PERGAMON

Journal of the Mechanics and Physics of Solids
48 (2000) 927–960

JOURNAL OF THE
MECHANICS AND
PHYSICS OF SOLIDS

www.elsevier.com/locate/jmps

Dynamic factor in impact, phase transition and fracture

L.I. Slepyan*

Department of Solid Mechanics, Materials and Structures, Faculty of Engineering, Tel Aviv University, PO Box 39040, Ramat Aviv 69978, Tel Aviv, Israel

Received 7 June 1999; received in revised form 5 August 1999

Abstract

The related questions 'how to avoid oscillations under an impact' and 'why a crack or phase-transition wave can/cannot propagate slowly' are discussed. The underlying phenomenon is the dynamic overshoot which can show itself in deformation of a body under a load suddenly applied. The manifestation of this phenomenon in a unit cell of the material structure is shown to trigger a fast crack in fracture as well as a fast wave in phase transition. Two ways for the elimination of the overshoot, to obtain a static-amplitude response (SAR), are examined. The first is a proper control of the load in an initial portion of the loading time. This is illustrated by means of an example of elastic collision. In the case of fracture, such control can be envisioned as provided by a proper post-peak tensile softening of the material. Secondly, the SAR can be achieved under the influence of viscosity. In this connection, the following transient problems are considered: a viscoelastic-spring oscillator under a step excitation, a square-cell viscoelastic lattice with a crack and a two-phase viscoelastic chain as the phase-transition waveguide. For each problem, in the space of viscosity parameters, the SAR domain is separated from the dynamic-overshoot-response (DOR) domain. In the SAR domain, in contrast to the DOR domain, a slow crack or a slow phase-transition wave can exist. A structure-associated size effect in the SAR/DOR domains separation is noted. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: A. Dynamics; B. Crack mechanics; B. Viscoelastic material; Lattice; C. Integral transforms

* Tel.: +972-3640-6224; fax: +972-3640-7617.
E-mail address: leonid@eng.tau.ac.il (L.I. Slepyan).

PII: S0022-5096(99)00061-7

1. Introduction

Consider a conservative dynamic system suddenly loaded by a force, P, which then remains invariable: $P=P_0H(t)$, where $P_0=\text{const}$ and H is the unit step function. As a rule the maximal dynamic displacement related to this force (the displacement in the force direction at the point where the force is applied), $u_{\max}$, exceeds the static value, u_{static}, which is assumed here to exist. This is because the work of the constant force on the static displacement exceeds the potential energy of the system:

$$P_0u_{\text{static}} > \int_0^{u_{\text{static}}} P_{qs}(u)\mathrm{d}u, \tag{1}$$

where $P_{qs}(u)$ is the force corresponding to a quasi-static loading of the system when the force grows slowly. The excess of the work of the dynamically applied force causes oscillations relative to the static value with amplitude $u_{\max}-u_{\text{static}}$. In this case, the dynamic amplification factor $k_d=u_{\max}/u_{\text{static}} > 1$. In a linear system, in the case where under the condition $\mathrm{d}u/\mathrm{d}t = 0$ the kinetic energy of the body is zero (that is, the particle velocities vanish over the entire body), $k_d=2$ as follows directly from the energy consideration as (1). Indeed, in the linear case,

$$P_{qs}(u) = P_0u/u_{\text{static}} \tag{2}$$

and

$$P_0 2u_{\text{static}} = \int_0^{2u_{\text{static}}} P_{qs}(u)\mathrm{d}u. \tag{3}$$

The dynamic overshoot phenomenon is common for free systems, where u_{static} does not exist, as well. In this case, a rigid-body uniform acceleration and hydrostatic stress distributions correspond to the quasi-static loading, while dynamic distributions differ by oscillations relative to these quasi-static values as in the case where the static displacement exists.

This dynamic phenomenon can manifest itself in various fields, for example, in a collision of vehicles or in switching on of an electrical system. Another area of its manifestation is the strong influence of the dynamic factor on the rate of a process such as phase transformation or fracture.

Indeed, consider a system of interconnected distributed dynamic elements under a dynamic action. Because the dynamic amplitude of an element approaches its maximal value in a fixed time (for an oscillator it is half the period), a neighboring element is excited in a given time as well, and this dictates the speed of the propagation of the excitation. For instance, if the dynamic amplification factor exists for an element of the structure, $k_d > 1$, this can lead to a fast phase-transition wave or fast crack propagation even in the case where the load (or another action) does not considerably exceed a critical, phase-transition or fracture initiation value. The energy exchange between the elements of the structure leads to a decreased resistance to the propagation in comparison with that for the initiation, and slow propagation is prohibited. Concerning fracture, it means that the material is brittle. Thus, the questions considered below, 'how to avoid oscillations under an impact' and 'why a phase-transition wave or a crack can/cannot propagate slowly', are closely related to the dynamic factor manifestation.

In this paper, the conditions are examined which lead to elimination of the dynamic overshoot. First, it is proper control of the dynamic load in an initial portion of the loading time. The load is assumed to increase monotonically during a time-period and then remain constant. The minimal period is found which allows a solution to exist. In particular, a two-step loading is shown to lead to such a 'quasi-static' response to the dynamic action, and this solution is valid for some inelastic and nonlinear systems as well.

The static-amplitude response can also be achieved under the influence of viscosity. In this connection, the following transient problems are considered: a viscoelastic-spring oscillator under a step excitation, a square-cell viscoelastic lattice with a crack and a two-phase viscoelastic chain as the phase-transition waveguide. For each problem, in the space of viscosity parameters, the SAR domain is separated from the DOR domain. Using these models and a limiting-strain criterion it is shown that phase-transition waves and cracks can propagate slowly in the former domain, while only fast waves and cracks can exist in the latter.

The SAR/DOR domains separation is carried out below based on an analysis of the Laplace transform of a function of time. In the case of the SAR domain, this function, the original, remains non-negative, while it changes the sign in the case of the DOR domain. A dependence for the interface is derived by an asymptotic analysis of the Laplace transform for large time (logarithmic asymptotes are considered). Numerical calculations performed for a finite range of time show the sufficiency of this analysis. Thus, the SAR/DOR interface is expressed analytically as well as the monotonic response (MR) domain boundary.

Lattice models for fracture and phase transition have been used in a number of works. The dynamic Mode III elastic fracture of a square-cell lattice was considered by Slepyan (1981a, 1981b, 1982a) for the sub-critical and super-critical crack speeds. The fracture Modes I and II for an elastic triangular-cell lattice were studied by Kulakhmetova et al. (1984). In these works, the structure-dependent total energy dissipation was analytically found for the three fracture modes as functions of the crack velocity (Fig. 1). Similar relations were obtained by Slepyan (1986) and Marder and Gross (1995) for elastic lattice strips. Some general conclusions concerning the resistance to crack propagation in a complex medium are presented in Slepyan (1982b, 1984). The papers by Slepyan and Troyankina (1984, 1988) were devoted to phase-transition waves in piece-wise linear and nonlinear chain structures. Reviews of works devoted to the fracture of elastic lattices have been provided by Slepyan (1990, 1993, 1998). A number of works have been devoted to the stability of crack propagation in discrete elastic lattices (Fineberg et al., 1991, 1992; Marder, 1991; Marder and Xiangmin Liu, 1993;

Marder and Gross, 1995). Mode III dynamic fracture in the square-cell lattice made of a standard viscoelastic material was examined by Slepyan et al., 1999.

In the works devoted to the dynamic fracture in elastic lattices, the phenomenon of crack-speed-dependent dissipation has been discovered and described. Such dissipation in a purely elastic structure manifests itself on the macro-level due to the radiation of high-frequency structure-associated waves excited by the propagating crack. Dissipation, measured by the energy loss per unit length of the crack propagation, does not vanish when the crack speed tends to zero; on the contrary, it first decreases with the crack velocity and reaches a minimum approximately at half the critical speed (it is the shear wave velocity for the Mode III and the Rayleigh wave velocity for Modes I and II). For the probable case where the criterion of the bond breakage is a critical force or strain, these dependences give evidence that a slow crack cannot exist in such an elastic structure. This question has been examined in detail by Marder and Gross (1995); they have shown that in a slow-crack regime the strain reaches a maximum long before the bond breaks as prescribed by the expected solution. As discussed above and shown below, this is the dynamic overshoot manifestation. However, as has been found in the paper by Slepyan et al. (1999), slow cracks do exist in a viscoelastic lattice if the viscosity is high enough.

Similar phenomena are common for the phase-transition wave in an elastic chain (Slepyan and Troyankina, 1984, 1988). The corresponding SAR domain (where slow waves can exist) for a two-phase viscoelastic chain is found below as well.

