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ORIGINAL ARTICLE

Crack dynamics in a nonlinear lattice

L.I. Slepyan · M.V. Ayzenberg–Stepanenko

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Abstract A discrete two-dimensional square-cell lattice with a steady propagating crack is considered. The lattice particles are connected by massless bonds, which obey a piecewise-linear double-humped stress–strain relation. Initially, Hooke's law is valid as the first stable branch of the force–elongation diagram; then, as the elongation becomes critical, the transition to the other branch occurs. Further, when the strain reaches the next critical value, the bond breaks. This transition is assumed to occur only in a line of the breaking bonds; the bonds outside the crack line are assumed to be in the initial branch all the time. The formulation relates to the crack propagation with a 'damage zone' in front of the crack. An analytical solution is presented that allows to determine the crack speed as a function of the far-field energy release rate, to find the total speed-dependent dissipation, and to estimate the role of the damage zone. The analytical formulation and the solution present a development of the previous ones for the crack and localized phase transition dynamics in linear and bistable-bond lattices.

Keywords Dynamics · Fracture · Nonlinear-bond lattice · Integral transforms

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1 Introduction

The classical continuum model of the material can be considered as the slowly-varying approximation of a discrete or structured material. This accuracy is sufficient for the analysis of regular processes in which waves corresponding to the microstructural scales can be neglected. However, in fracture, the energy release through the propagating crack tip imposes no lower limit on the wave length, and the asymptotic approximation of this kind is not sufficient. Under the microstructural influence a part of the macrolevel energy release is spent on the excitation of the microlevel and this phenomenon cannot be observed within the framework of the homogeneous model. The most important fact is that the dissipation due to the energy transfer to the microlevel is crack speed- and structure-dependent. This is why the use of structured models is important for fracture dynamics. Analytical studies of discrete lattice models allow, in particular, to clarify the connections between the classical (macrolevel) description of fracture and the microlevel behavior. While the solution describes

the complete pattern of the lattice dynamics, its long-wave (low-frequency) asymptote describes the macrolevel dynamics. Comparing these solutions can determine the total dissipation and the speed-dependent local-to-global energy release ratio.

Fracture based on a linear homogeneous lattice model was first analytically considered in Slepyan (1981). References to the following works and main results on this topic can be found in Slepyan (2002). Also see Marder and Gross (1995) and Marder (2004). The localized crack-like transition wave in homogeneous bistable-bond lattices, as a more general problem containing the crack problem as a particular case, was considered in Slepyan and Ayzenberg-Stepanenko (2004). The well-known Frenkel–Kontorova model (Frenkel and Kontorova 1948; Braun and Kivshar 1998) relates to this theme as well. Recently, the lattice-net with bonds of nonzero density was considered in Slepyan (2005). Main concepts of the fracture process zone and the corresponding references can be found in Broberg (1999); we also note Botsis and Chudnovsky (1987).

In addition to the high-frequency radiation, in the lattice works some other phenomena were revealed, in particular:

(a) Possibilities of *supersonic crack propagation* in a lattice where the crack can take energy from the initially stressed neighboring layers or from a wave (Slepyan 2002).
(b) The role of a *structure-associated dynamic amplification factor*, which appears to be a governing phenomenon in the rate of fracture. In a viscoelastic lattice, cracks can grow slowly. This is valid if the relaxation and creep times belong to a static-amplitude-response domain where the dynamic factor does not manifest itself (Slepyan 2000, 2002). Otherwise, in particular in an elastic lattice, cracks cannot grow slowly (Marder and Gross 1995).
(c) The *instability of a fast straight-line crack* (Marder and Gross 1995) and some irregularities in mode II crack growth (Slepyan and Ayzenberg-Stepanenko 2002).
(d) A *size effect in fracture* as a strong influence of the cell size is revealed in the viscoelastic lattice model (Slepyan et al. 1999).

The present paper is based on a presentation at ICF11 (Turin 2005). We show that analytical methods remain applicable to a more sophisticated case of the stress–strain relation provided that the nonlinearity manifests itself only in the transition line—between the neighboring rows of the lattice particles.

The analytical approach is suitable for two-dimensional, periodic on x lattices of a general view; however, the following conditions are assumed to be satisfied. The lattice knots in a line parallel to the x-axis, the line $n = 0$, are connected with those in the neighboring line $n = -1$ by identical, physically nonlinear *transition-line bonds*. The force–elongation diagram is presented in Fig. 1.

