62}$$

in which $X = X^0/2h$ and $Y = Y^0/2h$. Before constructing the required solution, note that, for $(n = 0, 1, \ldots)$ and $(y \to \pm 0)$,

$$\mathrm{Re}\,z^{-n-1} \to x^{-n-1}, n!\,\mathrm{Im}\,x^{-n-1} \to \pm(-1)^{n+1}\pi\delta^{(n)}(x)$$

$$x^m\delta^{(n)}(x) = \frac{(-1)^m n!}{(n-m)!}\delta^{(n-m)}(x)(m \leqslant n), x^m\delta^{(n)}(x) = 0 \quad (m > n). \tag{63}$$

Therefore, as $y \to \pm 0$,

$$\begin{aligned} &\mathrm{Re}\,\bar{z}z^{-2} \to x^{-1}, \quad \mathrm{Im}\,\bar{z}z^{-2} \to \mp\pi\delta(x), \\ &\mathrm{Re}\,\bar{z}z^{-3} \to x^{-2}, \quad \mathrm{Im}\,\bar{z}z^{-3} \to \pm\pi\delta'(x). \end{aligned} \tag{64}$$

The solution is constructed by supposing that the following forces are applied near the origin:

parallel to the x-axis, $\pm P_x/2a$ at $(x = \pm a, 0)$;

parallel to the y-axis, $\pm P_y/2a$ at $(0, y = \pm a)$.

In the limit as $a \to 0$, the actual loading imposed is succinctly expressed in the form

$$P \to -P_x\delta'(x)\delta(y) - iP_y\delta(x)\delta'(y). \tag{65}$$

From (62)

$$\phi = \frac{P_x - P_y}{2\pi(\kappa+1)}\frac{1}{z}, \quad \psi = -\kappa\frac{P_x + P_y}{2\pi(\kappa+1)}\frac{1}{z}. \tag{66}$$

Now from (61), (63) and (64), the generalized limits of (66) as $y \to \pm 0$ are

$$u_y \to \pm\frac{P_x + (2\kappa-1)P_y}{4\mu(\kappa+1)}\delta(x), \quad \sigma_{xy} \to \pm\frac{(2+\kappa)P_x - (2-\kappa)P_y}{2(\kappa+1)}\delta'(x). \tag{67}$$

The required conditions are such that the coefficients of $\pm\delta(x)$ and $\pm\delta'(x)$ in (67) should be unity and zero, respectively. Therefore,

$$P_x = 2\mu(2-\kappa)/\kappa, \quad P_y = 2\mu(2+\kappa)/\kappa. \tag{68}$$

In this case

$$u_x + iu_y = -\frac{1}{\pi(1+\kappa)}\left(\frac{\kappa}{z} - \frac{2}{\bar{z}} + \frac{z}{\bar{z}^2}\right) \to -\frac{\kappa-1}{\pi(1+\kappa)}x^{-1} \pm i\delta(x),$$

$$\sigma_{xx} + \sigma_{yy} = \frac{8\mu}{\pi(1+\kappa)}\mathrm{Re}\left(\frac{1}{z^2}\right) \to 2\sigma_{xx} \to \frac{8\mu}{\pi(1+\kappa)}x^{-2},$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = \frac{8\mu}{\pi(1+\kappa)}\left(\frac{1}{z^2} - \frac{\bar{z}}{z^3}\right) \to 0. \tag{69}$$

The fundamental solution for the conditions stated in (3) can be constructed from a combination of second order derivatives of the solution to a free plate subjected to a concentrated lateral force P. The latter can be written as any function which satisfies the homogeneous equation of the plate bending, $\Delta^2 u_z = 0$ $(z \neq 0)$, subject to the "distributed" loading, $q(x, y) = P\delta(x)\delta(y)$. Such a solution is

$$u_z = \frac{P}{8\pi D}r^2 \ln r. \tag{70}$$

The required derivatives correspond to similar limits as for the plane problem but in this case for generalized moments. They may be introduced by means of three forces on the x-axis and the three forces on the y-axis:

$$\frac{P_x}{a^2}\delta(x+a)\delta(y) - 2\frac{P_x}{a^2}\delta(x)\delta(y) + \frac{P_x}{a^2}\delta(x-a)\delta(y),$$

$$\frac{P_y}{a^2}\delta(y+a)\delta(x) - 2\frac{P_y}{a^2}\delta(y)\delta(x) + \frac{P_y}{a^2}\delta(y-a)\delta(x). \tag{71}$$

In this case, the generalized limit, $a \to 0$ gives

$$P \to P_x\delta''(x)\delta(y) + P_y\delta(x)\delta''(y). \tag{72}$$

If the same linear operator as used in (72) is applied to the function in (70) and the resulting behavior is examined as $y \to \pm 0$ to construct the desired limiting behavior defined in (3) it is readily apparent that $P_x = 2\nu D$ and $P_y = 2D$ and that the fundamental solution is as stated in (4).