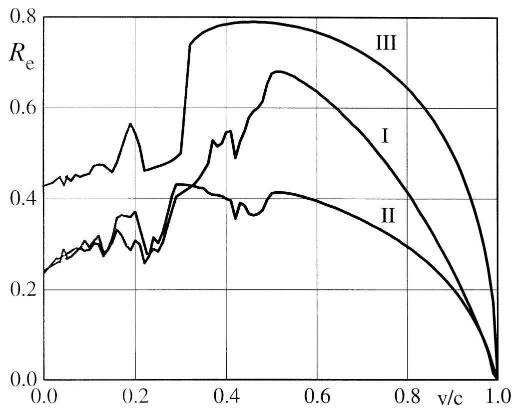

Fig. 1. The energy release ratios for elastic lattices. I: triangular-cell lattice, fracture Mode I; II: triangular-cell lattice, fracture Mode II, and III: square-cell lattice, fracture Mode III. Here R_e is the ratio of the energy lost in the breaking bonds to the total energy release corresponding to the continuous material as a long-wave approximation of the lattice, v is the crack speed and c is the continuous-material Rayleigh wave speed for Modes I and II and shear wave speed for Mode III.

2. How to avoid oscillations under an impact

2.1. Collision of different-in-length rods

To illustrate the phenomenon, consider the collision of two elastic rods moving along a horizontal axis, x, with velocities $v_{\pm} = \mp v_0$ (the subscripts '+' and '−' correspond to the right and left rods, respectively). Let the right rod be shorter than the left one: $l_+ < l_-$, where $l_{\pm}$ are the rod lengths. In other respects, the rods are assumed to be the same. The dynamics of each rod is assumed to satisfy the one-dimensional wave equation

$$u'' - \frac{1}{c^2}\ddot{u} = 0, \tag{4}$$

where $u(x, t)$ is the longitudinal displacement, c is the wave speed, $u' = \partial u/\partial x$, $\dot{u} = \partial u/\partial t$ and t is time. The solution to this problem shows that the collision period is defined by the shorter rod as $2\ l_+/c$. After the collision the right rod moves as a rigid body with the velocity $\dot{u} = v_0$, that is, with the same kinetic energy as before the collision. Thus, the collision is purely elastic regarding the shorter rod.

Now, using the momentum and energy conservation laws, the averaged-velocity-based kinetic energy of the left rod, $\mathscr{K}$, and its total energy, W can be calculated. The momentum conservation law gives us the equality

$$l_-\langle v_-\rangle + l_+ v_0 = l_- v_0 - l_+ v_0 \tag{5}$$

and hence, the left rod averaged velocity, $\langle v_-\rangle$, and the corresponding kinetic energy, $\mathscr{K}$, after the collision are

$$\langle v_-\rangle = v_0\left(1 - \frac{2l_+}{l_-}\right), \quad \mathscr{K} = \frac{1}{2}\left(1 - \frac{2l_+}{l_-}\right)^2 \rho A l_- v_0^2, \tag{6}$$

where ρ is density and A is the cross-section area of the rod. The total energy of the left rod after the collision is the same as before the collision because the right rod does not change its energy. (This statement also follows from the fact that there is no energy flux through the collision cross-section, $x = 0$, since it is unmoving during the collision, and the energy of each rod remains invariable with time.) So, the total energy of the left rod is

$$W = \frac{1}{2}\rho A l_- v_0^2. \tag{7}$$

Consequently, the energy of oscillations in the rod is

$$W - \mathscr{K} = 2v_0^2 A l_+\left(1 - \frac{l_+}{l_-}\right), \quad \frac{W - \mathscr{K}}{W} = 4\frac{l_+}{l_-}\left(1 - \frac{l_+}{l_-}\right). \tag{8}$$

Note that if the left rod is twice as large as the right one the former is stopped by the collision, and the oscillations only remain without any rigid body motion. Thus, after the collision, the energy of oscillations can represent any part of the total energy of the larger rod. We now consider the possibilities to avoid oscillations under the impact.

2.2. Loading of an elastic oscillator

We begin with an oscillator as the simplest dynamic system. The equation of its motion is

$$M\ddot{u} + \kappa u = P(t), \tag{9}$$

where M, κ and P are the mass, stiffness and external force, respectively. Let P be a suddenly applied constant force, $P = P_0 H(t)$, where $H(t)$ is the unit step function. Then

$$u = \frac{P_0}{\kappa}(1 - \cos\ \omega t), \quad \omega = \sqrt{\frac{\kappa}{M}}. \tag{10}$$

Thus, in this case, the maximal dynamic amplitude, $2P_0/\kappa$, is twice as much as the static one: $k_d = 2$.

We can consider two problems. The first is how to avoid oscillations after the impact which concerns the finite duration of the action. The solution is obvious: the duration, T, of the constant force action should be

$$T = nT_0, \quad T_0 = \frac{2\pi}{\omega}, \quad n = 1, 2, \ \ldots \tag{11}$$

because the oscillator returns to the initial state with zero displacement and velocity at $t = nT_0$.

The second problem is how to load the oscillator to obtain an SAR, that is, to satisfy the requirement $k_d = 1$. The force is assumed to be monotonically increasing during the loading time and invariable after that time.

A solution can be achieved by a force linearly increasing during the same time (11). Such a force can, in fact, be represented as

$$P = P_0 \frac{t}{T} H(t) - P_0 \frac{t - T}{T} H(t - T). \tag{12}$$

The displacement of the oscillator under this force is

$$u = \frac{P_0}{\kappa T}\left(t - \frac{\sin\ \omega t}{\omega}\right) H(t) - \frac{P_0}{\kappa T}\left(t - T - \frac{\sin\ \omega t}{\omega}\right) H(t - T) \tag{13}$$

since $\sin[\omega(t-T)] = \sin\ \omega t$. It can be seen that the displacement increases during the loading time when the second term in this expression is zero; it follows that

$$u = \frac{P_0}{\kappa} = u_{\text{static}} \quad (T \le t). \tag{14}$$

Thereafter, at any time $t \geq T$, the oscillator can be unloaded in the same way during the same time-period, T, and this process continues, as well with no dynamic effects.

This result is still valid in the case of a multi-degree-of-freedom system if the frequencies of free oscillations, $\omega < \omega_1 < \cdots < \omega_n < \cdots$, satisfy the condition

$$\omega_n = p_n \omega \quad (p_n = 2, 3, \ldots). \tag{15}$$

In this case, the condition (11) is satisfied with respect to each mode of oscillations. Also note that the result is applicable to a free body since such a body, in addition to oscillation modes, has only a rigid-motion mode which is monotonic under the constant and linearly increasing forces.

The static-amplitude response of the oscillator can also be achieved by a piecewise constant force, namely

$$P = \frac{1}{2} P_0 H(t) + \frac{1}{2} P_0 H\left(t - \frac{T}{2}\right). \tag{16}$$

In this case, at $t = T/2 = \pi/\omega$ the dynamic displacement corresponding to the force $P_0/2$ reaches the static value which corresponds to the total force, P_0. At this moment, when the velocity is zero, there is a jump in the force which becomes equal to P_0, and the oscillator is then in equilibrium:

$$u = \frac{P_0}{2\kappa}(1 - \cos \omega t)H(t) + \frac{P_0}{2\kappa}(1 + \cos \omega t)H(t - \pi/\omega). \tag{17}$$

2.3. Oscillation-free collision of the different-in-length rods

We can now return to the collision problem for the elastic rods of different lengths. Consider the case $l_- = 2\, l_+$. Let us introduce a nonlinear elastic shock absorber between the rods, such that its resistance to compression is invariable. Based on the above condition (11) we take the collision time to be equal to the main period of oscillations for the left rod: $T = 2\, l_-/c$. In this case, in the condition (11), $n = 1$ for the left rod and $n = 2$ for the right one.

The contact force can be found using the momentum and energy conservation laws. Since there are no oscillations after the collision and the shock absorber is elastic, it follows from these laws that the velocities of the right (v_+) and the left (v_-) rods and the contact force, P_0 are as follows:

$$v_+ = \frac{3l_- - l_+}{l_- + l_+} v_0 = \frac{5}{3} v_0, \quad v_- = -\frac{l_- - 3l_+}{l_- + l_+} v_0 = -\frac{1}{3} v_0,$$

$$P_0 = \rho v_0 c A \frac{2l_+}{l_- + l_+} = \frac{2}{3}\rho v_0 c A. \tag{18}$$

It can be seen that the contact force is less than that for the collision without the shock absorber. Four waves propagate along the right rod during the collision period, and each adds $2/3v_0$ to the initial particle velocity, $-v_0$. At the same time, only two waves with the same but opposite particle velocities propagate in the left rod (because it is twice as long) adding, in total, $-4/3v_0$ to the initial particle velocity, v_0.

Clearly, such an oscillation-free solution is valid for any rational l_+/l_--ratio: the collision time should be a common multiple of 2 l_+/c and 2 l_-/c.