The lattice half-planes, $n \geq 0$ and $n \leq -1$, are connected only by these bonds, that is, the local interaction between the half-planes is assumed. The connections between the knots in the half-planes can be, in principle, any, but each half-plane is assumed to represent a linear system. The latter condition can also follow from the assumption that relatively large strains may occur in the transition-line bonds only. For simplicity it is assumed that the tensile force, T, in the transition-line bond depends on the elongation, q, only. With the goal to derive an analytical solution, the dependences, $T(q)$, shown in Fig. 1 are accepted. These dependences are characterized by two branches, each of a constant tangent modulus. If the ratio q_{**}/q_* is

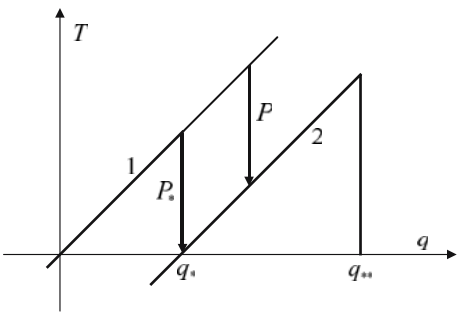

Fig. 1 The piecewise linear force–elongation diagram. 1 is the first, initial branch, $T = q$. 2 is the second branch, $T = q - P_*$; it comes in force at $t = t_*$ when the elongation reaches the critical value, $q = q_*$. The final break of the bond occurs at $q = q_{**}$ ($t = t_{**}$): $T = 0$ for $t > t_{**}$. The case of a free crack corresponds to $q_{**} = q_*$, while there is a line 'damage zone' if $q_{**} > q_*$. The vertical distance between the branches is denoted by P; $P = P_*$ for $t_* < t < t_{**}$ and $P = q$ for $t > t_{**}$

large enough, such that $q < q_{**}$ all the time, and also in some other cases, a closed analytical solution can be obtained. In the case of a diagram with a general path from the first branch to the other, the problem can be reduced to an integral equation. The infinite two-dimensional lattice or a lattice strip is assumed to be under self-equilibrated loads.

Specifically, in the present paper, a square-cell periodic lattice is considered. The steady-state dynamic problem is formulated for a crack propagating at a given speed, v, along the x-axis. In this case, the transition-line bond elongation is $q = q(\eta)$, where $\eta = x - vt$. It is assumed that the transition-line bonds are in the initial phase for $q < q_*$; the first transition occurs at $q = q_*$, while the final break takes place at $q = q_{**} > q_*$. The solution defines the crack speed as a function of the far-field (global) energy release rate. It also defines the total speed-dependent dissipation caused by the structure-associated radiation. Finally, it elucidates the role of the second branch of the diagram.

Note that the square-cell lattice is intended to model anti-plane shear; however, a hypothetical plane deformation with only vertical displacements—with the same formulation and results—can also be assumed. We use the latter viewpoint when it is more convenient; for example, when displacements and forces are to be shown in a plane figure.

2 Formulation

2.1 Equations

Consider a square-cell lattice, Fig. 2, whose bonds between the lines $n = 0$ and $n = -1$ obey the piecewise-linear double-humped stress-strain relation presented in Fig. 1, while the other bonds follow the first linear branch of the diagram. The dynamic equation of the intact square-cell lattice is

$$M\frac{\mathrm{d}^2 u_{m,n}}{\mathrm{d}t^2} = \mu_0\big(u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n}\big), \tag{1}$$

where $u_{m,n}$ are displacements, m and n are horizontal and vertical numbers of particles, respectively (their coordinates are $x = ma$, $y = na$; a is the cell size), and μ_0 is the bond stiffness. This equation is

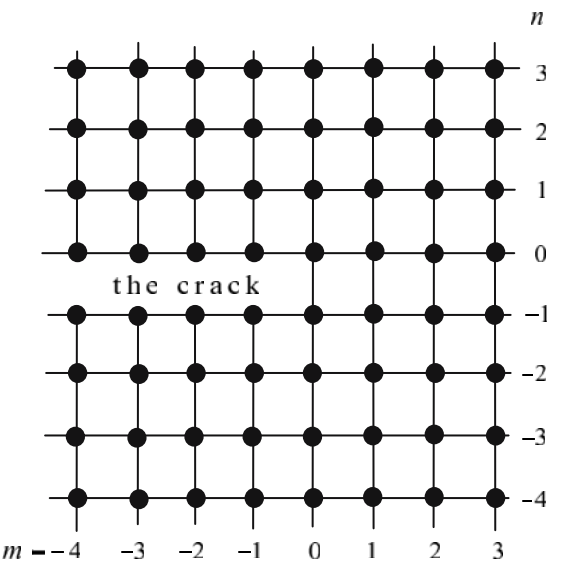

Fig. 2 The square-cell lattice

valid outside the crack, that is, for the left-hand side with $n > 0$ and $n < -1$. The symmetry condition is

$$u_{m,-n-1} = -u_{m,n}\,. \tag{2}$$

Under this condition, the equation for the particles with $n = 0$ is

$$M\frac{\mathrm{d}^2 u_{m,0}}{\mathrm{d}t^2} = \mu_0(u_{m+1,0} + u_{m-1,0} + u_{m,1} - 3u_{m,0}) - 2\mu u_{m,0}\,, \tag{3}$$

where μ is not a constant; it follows from the stress–strain diagram, Fig. 1. For simplicity we here consider the diagram with parallel branches

$$\mu = \mu_0 \quad (t < t_*), \qquad \mu = \mu_0 - P_* \quad (t_* < t < t_{**})$$
$$\mu = 0 \quad (t > t_{**})\,, \tag{4}$$

where $t = t_*$ is the moment when the strain of the considered bond, $q_m = 2u_{m,0}$, reaches the first critical value, $q_m = q_*$; $t = t_{**}$ is the the moment when the strain reaches the second critical value, $q_m = q_{**}$; the bond breaks at this moment.

In the following we use M, μ_0, and a as the natural units of mass, stiffness, and size, respectively. Further, we consider a steady-state regime

$$u_{m,n} = u_{m,n}(\eta)\,, \quad q_m = q(\eta)\,, \quad \eta = m - vt\,, \tag{5}$$

where $v = \mathrm{const}$ is the nondimensional crack speed (the ratio of the crack speed to the long-wave speed $c = a\sqrt{\mu/M}$). In this formulation, the crack is assumed to be at the left, $\eta < 0$, and $q(0) = q_{**}$.

For our goal it is convenient to reformulate this nonlinear problem as follows. Consider the linear lattice all the bonds of which are in the initial phase for any q. Introduce self-equilibrated pairs of external forces, $P(\eta)$, applied to the particles connected by the transition-line bonds. The forces are directed along the corresponding bond; they must compensate the difference in the tensile forces between the real and linear dependences. The intact bonds together with the external forces act on the transition-line particles, $n = 0$ and $n = -1$, in the same way as if the bonds followed the given nonlinearity. Hence, such a reformulation does not influence the lattice dynamic behavior. These forces are

$$P(\eta) = 0 \quad (\eta > l)\,, \quad P(\eta) = P_* \quad (0 < \eta < l)\,,$$
$$P(\eta) = q \quad (\eta < 0)\,, \tag{6}$$

where the point $\eta = l$ is defined by the condition $q(l) = q_*$.

The elongation caused by these forces is expressed in terms of the corresponding fundamental solution, $Q(\eta)$, which reflects the structure of the intact lattice as a whole. The fundamental solution properties and their connections with the lattice structure are of the most interest. Below the main points in deriving analytical solutions are discussed. Nondimensional variables are used.

2.2 Causality principle for steady-state solution

A steady-state solution is not unique if a homogeneous solution exists related to a free wave. Uniqueness can be achieved in various ways, in particular, by the use of a rule based on the causality principle. Under this principle, the solution is considered as the limit of the solution to the corresponding transient problem with zero initial conditions. In terms of the Fourier transform on η with parameter k, this consideration results in the following rule. The transformed steady-state solution can be presented as a function of two variables, k and ikv, and the latter must be treated as the limit: $0 + ikv \equiv \lim(s + ikv)$ $(s \to +0)$, where the parameter s reflects the Laplace transform on t (the transient solution is considered as a function of two independent variables: t and η). Details can be found in Slepyan (2002).