2.4. Acceleration of a free rod

Consider a free rod under an axial compressive force, P_0, suddenly applied at its left end at $t = 0$. The solution which satisfies the boundary conditions at $x = 0$ ($u' = -P_0/EA$, where E is the elastic modulus) and $x = l$ ($u' = 0$) can be found using the Laplace transformation

$$u^L(x, s) = u_0^L = \frac{P_0 c}{AEs^2} \frac{\mathrm{e}^{(l-x)s/c} + \mathrm{e}^{-(l-x)s/c}}{\mathrm{e}^{ls/c} - \mathrm{e}^{-ls/c}} = \frac{P_0 c}{AEs^2}(\mathrm{e}^{-xs/c} + \mathrm{e}^{-(2l-x)s/c})\sum_{n=0}^{\infty}\mathrm{e}^{-2nls/c}$$

$$\left(u_0^L(s) = \int_0^\infty u_0(t)\mathrm{e}^{-st}\,\mathrm{d}t\right) \tag{19}$$

with

$$\dot{u}_0(x, t) = \frac{P_0 l}{Mc}\sum_{n=0}^{\infty} H[ct - x - 2nl] + H[ct + x - 2(n+1)l]. \tag{20}$$

This expression represents a piece-wise constant particle velocity oscillating relative to the rigid-body velocity, $v = P_0 t/M$.

Thus, the particle velocity oscillates. The question is how to apply the force (which should be constant after the loading time) to exclude these oscillations. We have already found the solution: the force must be linearly increasing during a period of free oscillations of the rod. Let it be the main (the minimal) period, 2 l/c. In this case, the force can be represented as

$$P = P_0\frac{ct}{2l}H(t) - P_0\frac{ct - 2l}{2l}H(ct - 2l). \tag{21}$$

Consequently, the Laplace transform (19) becomes

$$u^L(x, s) = \frac{c}{2ls}u_0^L(1 - \mathrm{e}^{-2ls/c}). \tag{22}$$

The exponent-series expansion of this expression contains only two terms:

$$u^L(x, s) = \frac{P_0 c^2}{2lAEs^3}[\mathrm{e}^{-xs/c} + \mathrm{e}^{-(2l-x)s/c}]. \tag{23}$$

The original and its derivatives are

$$u(x, t) = \frac{P_0}{4Mc^2}[(ct\text{-}x)^2 H(ct-x) + (ct+x-2l)^2 H(ct+x-2l)],$$

$$\dot{u}(x, t) = \frac{P_0}{2Mc}[(ct-x)H(ct-x) + (ct+x-2l)H(ct+x-2l)],$$

$$\ddot{u}(x, t) = \frac{P_0}{2M}[H(ct-x) + H(ct+x-2l)] \geq 0, \tag{24}$$

which give an x-independent particle velocity for post-collision time, $t > 2\ l/c$:

$$\dot{u}(x, t) = \frac{P_0}{M}\left(t - \frac{l}{c}\right) \tag{25}$$

with the hydrostatic type of the time-dependent stress distribution

$$\sigma_{xx} = -\frac{P_0}{A}\left(1 - \frac{x}{l}\right). \tag{26}$$

Thus, the hydrostatically stressed rod moves as a rigid body as it should. Note that such a non-oscillatory motion can be obtained by a quasi-static, very slowly growing load as well. However, as shown, a dynamic loading properly controlled in an initial portion of the loading time leads to the same result.

2.5. Nonlinear oscillator

Let the displacement amplitude, $u_{\max}$, corresponding to a suddenly applied load, P, be an increasing function of P and let $u_{\max}(P) > u_{\text{static}}(P)$, where u_{static} is the corresponding static value. For the total load, P_{total}, we choose the first step of the loading to be $P = P_1$, such that

$$u_{\max}\ (P_1) = u_{\text{static}}(P_{\text{total}}), \quad 0 < P_1 < P_{\text{total}}. \tag{27}$$

Under this suddenly applied force the displacement reaches its maximal value, $u_{\max}(P_1)$, at a moment, $t = t_*$. At this time, the velocity $\dot{u}(t_*) = 0$ and the second step of the loading, $P_2 = P_{\text{total}} - P_1$, leads to the static state, $u = u_{\text{static}}(P_{\text{total}})$ $(t \geq t_*)$.

Consider the following example where we let

$$u_{\text{static}}(P) = u_0\left(\frac{P}{P_0}\right)^{\nu}, \quad \nu > 0. \tag{28}$$

The dynamic amplitude, $u_{\max}(P)$, can be easily found by equating the work of the force and the potential energy of the oscillator. The equalities

$$Pu_{\max} = \int_0^{u_{\max}} P(u_{\text{static}})\mathrm{d}u_{\text{static}} = \frac{\nu}{1+\nu} P_0 u_{\max} \left(\frac{u_{\max}}{u_0}\right)^{1/\nu} \tag{29}$$

lead to the relations

$$u_{\max} = u_0 \left[\frac{1+\nu}{\nu} \frac{P}{P_0}\right]^{\nu} \tag{30}$$

and

$$k_d = \frac{u_{\max}}{u_{\text{static}}} = \left(1 + \frac{1}{\nu}\right)^{\nu}, \quad k_d \rightarrow 1 \quad (\nu \rightarrow 0), \quad k_d \rightarrow e \approx 2.72 \quad (\nu \rightarrow \infty). \tag{31}$$

Thus, in this specific case, the dynamic factor is P-independent. The first-step force can now be found from the equality (27) as

$$P_1 = \frac{\nu}{1+\nu} P_{\text{total}}. \tag{32}$$

To find the loading time, T_{load}, that is, the time at which the second step is turned on, we consider the oscillator equation

$$M\ddot{u} + P_0 \left(\frac{u}{u_0}\right)^{1/\nu} = P_1, \tag{33}$$

or

$$\frac{M}{2} \frac{\mathrm{d}\dot{u}^2}{\mathrm{d}u} + P_0 \left(\frac{u}{u_0}\right)^{1/\nu} = P_1. \tag{34}$$

From this it can be found that the loading time is

$$T_{\text{load}} = \sqrt{\frac{M}{2P_0}\left(1 + \frac{1}{\nu}\right)} \int_0^{u_{\text{static}}} \left[\left(\frac{u_{\text{static}}}{u_0}\right)^{1/\nu} - \left(\frac{u}{u_0}\right)^{1/\nu}\right]^{-1/2} \frac{\mathrm{d}u}{\sqrt{u}}. \tag{35}$$

2.6. Damped oscillator

Consider a generalization of this solution for an oscillator with dissipation:

$$M\ddot{u} + \alpha\kappa\dot{u} + \kappa u = P(t). \tag{36}$$

Nondimensional time $t' = t\omega$ ($\omega = \sqrt{\kappa/M}$), the creep time $\alpha' = \alpha\omega$ and the force $P' = P/\kappa$ are used below, but the primes are dropped. This concerns the relaxation time, β, as well (the latter is introduced in the next section). In these terms, Eq. (36) becomes

$$\ddot{u} + \alpha\dot{u} + u = P(t). \tag{37}$$

The Green function which corresponds to a unit pulse, $P=\delta(t)$, is

$$u=u_0=\frac{\sin\ \Omega t}{\Omega}\mathrm{e}^{-\alpha t/2},\quad u_0^L=\frac{1}{(s+\alpha/2)^2+\Omega^2}, \tag{38}$$

where

$$\Omega=\sqrt{1-\alpha^2/4}. \tag{39}$$

This expression for u_0 is valid for the case of sub-critical damping, $0\le\alpha<2$. Note that $u_0>0$ during the first half-period, $0<t<\pi/\Omega$. We take the force as

$$P=P_0[1+\exp(-\pi\alpha/(2\Omega))]^{-1}[H(t)+\exp(-\pi\alpha/(2\Omega))H(t-\pi/\Omega)],$$

$$P^L=\frac{P_0}{s}[1+\exp(-\pi\alpha/(2\Omega))]^{-1}[1+\exp(-[\pi\alpha/(2\Omega)+s\pi/\Omega])], \tag{40}$$

where the two-step force is invariable after the loading period: $P=P_0$ $(t\ge\pi/\Omega)$. We thus obtain a displacement increasing monotonically during the loading period, $0<t<\pi/\Omega$, which thereafter remains constant. It is easier to see this by considering the Laplace transform

$$u^{\mathrm{L}}=P^{\mathrm{L}}u_0^{\mathrm{L}}. \tag{41}$$

In the region $t<\pi/\Omega$, the second term in the expression for P^{L} does not influence the original, $u(t)$, which is a monotonically increasing function since the derivative is positive (it is proportional to u_0). The Laplace transform, $u_0^L(s)$, has the complex poles, $s=-\alpha/2\pm i\Omega$ which suggests an oscillating original as it is. However, for $t>\pi/\Omega$ the complete transform must be considered where the multiplier P^{L} has zeros at the same points. The only remaining pole is $s=0$ and this yields the static displacement, $u=P_0$, for $t>\pi/\Omega$.