From a physical point of view, the causality principle as stated in the above *narrow sense* says that the solution should not contain waves carrying energy from infinity (nor waves exponentially growing to infinity). This is also called the Mandelshtam principle (see Bolotovsky and Stolyarov 1972). It corresponds to the case where no energy source at infinity is assumed. In a *broad sense*, the causality principle permits all waves to occur whose sources are prescribed by the problem formulation. In particular, some remote sources may be assumed to exist at infinity and the corresponding waves carrying energy from infinity can be present in the solution. These latter waves do not obey the above-mentioned rule. Note that such waves always appear in fracture if the energy flux to the propagating crack tip is caused by remote forces. In this case, the rule serves to guide the derivation of the other part of the solution.

3 Solution

3.1 Governing equation

Using the Fourier transform on η we find

$$\begin{aligned}&(h(k)+2)u_n^F(k) - u_{n+1}^F(k) - u_{n-1}^F(k)\\ &\quad = 0 \quad (n > 0)\,,\\ &(h(k)+3)u_n^F(k) - u_{n+1}^F(k)\\ &\quad = P^F(k) = q_-(k) + P_*\frac{\exp(ikl)-1}{ik} \quad (n = 0)\end{aligned} \tag{7}$$

with

$$\begin{aligned}&q^F = 2u_0^F = q_+ + q_-, \quad h(k) = 2(1-\cos k)\\ &\qquad +(0+ikv)^2, \quad r(k) = h(k) + 4,\\ &q_+ = \int_0^\infty q(\eta)\exp(ik\eta)\,\mathrm{d}\eta,\\ &q_- = \int_{-\infty}^0 q(\eta)\exp(ik\eta)\,\mathrm{d}\eta\,.\end{aligned} \tag{8}$$

From this and the condition $u_n^F \to 0$ $[n \to \infty, h \neq 0, h \neq -4)]$ the governing equation follows as

$$\begin{aligned}&L(k)q_+ + q_- = \frac{P_*}{ik}[L(k)-1]\left[\exp(ikl)-1\right]\,,\\ &L(k) = \sqrt{r(k)/h(k)}\,.\end{aligned} \tag{9}$$

3.2 Factorization

With the purpose to use the Wiener–Hopf technique to resolve Eq. 9 consider the functions $h(k)$ and $r(k)$. For $0.31584619 \approx v_* < v < 1$ there is only one positive root of $h = 0$: $k = h_1(v) + i0$, and one positive root of $r = 0$: $k = r_1(v) + i0$. Just this range of v is considered [the lower speeds are usually unrealizable (Marder and Gross 1995)]. These functions also contain infinite sets of complex zeros placed symmetrically relative to the real and imaginary axes, and the function r, in addition, has two purely imaginary zeros: $k = \pm i r_2(v)$. For the following it is useful to represent (similarly to Slepyan 2002, p399)

$$\begin{aligned}
&\frac{r}{h} = \frac{R_1 R_2}{H_1} S^2(k) \quad \left(S(k) = \sqrt{\frac{rH_1}{hR_1R_2}} \right), \\
&H_1(k) = H_{1-}(k) = 1 + \left(\frac{h_1}{0 + ik} \right)^2 , \\
&R_1(k) = R_{1-}(k) = 1 + \left(\frac{r_1}{0 + ik} \right)^2 , \qquad (10) \\
&R_2(k) = R_{2+}(k) R_{2-}(k) , \\
&R_{2+}(k) = 1 + \frac{r_2}{0 - ik} , \qquad R_{2-}(k) = 1 + \frac{r_2}{0 + ik} .
\end{aligned}$$

The function $S(k) = S(-k)$ is positive and finite for any real k, and $S(k) \to 1$ for $k \to \infty$.

The required factorization is now given as

$$\begin{aligned}
&L(k) = L_+(k) L_-(k) , \\
&L_+(k) = \sqrt{R_{2+}(k)}\, S_+(k) , \\
&L_-(k) = \sqrt{\frac{R_1(k) R_{2-}(k)}{H_1(k)}}\, S_-(k) , \\
&S_\pm(k) = \exp\left[\pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln S(\xi)}{\xi - k}\, \mathrm{d}\xi \right] , \\
&\quad (\pm \Im k > 0) . \qquad (11)
\end{aligned}$$

For real k

$$\begin{aligned}
&S_\pm = \sqrt{S(k)}\, \exp[\mp i \gamma(k)], \\
&\gamma(k) = \frac{1}{\pi} V.p. \int_0^{\infty} \frac{\ln S(\xi k)}{\xi^2 - 1}\, \mathrm{d}\xi . \qquad (12)
\end{aligned}$$

Note that

$$\gamma(0) = \gamma(\pm\infty) = 0 . \qquad (13)$$

Using the above factorization we can present the governing equation in the form

$$\begin{aligned}
&L_+(k) q_+ + \frac{q_-}{L_-} = Q(k) , \qquad (14) \\
&Q(k) = \frac{P_*}{ik} \left[L_+(k) - \frac{1}{L_-(k)} \right] \left[\exp(ikl) - 1 \right] .
\end{aligned}$$

The problem is now to divide the right-hand side of this equation into two terms, one marked by $+$ and the other by $-$, that is, the terms analytical in the upper and the lower half-planes of k (including the real axis), respectively.

3.3 Simplification

In fact, in the considered range of the crack speed, the factorization of the function $S(k)$, Eq. 11, can be avoided. A simplified presentation appears if we put in Eq. 10

$$r_2 = \alpha = \frac{2h_1/r_1}{\sqrt{1 - v^2}} . \qquad (15)$$

It results in $S(0) = 1$ and in a small difference $S(k) - 1$ and small derivative $\mathrm{d}S(k)/\mathrm{d}k$ (at least for $v \geq 0.5$ – see Fig. 3). So, in the determination of functions for real k we keep equality (15) below and put (for real k): $S_+ = S_- = S \equiv 1$.

3.4 Division of the right-hand side

Note that multiplication of a Fourier transformed function by e^{ikl} corresponds to the shift of the

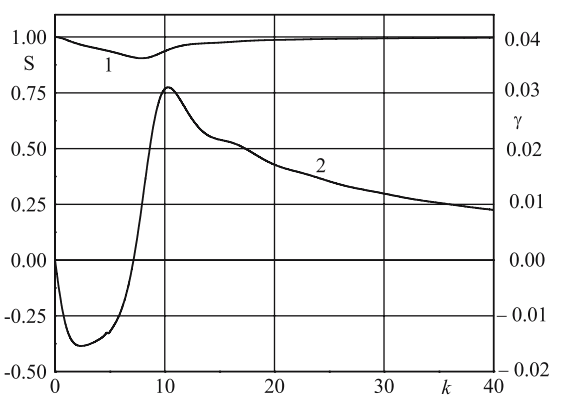

Fig. 3 Functions $S(k)$ (*curve 1*) and $\gamma(k)$ (*curve 2*) for $r_2 = \alpha$ and $v = 0.5$

original function to the right by l. So, the product

$$\frac{L_+(k)}{ik}\left[\exp(ikl)-1\right]=\frac{L_+(k)}{0-ik}\left[1-\exp(ikl)\right]$$

can be marked by the subscript + since the support of the original function is $0<l<\eta<\infty$. As to the term

$$\frac{1}{ikL_-(k)}\left[\exp(ikl)-1\right]$$
$$=\frac{1}{(0+ik)L_-(k)}\left[\exp(ikl)-1\right]$$

the corresponding support is $-\infty<\eta<l$, and hence it contains both the 'minus' and the 'plus' functions. The latter (with the support of its original $0<\eta<l$) can be determined as

$$B_+(k)=\int_0^l\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(0+i\xi)L_-(\xi)}\times\exp[i\xi(l-\eta)\,\mathrm{d}\xi\right]\mathrm{e}^{ik\eta}\,\mathrm{d}\eta$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(\xi-i0)(\xi-k)L_-(\xi)}\times\left[\mathrm{e}^{ikl}-\mathrm{e}^{i\xi l}\right]\mathrm{d}\xi\,. \tag{16}$$

This is the Cauchy type integral modified for the separation of a function having a compact support (here—for the support $0,l$). Note that the point $\xi=k$ (this equality is realized for real k) is regular, while the integrant at point $\xi=i0$ is integrable.