Thus, in the case of a damped oscillator, the two-step, piece-wise constant, dynamically applied force (40) leads to the non-oscillatory SAR response. The loading period, π/Ω, increases beginning from π $(\alpha=0)$ and tends to infinity when $\alpha\to 2$, while the second-step force tends to zero as $\exp[-(\pi\alpha/(2\Omega))]$ $(\Omega\to 0)$.

For $\alpha>2$ the Laplace transform, $u_0^{\mathrm{L}}(s)$, has two real poles

$$s=-\frac{1}{2}\alpha\pm\Lambda^0,\quad \Lambda^0=\sqrt{\frac{1}{4}\alpha^2-1}, \tag{42}$$

the original is

$$u_0=\frac{1}{\Lambda^0}\exp(-\alpha t/2)\sinh\ \Lambda^0 t>0\quad(0<t<\infty) \tag{43}$$

and hence, $u(t)$, corresponding to a one-step suddenly applied force, as an integral of u_0, increases monotonically and tends to the static value. For a large α, the displacement

$$u \sim \frac{P_0}{k}(1 - \mathrm{e}^{-t/\alpha}) \tag{44}$$

increases more slowly as damping increases.

2.7. Spring of a standard viscoelastic material

Consider now an oscillator with the spring made of a standard viscoelastic material. In terms of the Laplace transform, its dynamic equation is

$$Su^{\mathrm{L}} = (1 + \beta s)P^{\mathrm{L}}, \quad S = (\beta s + 1)s^2 + \alpha s + 1, \tag{45}$$

where α and β are nondimensional creep and relaxation times, respectively, and $0 \le \beta \le \alpha$ as follows from energy considerations. (This inequality corresponds to the case where only dissipation exists with no energy release from the material.) The function $1/S$ has three poles, at least one of which, $s = s_0$, is real. It can be seen that

$$S < 0 \quad (s \le -1/\beta), \quad S > 0 \quad (s \ge -1/\alpha), \quad -1/\beta < s_0 < -1/\alpha. \tag{46}$$

There is a critical-damping boundary at the α, β-plane which separates an 'oscillatory domain' (the two-complex-poles domain) and a 'monotonic-response' (MR) domain, or the three-negative-poles domain. The one-step loading does not excite oscillation if the viscosity parameters belong to the latter and does excite otherwise.

In crossing this boundary, two roots of the function S (45) merge with each other, and the roots satisfy two equations

$$S(s) = (\beta s + 1)s^2 + \alpha s + 1 = 0,$$

$$S'(s) = 3\beta s^2 + 2s + \alpha = 0. \tag{47}$$

Substituting the roots of the latter,

$$s = -\frac{1}{3\beta}(1 \pm \sqrt{1 - 3\alpha\beta}), \tag{48}$$

into the former we obtain the following equation with respect to the viscosity parameters:

$$\beta^2 = \frac{2}{27}[\pm(1 - 3\alpha\beta)^{3/2} + \frac{9}{2}\alpha\beta - 1]. \tag{49}$$

This equation defines the above-mentioned boundary consisting of two branches. In particular, for $\beta = 0$ it yields $\alpha = 2$ as it should. A point belonging to both branches is $\alpha = \alpha_{\min} = \sqrt{3}$, $\beta = \beta_{\max} = 1/(3\sqrt{3})$. The MR domain is bounded by the lower branch [the positive radical in (49) and the half-axis $\alpha \geq 2$] from below, and by the upper branch [the negative radical in (49)] from above.

The MR domain is a part of the SAR domain where oscillations can exist while the dynamic amplification factor is equal to unity. To find the SAR domain boundary consider the difference between the static and dynamic displacements as

$$U^{\mathrm{L}}(s) = [u(\infty) - u(t)]^{\mathrm{L}} = \frac{P_0}{s} - \frac{P_0(1+\beta s)}{sS(s)} = \frac{P_0(\beta s^2 + s + \alpha - \beta)}{S(s)}. \tag{50}$$

Note that this relation can also be expressed as

$$U^{\mathrm{L}}(s) = \frac{P_0(s^2 + E - 1)}{s(s^2 + E)}, \quad E = \frac{1+\alpha s}{1+\beta s}. \tag{51}$$

The SAR domain corresponds to the condition

$$U(t) \geq 0. \tag{52}$$

We represent the function S as

$$S = (\beta s + s_1)(s + s_2 + i\Omega)(s + s_2 - i\Omega), \tag{53}$$

where $s_1 = -s_0\beta$ and s_2 are positive values, and consider the oscillatory case, $\Omega >$ 0. Comparing this with the expression (45) we obtain the following equations:

$$s_1 + 2\beta s_2 = 1,$$

$$2\, s_1 s_2 + \beta(s_2^2 + \Omega^2) = \alpha,$$

$$s_1(s_2^2 + \Omega^2) = 1. \tag{54}$$

From this is follows that

$$s_1 = 1 - 2\beta s_2,$$

$$\Omega^2 = \frac{1}{1 - 2\beta s_2} - s_2,$$

$$\alpha = \frac{\beta}{1 - 2\beta s_2} + 2s_2(1 - 2\beta s_2). \tag{55}$$

The original, $U(t)$, can then be represented as

$$U(t) = C_1\, \mathrm{e}^{-s_1 t/\beta} + C_2\, \mathrm{e}^{-s_2 t}\, \sin(\Omega t + \varphi), \tag{56}$$

where C_1 and C_2 are nonzero constants and φ is an initial phase.

Clearly, $U(t)$ cannot be non-negative if $s_2 < s_1/\beta$ ($\Omega > 0$). This asymptotic-behavior-based consideration represents a necessary condition for the SAR. Using expressions (55) the limiting dependence, corresponding to the equality $s_2 = s_1/\beta$, can be obtained in the form

$$\alpha = 3\beta + \frac{2}{9\beta}. \tag{57}$$

The domain with the left boundary (57) should be united with the MR domain with the left boundary (49). The dependence (57) for $\beta \geq 1/(3\sqrt{3})$ represents an upper part of the boundary of the united domain, while the lower part of its boundary, $\beta \leq 1/(3\sqrt{3})$, coincides with the lower boundary of the MR domain [see (49) and the conclusions following this equation]. An analysis shows that this two-branch boundary does separate the DOR and SAR domains with the latter lying to the right. The domains considered, MR, SAR and DOR, are shown in Fig. 2. The normalized response for a set of α, β-values is presented in Fig. 3 where $u = u(t)/u(\infty)$.

Note that the point in time when $U(t)$ first becomes negative tends to infinity when the corresponding point in the DOR domain approaches the SAR/DOR interface. Thus, the DOR-to-SAR transition is continuous. This and other results obtained here, such as the shape of the SAR/DOR interface and the validity of the asymptotic analysis for its determination, are similar to that for the fracture and phase transition problems examined below.

3. Crack growth in a viscoelastic square-cell lattice

Consider a square-cell lattice, Fig. 4, consisting of point particles of mass M

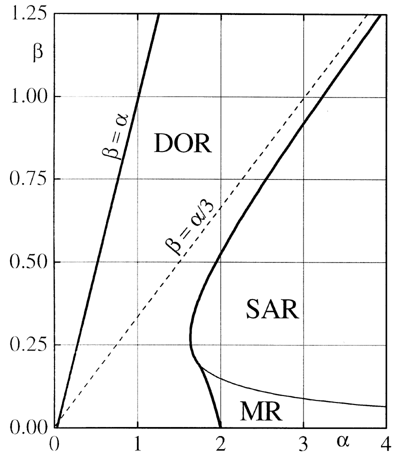

Fig. 2. The DOR, SAR and MR domains for the standard-material-spring viscoelastic oscillator (α and β are nondimensional creep and relaxation times, respectively).