In general, if the support of function $g(x)$ is $x_1<x<x_2$, where $x_{1,2}$ are finite, and $f(x)=g(x)$ for $x_1<x<x_2$, then

$$g^F(k)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}f^F(\xi)\times\frac{\exp[ix_1(k-\xi)]-\exp[ix_2(k-\xi)]}{\xi-k}\,\mathrm{d}\xi. \tag{17}$$

Note that the integral in (16) by means of the deformation of the integration path [we make three branch cuts $(-r_1+i0,-h_1+i0)$, (h_1+i0,r_1+i0), and $(+i0,i\alpha)$] can be presented as a sum of two finite integrals:

$$B_+(k)=\frac{1}{\pi}\int_0^{\alpha}\sqrt{\frac{h_1^2+x^2}{x(\alpha-x)(r_1^2+x^2)}}\times\left[\exp(ikl)-\exp(-lx)\right]\frac{\mathrm{d}x}{x+ik}$$
$$+\frac{i}{\pi}\int_{h_1}^{r_1}\sqrt{\frac{x^2-h_1^2}{x(r_1^2-x^2)}}\,\Phi(x,k)\,\mathrm{d}x\,,$$
$$\Phi(x,k)=\frac{\exp(ikl)-\exp(ixl)}{(x-k)\sqrt{x-i\alpha}}-\frac{\exp(ikl)-\exp(-ixl)}{(x+k)\sqrt{x+i\alpha}}\,. \tag{18}$$

We also note that for real k the symmetry is held as $\Re B_+(-k)=\Re B_+(k)$, $\Im B_+(-k)=-\Im B_+(k)$.

From the above considerations we get

$$Q_+(k)=\frac{P_*L_+(k)}{ik}\left(\mathrm{e}^{ikl}-1\right)-P_*B_+(k)\,,$$
$$Q_-(k)=-\frac{P_*}{ikL_-(k)}\left(\mathrm{e}^{ikl}-1\right)+P_*B_+(k)\,. \tag{19}$$

In particular, it follows that

$$\lim_{p\to\infty}p\,Q_+(ip)$$
$$=P_*(1-M)\,,M=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\exp(i\xi l)}{L_-(\xi)}\,\frac{\mathrm{d}\xi}{\xi}$$
$$=\frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin\xi l}{\xi}\Re\frac{1}{L_-(\xi)}+\frac{\cos\xi l}{\xi}\Im\frac{1}{L_-(\xi)}\right]\mathrm{d}\xi\,. \tag{20}$$

Note that for $\Im k<0$ the first-term integral in (16) is given by the residue at $\xi=k$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(\xi-i0)(\xi-k)L_-(\xi)}\,\mathrm{d}\xi\,\mathrm{e}^{ikl}=\frac{1}{(0+ik)L_-(k)}\,\mathrm{e}^{ikl} \tag{21}$$

and the exponentially growing terms in the expression (19) for $Q_-(k)$ [in the first term and in the expression for $B_+(k)$] cancel each other. As a result

$$\lim_{p\to\infty}p\,Q_-(-ip)=\lim_{p\to\infty}p\,Q_+(ip)=P_*(1-M)\,. \tag{22}$$

3.5 Final relations

From (14) and (19) it follows that

$$q_+ = \frac{C}{(0-ik)L_+(k)} + \frac{Q_+(k)}{L_+(k)}\,,$$
$$q_- = \frac{CL_-(k)}{0+ik} + Q_-(k)L_-(k)\,, \tag{23}$$

where C is an arbitrary real constant [the corresponding terms represent the homogeneous solution of (14) which reflects a remote loading]. For $k = \pm ip$, $p \to \infty$ (for $q_\pm$, respectively)

$$q_\pm \sim \tfrac{C+P_*(1-M)}{p}[L_\pm(\pm ip) \to 1 \ \ (p \to \infty)]\,. \tag{24}$$

It follows

$$q(0) = q(\pm 0) = C + P_*(1-M)\,. \tag{25}$$

Next we need the value of $q(l)$; it is

$$\begin{aligned} q(l) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} q_+(k)\mathrm{e}^{-ikl}\,\mathrm{d}k \\ &= \frac{P_*}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{C}{(0-ik)L_+(k)}\mathrm{e}^{-ikl}\,\mathrm{d}k \\ &\quad -\frac{P_*}{2\pi}\int_{-\infty}^{\infty}\frac{B_+(k)}{L_+(k)}\mathrm{e}^{-ikl}\,\mathrm{d}k \\ &= \frac{P_*}{2} + \frac{C}{\pi}\int_0^{\infty}\Bigg[\Re\sqrt{k^2-ik\alpha}\,\sin kl \\ &\quad -\Im\sqrt{k^2-ik\alpha}\,\cos kl\Bigg]\frac{\mathrm{d}k}{\sqrt{k^4+k^2\alpha^2}} \\ &\quad -\frac{P_*}{\pi}\int_0^{\infty}\Bigg[\Re\frac{B_+(k)}{L_+(k)}\cos kl \\ &\quad +\Im\frac{B_+(k)}{L_+(k)}\sin kl\Bigg]\mathrm{d}k\,. \end{aligned} \tag{26}$$