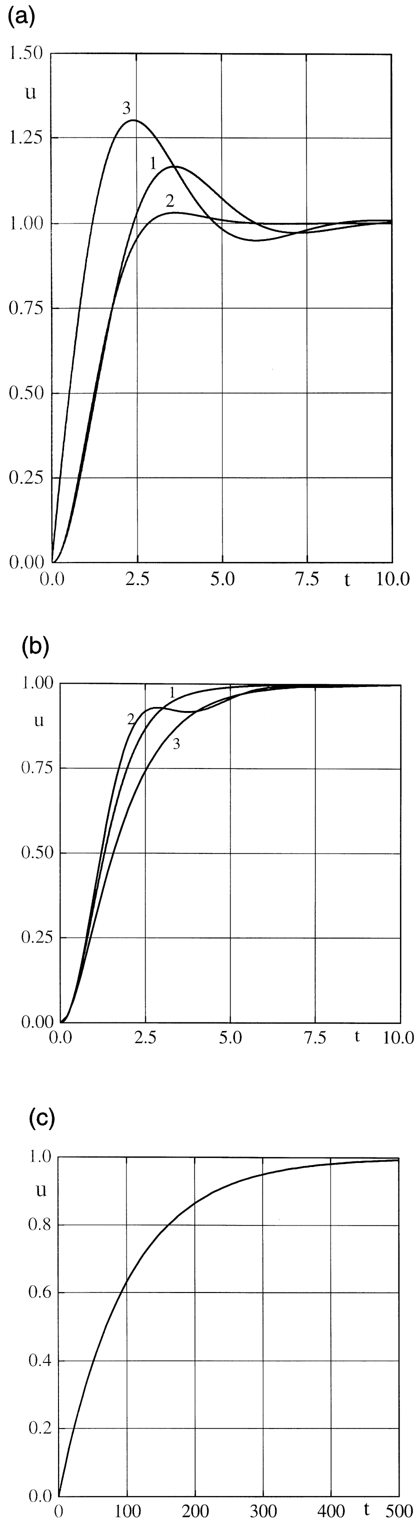

Fig. 3. The normalized response of the standard-material viscoelastic oscillator. (a) The DOR domain: 1. $\alpha=1$, $\beta=0$; 2. $\alpha=1.5$, $\beta=0.25$; 3. $\alpha=2$, $\beta=0.6$. (b) The SAR domain: 1. $\alpha=1.75$, $\beta=0.25$; 2. $\alpha=2$, $\beta=0.5$; 3. $\alpha=2$, $\beta=0.1$. (c) The SAR domain: $\alpha=100$, $\beta=0$.

connected by massless standard-viscoelastic-material bonds of the length a. The particles are numbered by two integers, m and n ($x=ma,\ y=na$). The crack propagation is a consequence of breakage of bonds between particles with $n=0$ and $n=-1$. The fracture Mode III where displacements, $u_{m,n}(t)$, are perpendicular to the lattice plane is considered.

The dynamic equation of the lattice is

$$M\left(1+\beta\frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\mathrm{d}^2u_{m,\,n}}{\mathrm{d}t^2}$$
$$=\left(1+\alpha\frac{\mathrm{d}}{\mathrm{d}t}\right)\kappa(u_{m+1,\,n}+u_{m-1,\,n}+u_{m,\,n+1}+u_{m,\,n-1}-4u_{m,\,n}). \tag{58}$$

This equation is valid for the intact lattice, that is, for particles with $n>0$ and $n<-1$ which are not connected by the breaking bonds.

Via a long-wave (low-frequency) approximation for the anti-plane deformation, the lattice corresponds to a homogeneous body of density M/a^2 and shear modulus κ. Accordingly, the shear wave velocity is given by $c=a\sqrt{\kappa/M}$. Note that a steady-state problem for the same lattice was considered in the paper by Slepyan et al. (1999).

In the following, a slowly growing crack is considered, such that the lattice approaches a static state before the next bond breaks. This leads to a transient problem described below. In the determination of conditions which permit a slow crack, the limiting strain criterion of the bond breakage is used, and finally the problem is reduced to the corresponding SAR/DOR domains separation.

Note that the same equations are valid in the case where only vertical displacements in the lattice plane are allowed instead of the normal ones. For convenience this mode of the lattice dynamics will be referred as well.

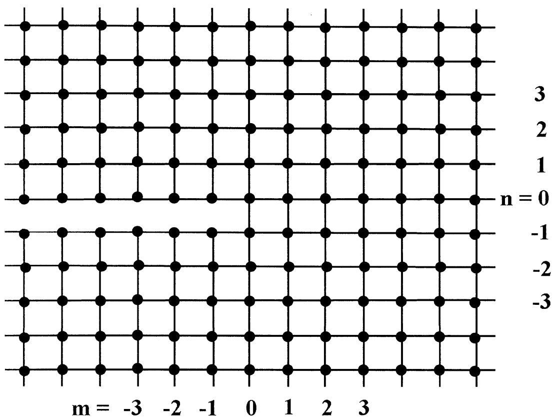

Fig. 4. The square-cell lattice consisting of point particles connected by standard-viscoelastic-material massless bonds.

3.1. Superposition

We now consider two problems: the first is that of a homogeneous static state of a stressed lattice with a crack at $m=-2, -3, \ldots$. The crack surfaces are assumed to be free. We denote internal forces in bonds in front of the crack by σ_{-1} and σ_0, respectively, to the bond numbers. The second, dynamic problem is for the same, but initially unstressed lattice with the extended crack, $m=-1, -2, \ldots$. The lattice is suddenly loaded by the forces $\mp\sigma_{-1}$ applied at the upper, $n = 0$, and the lower, $n=-1$, crack surfaces, respectively, at $m=-1$, $t = 0$. In this problem, let the internal force in the bond $m = 0$ be $\sigma(t)$.

Further, consider the problem in total. It corresponds to the first one where the crack-front bond $m=-1$ breaks at $t = 0$ because the total crack-surface forces acting at $m=-1$ are zero for $t > 0$. This problem corresponds to a very slow crack growth when the time-interval between the break of neighboring bonds is large enough to permit the lattice to approach the static state before the next break. The crack-tip force is

$$\sigma_{\text{total}}(0, t) = \sigma_0 + \sigma(t) \tag{59}$$

and therefore,

$$\sigma_{\text{total}}(0, t) \leq \sigma_{\text{total}}(0, \infty) = \sigma_{-1} \quad \text{if } \sigma(t) \leq \sigma(\infty) \tag{60}$$

and vice versa. Thus, the SAR/DOR interfaces for the second and the total problems are the same, and such an interface for the slow crack in the lattice can be found based on the examination of the second problem.

3.2. Derivation of a governing equation

We first denote the nondimensional values

$$x' = \frac{x}{a}, \quad t' = t\omega = \frac{ct}{a}, \quad u' = \frac{u}{a}, \quad \sigma' = \frac{\sigma}{\kappa}, \quad \alpha' = \alpha\omega = \frac{\alpha c}{a}, \\ \beta' = \beta\omega = \frac{\beta c}{a}. \tag{61}$$

In the following, as before the primes are dropped (the normalization of t, α and β remains the same; recall that $\omega = \sqrt{\kappa/M}$). Eq. (58) becomes

$$\left(1 + \beta\frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\mathrm{d}^2 u_{m,\,n}}{\mathrm{d}t^2} = \left(1 + \alpha\frac{\mathrm{d}}{\mathrm{d}t}\right)(u_{m+1,\,n} + u_{m-1,\,n} + u_{m,\,n+1} + u_{m,\,n-1} \\ - 4u_{m,\,n}). \tag{62}$$

The Laplace transformation under zero initial conditions leads to

$$s^2 u^{\mathrm{L}}_{m,\,n}(s) = E[u^{\mathrm{L}}_{m+1,\,n}(s) + u_{m-1,\,n}(s) + u_{m,\,n+1}(s) + u_{m,\,n-1}(s) - 4u_{m,\,n}(s)], \tag{63}$$

where

$$E = \frac{1 + \alpha s}{1 + \beta s}. \tag{64}$$

Further, the Fourier discrete transform

$$u_n^{\mathrm{LF}}(s, k) = \sum_{m=-\infty}^{\infty} u_{mn}^{\mathrm{L}}(s) \mathrm{e}^{ikm} \tag{65}$$

of this equation leads to the equality

$$(h^2 + 2E) u_n^{\mathrm{LF}} - E(u_{n+1}^{\mathrm{LF}} + u_{n-1}^{\mathrm{LF}}) = 0. \tag{66}$$

Here and below the following notations are used:

$$h^2 = 2E(1 - \cos\ k) + s^2, \quad r^2 = h^2 + 4E. \tag{67}$$

Eq. (66) and zero conditions at infinity are satisfied by the expressions valid for symmetric strain of the lattice

$$u_n^{\mathrm{LF}} = u^{\mathrm{LF}} \lambda^n \quad (n > 0), \quad u^{\mathrm{F}} = u_0^{\mathrm{F}},$$

$$u_n^{\mathrm{LF}} = -u^{\mathrm{LF}} \lambda^{-n-1} \quad (n < -1), \quad \lambda = \frac{r - h}{r + h}. \tag{68}$$

Consider now the line $n = 0$. Let σ_m be the nondimensional stress that acts on the particle $(m, 0)$ from below. The dynamic equation for a particle lying on this line is

$$\left(1 + \beta \frac{\mathrm{d}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^2 u_{m,\,0}}{\mathrm{d}t^2} + \sigma_m\right)$$
$$= \left(1 + \alpha \frac{\mathrm{d}}{\mathrm{d}t}\right)(u_{m+1,\,0} + u_{m-1,\,0} + u_{m,\,1} - 3u_{m,\,0}). \tag{69}$$

From this it follows that

$$(\sigma_m)^{\mathrm{LF}} = \sigma_+ + \sigma_- = -(h^2 + E) u^{\mathrm{LF}} - E u_1^{LF} \tag{70}$$

or, using (68),

$$\sigma_+ + \sigma_- = -\frac{h(r + h)}{2} u^{\mathrm{LF}} = -\frac{h(r + h)}{2}(u_+ + u_-). \tag{71}$$

Here and below σ_+, u_+ are the double, Laplace and Fourier, transforms of the functions with the support at $m = 0, 1, \ldots$, while σ_-, u_- are those of the functions with the support at $m = -1, -2, \ldots$.

The viscoelastic law is

$$\sigma_+ = E\epsilon_+ = 2Eu_+. \tag{72}$$

Substituting u_+ from this into (71) we get the governing equation

$$\frac{L}{2E}\sigma_+ + u_- = -\frac{L-1}{2E}\sigma_-. \tag{73}$$

with

$$L(s,k) = \frac{r}{h}. \tag{74}$$

3.3. Factorization

The following explicit factorization of $L(s, k)$ is valid:

$$L(s,k) = L_+(s,k)L_-(s,k),$$

$$L_+ = \left[\frac{\sin(k/2 + i\ \mathrm{arcsinh}\sqrt{1+s^2/(4E)})}{\sin(k/2 + i\ \mathrm{arcsinh}(s/(2\sqrt{E}))}\right]^{1/2}$$

$$= \left[\frac{\sqrt{s^2+8E}\ \sin\ k/2 + i\sqrt{s^2+4E}\ \cos\ k/2}{\sqrt{s^2+4E}\ \sin\ k/2 + is\ \cos\ k/2}\right]^{1/2},$$

$$L_- = \left[\frac{\sin(k/2 - i\ \mathrm{arcsinh}\sqrt{1+s^2/(4E)})}{\sin(k/2 - i\ \mathrm{arcsinh}(s/(2\sqrt{E}))}\right]^{1/2}$$

$$= \left[\frac{\sqrt{s^2+8E}\ \sin\ k/2 - i\sqrt{s^2+4E}\ \cos\ k/2}{\sqrt{s^2+4E}\ \sin\ k/2 - is\ \cos\ k/2}\right]^{1/2}. \tag{75}$$

It can be seen that L_+ is a regular function in the upper half-plane of the complex variable k, while L_- is a regular function in the lower half-plane. Eq. (73) can now be presented as follows:

$$L_+\sigma_+ + \frac{2E}{L_-}u_- = C_+ + C_- + \frac{\sigma_-}{L_-}, \tag{76}$$

where C_+ and C_- correspond to functions with the support $m = 0, 1, \ldots$ and $m = -1, -2, \ldots$, respectively, and the sum is

$$C_+ + C_- = C = -L_+\sigma_-. \tag{77}$$

3.4. Division of the right-hand part

For such a division of C there exists a formula as a generalization of the Cauchy-type integral for a 2π-periodic function (see Eatwell and Willis, 1982; Slepyan, 1982a)

$$C_{\pm}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(\xi)\delta_{\pm}(\xi - k)\mathrm{d}\xi$$

$$\delta_{+}(k) = \sum_{n=0}^{\infty}\mathrm{e}^{-ikn} = (1 - \mathrm{e}^{-ik})^{-1} \quad (\mathrm{Im}\, k > 0),$$

$$\delta_{-}(k) = \sum_{n=-\infty}^{-1}\mathrm{e}^{-ikn} = \mathrm{e}^{ik}(1 - \mathrm{e}^{ik})^{-1} \quad (\mathrm{Im}\, k < 0). \tag{78}$$

It can be seen that in the limit, $\mathrm{Im}\, k \to 0$, the sum $\delta_{+}(k)+\delta_{-}(k)=\delta(k)$, while the functions $\delta_{+}(\xi-k)$ and $\delta_{-}(\xi-k)$ separately satisfy the required conditions at $k \to \pm i\infty$, respectively. When the separation is made, the Laplace transform of the strain of the bond, which connects the particles $m=n=0$ and $m=0$, $n=-1$, can be obtained as

$$\epsilon^{\mathrm{L}} = \lim_{k\to i\infty}\frac{C_{+}}{EL_{+}}. \tag{79}$$

As discussed above, constant external forces should be applied (say, at $t=0$) to the particles $n=0$, $m=-1$ and $n=-1$, $m=-1$ in opposite directions; let the force be $-\sigma_0$ for $n=0$. Thus,

$$\sigma_{-} = -\frac{\sigma_0}{s}\mathrm{e}^{-ik}. \tag{80}$$

Formula (78) for $k \to i\infty$ gives

$$C_{+} \sim \frac{1}{2\pi}\int_{-\pi}^{\pi} C(k)\mathrm{d}k = \frac{\sigma_0}{s}I, \quad I = \frac{1}{2\pi}\int_{-\pi}^{\pi} L_{+}(s, k)\mathrm{e}^{-ik}\,\mathrm{d}k. \tag{81}$$

The function L_{+} can be presented by a series as

$$L_{+} = \sum_{n=0}^{\infty} l_n(s)\mathrm{e}^{ikn} \tag{82}$$

and hence,

$$I = l_1 = \lim_{k\to i\infty}\mathrm{e}^{-ik}[L_{+}(s, k) - L_{+}(s, i\infty)]. \tag{83}$$

Further, for $k \to i\infty$,

$$L_+(s, k) - L_+(s, i\infty)$$

$$\sim \frac{L_-(s, i\infty)}{2}\left(\frac{\sqrt{s^2+4E}-s}{\sqrt{s^2+4E}+s} - \frac{\sqrt{s^2+8E}-\sqrt{s^2+4E}}{\sqrt{s^2+8E}+\sqrt{s^2+4E}}\right)e^{ik}. \quad (84)$$

3.5. Solution

The required Laplace transform can now be expressed as follows:

$$\epsilon^{\mathrm{L}}(s) = \frac{\sigma_0}{4E^2 s}[\sqrt{s^2+4E}(\sqrt{s^2+8E}-s) - 4E], \quad E = \frac{1+\alpha s}{1+\beta s}. \quad (85)$$

The limiting strain is

$$\lim_{t\to\infty}\epsilon \equiv \epsilon_\infty = \lim_{s\to 0} s\epsilon^{\mathrm{L}}(s) = (\sqrt{2}-1)\sigma_0. \quad (86)$$

Note that in terms of dimensional values it is

$$\epsilon_\infty = (\sqrt{2}-1)\frac{\sigma_0}{\kappa a}. \quad (87)$$

The SAR/DOR boundary can be found as that which separates a non-negative difference, $\Lambda = [\epsilon_\infty - \epsilon(t)]/\sigma_0$, namely, $\Lambda \geq 0$ $(0 \leq t < \infty)$ in the SAR domain, and Λ does not satisfy the inequality in the DOR domain. The Laplace transform of Λ is

$$\Lambda^{\mathrm{L}}(s) = \frac{1}{s}\left\{\sqrt{2} - 1 - \frac{1}{4E^2}[\sqrt{s^2+4E}(\sqrt{s^2+8E}-s) - 4E]\right\}. \quad (88)$$

This expression has the following singular points:

$$s = -s_\alpha, \quad s_\alpha = 1/\alpha,$$

$$s = -s_\beta, \quad s_\beta = 1/\beta \quad (89)$$

and the roots of the equations

$$(1+\beta s)s^2 + 4(1+\alpha s) \equiv (\beta s + s_{1,4})(s + s_{2,4} + i\Omega_4)(s + s_{2,4} - i\Omega_4),$$

$$(1+\beta s)s^2 + 8(1+\alpha s) \equiv (\beta s + s_{1,8})(s + s_{2,8} + i\Omega_8)(s + s_{2,8} - i\Omega_8). \quad (90)$$

Consider the case of real frequencies Ω_4 and Ω_8. The singular points

$$s_\alpha < s_{1,8}/\beta < s_{1,4}/\beta < s_\beta \quad (91)$$

correspond asymptotically ($t \to \infty$) to non-oscillatory exponentials, while the real parts of the remaining singular points

$$s_{2,\,4} < s_{2,\,8} \tag{92}$$

correspond to oscillatory exponentials. Note that inequalities (91) are still true in the MR domains, that is in the cases of imaginary frequencies, Ω_4 or/and Ω_8, as well as $-s_{1,4}$ and $-s_{1,8}$ are the minimal roots of the corresponding polynomials.

The necessary condition for the SAR is

$$s_\alpha \leq s_{2,\,4}. \tag{93}$$

Indeed, in the opposite case, the oscillatory exponential associated with $s_{2,4}$ will manifest itself in oscillations relative to the axis $\Lambda = 0$, at least, when time is sufficiently large.