Lastly, for $k \to 0$

$$\begin{aligned} q_+(k) &\sim Q_{0+}(k) = C\sqrt{\frac{r_1\sqrt{1-v^2}}{2h_1(0-ik)}}\,, \\ q_-(k) &\sim Q_{0-}(k) = C\sqrt{\frac{2r_1}{h_1\sqrt{1-v^2}}}(0+ik)^{-3/2} \end{aligned} \tag{27}$$

and the far-field energy release rate is

$$G = p^2 Q_{0+}(ip)Q_{0-}(-ip) = C^2\frac{r_1}{h_1}\,. \tag{28}$$

For the determination of two unknowns, C and l, we have two equalities

$$q(0) = q_{**}, \quad q(l) = q_*\,. \tag{29}$$

The first one and (25) give us

$$C = q_{**} - P_*(1-M)\,, \tag{30}$$

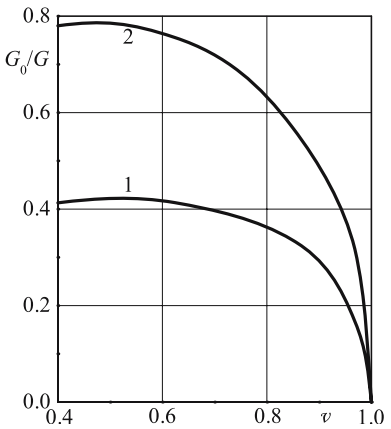

Fig. 4 Energy release ratio for $P_* = q_*$. The considered diagram for the crack with a damage zone, $q_{**} = 2q_*$ (*1*), and the diagram for the crack without such a zone, $q_{**} = q_*$ (*2*)

where M depends on v and l. The other serves for the determination of $l = l(v)$. So, if the critical elongations, q_*, q_{**}, and the constant P_* are given, the far-field energy release, G, can be determined as a function of the crack speed. The energy required for a bond to be broken is

$$G_0 = \frac{1}{2}q_{**}^2 - P_*(q_{**} - q_*)\,. \tag{31}$$

The difference, $G - G_0$, is radiated by high-frequency waves excited by the moving crack; it is the dissipation. The energy release ratio G_0/G calculated for $P_* = q_*$, $q_{**} = 2q_*$ as a function of the crack speed, v, is presented in Fig. 4 (curve 1). The ratio for a 'regular' lattice ($q_{**} = q_*$) is also shown (curve 2). It can be seen that the two-humped diagram leads to an increase of the dissipation.

4 Conclusion

In this paper, an idealized formulation for a dynamic crack in a nonlinear discrete lattice is considered. It is assumed that fracture takes place on the crack line as a two-step event. The first step is an instantaneous decrease of the interparticle tensile force, while the tangent modulus is still invariable. The other step is the final fracture when the interparticle interaction disappears. In contrast to the one-step fracture in a lattice considered earlier,

there exist two critical values of the strain, q_* and q_{**} (these values are assumed to be given), while the time-interval between the two steps of fracture is unknown. This creates an additional difficulty in the application of analytical methods; however, the Wiener–Hopf technique is still applicable.

The calculations are conducted for the case where the tensile force drops from a critical value to zero in the first step of fracture further increasing till the same critical value. Mathematically, any other case can be considered using the same technique. Two evident asymptotes can be noted. Clearly, the case $q_{**} = q_*$ corresponds to the one-step fracture, that is, when $q_{**} \to q_* + 0$ the solution must approach that for the crack without any damage zone. In the other case, $q_{**} \gg q_*$, the two steps of fracture are separated: the first step corresponds to the transition considered in Slepyan and Ayzenberg-Stepanenko (2004), while the second one corresponds to the one-step fracture.

For a more general fracture process, in the case where it develops on a line, the two-dimensional lattice problem can be reduced to an integral equation based on the fundamental solution considered in Slepyan and Ayzenberg-Stepanenko (2004). In reality, the fracture process is much more complicated (see, e.g. Marder and Gross 1995, Broberg 1999), and not only one lattice line is involved in it. In this respect, analytical solutions can serve for better understanding of some important phenomena in fracture (as it is discussed in Sect. 1); they also can serve as the analytical framework for numerical simulations.

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