For the determination of the interface in α, β-plane corresponding to the equality in (93) the following equations can be used [compare with (54)]:

$$s_{1,\,4} + 2\beta s_{2,\,4} = 1,$$

$$2\,s_{1,\,4}s_{2,\,4} + \beta(s_{2,\,4}^2 + \Omega_4^2) = 4\alpha,$$

$$s_{1,\,4}(s_{2,\,4}^2 + \Omega_4^2) = 4. \tag{94}$$

Substituting $s_{2,4} = s_\alpha$, we find the relation required as

$$\alpha = \frac{1 - 2\phi}{\sqrt{2(1 - 3\phi)}}, \quad \phi = \frac{\beta}{\alpha}. \tag{95}$$

In this dependence

$$\alpha = 1/\sqrt{2} \quad (\beta = 0), \quad \alpha_{\min} = 2/3 \quad (\beta = 1/9), \quad \phi \to 1/3 \quad (\alpha \to \infty). \tag{96}$$

The corresponding MR domain boundary ($\Omega_4 = 0$) can be determined in the same way as for the oscillator. The lower and the upper branches are described as [compare with (49)]

$$\beta^2 = \frac{1}{54}[\pm(1 - 12\alpha\beta)^{3/2} + 18\alpha\beta - 1], \tag{97}$$

respectively. In this dependence

$$\alpha = 1 \quad (\beta = 0), \quad \alpha_{\min} = \sqrt{3}/2 \quad (\beta = \beta_{\max} = 1/(6\sqrt{3})). \tag{98}$$

Dependencies (95) and (97) are shown in Fig. 5.

The MR domain boundary for Ω_8 follows from (97) by way of the uniform compression of the α, β-plane, namely, $\alpha_8 = \alpha_4/\sqrt{2}$, $\beta_8 = \beta_4/\sqrt{2}$, where the subscripts are used in accordance with the frequency subscripts. However, this boundary position does not influence the SAR/DOR domains interface, and only the fact that the MR domain for Ω_4 is contained in the above-determined SAR domain is important.

Numerical calculations show sufficiency of the above asymptotic analysis for the SAR/DOR interface determination and hence, the dependence (95) does represent this interface. Note that the difference $s_{2,4}-s_\alpha$ increases together with α, and hence, the SAR domain lies to the right from the interface. This concerns the phase-transition problem considered below as well. Some results obtained by the numerical inversion of the expression (88) are presented in Fig. 6 where $\Lambda=[\epsilon_\infty-\epsilon(t)]/\sigma_0$.

For the point $\alpha=1/\sqrt{2}$, $\beta=0$ of the interface (95), the expression (88) can be simplified dramatically. In this case, the original, $\Lambda(t)$, is

$$\Lambda=\int_0^t\left[\int_0^t\frac{J_1(\sqrt{2}\tau)}{\tau}(t_1-\tau)\mathrm{d}\tau\right]\mathrm{e}^{-\sqrt{2}t_1}\,\mathrm{d}t_1. \tag{99}$$

It can be seen that $\Lambda>0$ $(0\le t<\infty)$ and hence, this point belongs to the SAR domain.

The original of Λ^{L} for the elastic lattice ($E=1$) can be expressed explicitly as

$$\Lambda(t)=\sqrt{2}-\frac{J_1(\sqrt{8}t)}{\sqrt{2}t}+\sqrt{2}\int_0^t\left[J_1(2\tau)-2\int_0^\tau J_0(2\tau')\mathrm{d}\tau'\right]\frac{J_1(\sqrt{8}(t-\tau))}{t-\tau}\mathrm{d}\tau, \tag{100}$$

where J_0 and J_1 are the Bessel functions. This dependence is shown in Fig. 6(a) (curve 1). It can be seen that $\Lambda(t)$ becomes negative at a finite time and this does

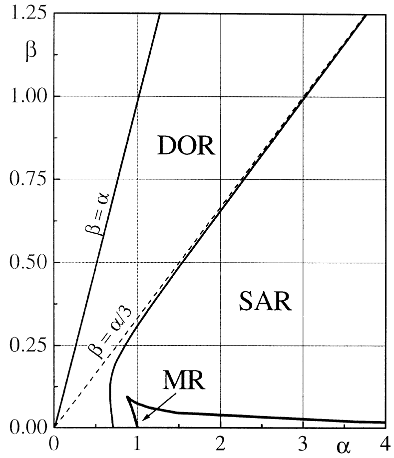

Fig. 5. The DOR, SAR and MR domains for the standard-material viscoelastic square-cell lattice.

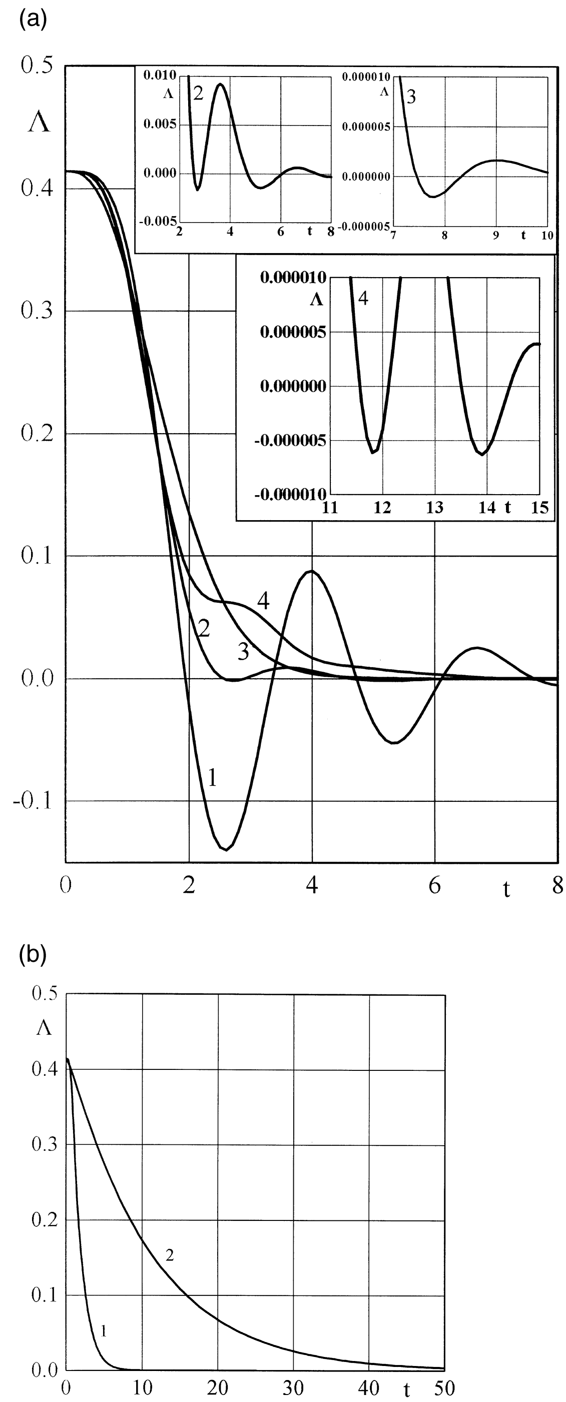

Fig. 6. Normalized strain of the crack-front bond in the viscoelastic lattice. (a) The DOR domain: 1. the elastic lattice response, $\alpha=\beta$; 2. $\alpha=0.5$, $\beta=0.25$; 3. $\alpha=0.5$, $\beta=0$; 4. $\alpha=1$, $\beta=0.5$. (b) The SAR domain: 1. $\alpha=1$, $\beta=0.25$; 2. $\alpha=10$, $\beta=0$.

not permit a crack to propagate slowly. In contrast, for a high viscosity, within the SAR domain but far away from its boundary, the crack-tip bond elongates slowly and approaches the limiting value long after the previous bond is broken [Fig. 6(b), curve 2].

4. Slow phase-transition wave in a chain

Consider a two-phase chain consisting of point particles of mass m, connected by massless standard-viscoelastic-material bonds, Fig. 7. In terms of Laplace transforms, the connection between the internal force, σ^{L}, and strain, ϵ^{L}, for an intact bond is

$$\sigma^{\mathrm{L}} = \mu_+ E\epsilon^{\mathrm{L}},\ \mu_+ = \kappa a, \quad E = \frac{1+\alpha s}{1+\beta s}. \tag{101}$$

At the moment when the strain first exceeds a critical value, ϵ_*, the modulus κ drops and the relation becomes

$$\sigma^{\mathrm{L}} = \mu_- E\epsilon^{\mathrm{L}}, \quad \mu_- = \gamma\kappa a, \quad \gamma < 1. \tag{102}$$

Equalities (101) and (102) reflect the two possible phases of the chain state.

As for the case of the crack in the lattice a slow phase-transition wave is considered here, the conditions which permit such a propagation are determined and the SAR/DOR domains are separated.

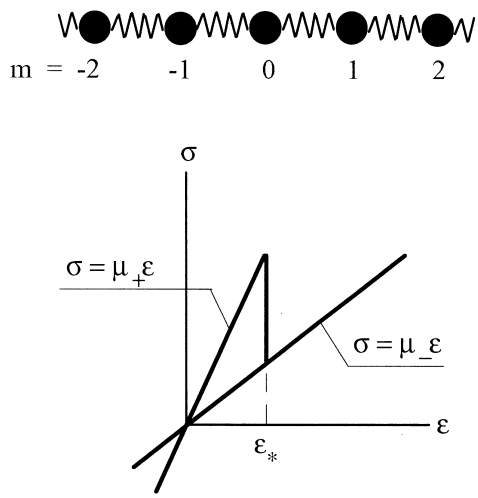

Fig. 7. The chain and the two-phase-bond force–strain relation shown for quasi-static deformation ($E = 1$).

4.1. Superposition

Consider a static state of the chain with the intact bonds in front of the particle $m=-1$ and the dropped-modulus bonds behind this particle. Let the chain be under the limiting force

$$\sigma = \kappa a \epsilon_*, \tag{103}$$

while the strain is

$$\epsilon = \epsilon_* \quad \text{(in front of the particle } m = -1\text{)},$$

$$\epsilon = \epsilon_*/\gamma \quad \text{(behind the particle } m = -1\text{)}. \tag{104}$$

We refer to this as state A. In parallel, consider the same stressed chain but with dropped modulus of the bonds to the left of the particle $n = 0$. Its state is the same as in the above chain if the force

$$p = (1-\gamma)\kappa a \epsilon_* \tag{105}$$

directed to the right is applied to the particle $m=-1$ and the same but opposite force is applied to the particle $m = 0$. We refer to this as state B. The elimination of these forces means that the jump in the modulus of the bond between the particle $m=-1$ and $m = 0$ in the state A occurs.

The dynamics of the chain under this jump can be presented by the superposition of solutions of two problems. The first problem is the initial state B of the stressed chain, while the second problem is the dynamics of the initially unstressed chain with dropped modulus behind the particle $m = 0$; this last chain is under the force (105) suddenly applied to the particle $m=-1$ and directed to the left, and the same but opposite force applied to the particle $m = 0$.

The problem in total corresponds to a very slow phase-transition wave when the time-interval between the jumps in the modulus of neighboring bonds is large enough to permit the lattice to approach the static state before the jump. The main question is whether such a wave can exist. The answer depends on the behavior of the next bond which connects the particles $m = 0$ and $m = 1$. A slow wave can exist only in the case where parameters α and β belong to the SAR domain. Clearly, the boundary of this domain is the same for the second and in-total problems and hence the solution can be obtained by means of the examination of the former.

4.2. Solution

In terms of the dimensional values, dynamic equations for the left, $m < -1$, and the right, $m > 0$, parts of the chain are

$$M\left(1+\beta\frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\mathrm{d}^2u_n}{\mathrm{d}t^2}=\gamma\kappa\left(1+\alpha\frac{\mathrm{d}}{\mathrm{d}t}\right)(u_{n+1}+u_{n-1}-2u_n)\quad(n<-1),$$

$$M\left(1+\beta\frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\mathrm{d}^2u_n}{\mathrm{d}t^2}=\kappa\left(1+\alpha\frac{\mathrm{d}}{\mathrm{d}t}\right)(u_{n+1}+u_{n-1}-2u_n)\quad(n>0),\tag{106}$$

while for $n=-1$ and $n=0$ they take the form

$$M\left(1+\beta\frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\mathrm{d}^2u_{-1}}{\mathrm{d}t^2}=\kappa\left(1+\alpha\frac{\mathrm{d}}{\mathrm{d}t}\right)[\gamma(u_{-2}-u_{-1})+u_0-u_{-1}]-p\quad(n=-1),$$

$$M\left(1+\beta\frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\mathrm{d}^2u_0}{\mathrm{d}t^2}=\kappa\left(1+\alpha\frac{\mathrm{d}}{\mathrm{d}t}\right)(u_1+u_{-1}-2u_0)+p\quad(n=0).\tag{107}$$

Below the nondimensional values are used. The time-unit is defined similarly as above: $1/\omega=\sqrt{M/\kappa}$, $p'=p/(\kappa a)$ (with the prime dropped) and the same normalization of the remaining variables is used as for the lattice. After taking the Laplace transformation with respect to time, these equations become

$$s^2u_n^{\mathrm{L}}=\gamma E(u_{n+1}^{\mathrm{L}}+u_{n-1}^{\mathrm{L}}-2u_n^{\mathrm{L}})\quad(n<-1),$$

$$s^2u_n^{\mathrm{L}}=E(u_{n+1}^{\mathrm{L}}+u_{n-1}^{\mathrm{L}}-2u_n^{\mathrm{L}})\quad(n>0),$$

$$s^2u_{-1}^{\mathrm{L}}=E[u_0^{\mathrm{L}}-u_{-1}^{\mathrm{L}}+\gamma(u_{-2}^{\mathrm{L}}-u_{-1}^{\mathrm{L}})]-\frac{p}{s}\quad(n=-1),$$

$$s^2u_0^{\mathrm{L}}=E(u_{-1}^{\mathrm{L}}+u_1^{\mathrm{L}}-2u_0^{\mathrm{L}})+\frac{p}{s}\quad(n=0).\tag{108}$$

A general solution can be expressed as follows:

$$u_n^{\mathrm{L}}=u_{-1}^{\mathrm{L}}\lambda_-^{-n-1}\quad(n\le-1),\qquad u_n^{\mathrm{L}}=u_0^{\mathrm{L}}\lambda_+^n\quad(n\ge0),$$

$$\lambda_-=1-\frac{s}{2\gamma E}(\sqrt{s^2+4\gamma E}-s),$$

$$\lambda_+=1-\frac{s}{2E}(\sqrt{s^2+4E}-s).\tag{109}$$

Substituting this into the inhomogeneous equations of system (108) we find the solution as

$$u_1^{\mathrm{L}}-u_0^{\mathrm{L}}=-\frac{8p(\sqrt{s^2+4E\gamma}+s)}{(\sqrt{s^2+4E\gamma}+\sqrt{s^2+4E})(\sqrt{s^2+4E}+s)^3}.\tag{110}$$

The elongation considered first becomes negative and corresponds to the SAR domain if it remains non-positive all the time.

Singular points of expression (110) are $s = -s_\alpha = 1/\alpha$ and the roots of the equations [compare with (90)]:

$$(1+\beta s)s^2 + 4(1+\alpha s) \equiv (\beta s + s_{1,\,4})(s + s_{2,\,4} + i\Omega_4)(s + s_{2,\,4} - i\Omega_4),$$

$$(1+\beta s)s^2 + 4\gamma(1+\alpha s) \equiv (\beta s + s_{1,\,4\gamma})(s + s_{2,\,4\gamma} + i\Omega_{4\gamma})(s + s_{2,\,4\gamma} - i\Omega_{4\gamma}). \quad (111)$$

For the case of real frequencies, Ω_4 and $\Omega_{4\gamma}$,

$$s_\alpha < s_{1,\,4} < s_{1,\,4\gamma}, \quad s_{2,\,4\gamma} < s_{2,\,4}. \quad (112)$$

The roots satisfy the Eqs. (94) and the following ones:

$$s_{1,\,4\gamma} + 2\beta s_{2,\,4\gamma} = 1,$$

$$2\, s_{1,\,4\gamma} s_{2,\,4\gamma} + \beta(s_{2,\,4\gamma}^2 + \Omega_{4\gamma}^2) = 4\gamma\alpha,$$

$$s_{1,\,4\gamma}(s_{2,\,4\gamma}^2 + \Omega_{4\gamma}^2) = 4\gamma. \quad (113)$$

The SAR/DOR interface can be determined in the same way as above, namely, proceeding with the equality $s_{2,4\gamma} = s_\alpha = 1/\alpha$. The following parametric dependence for the interface follows from this and Eqs. (113):

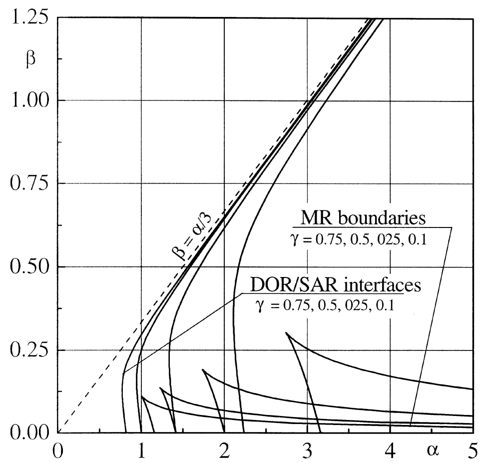

Fig. 8. The SAR, DOR and MR domains for the standard-material-spring viscoelastic chain.

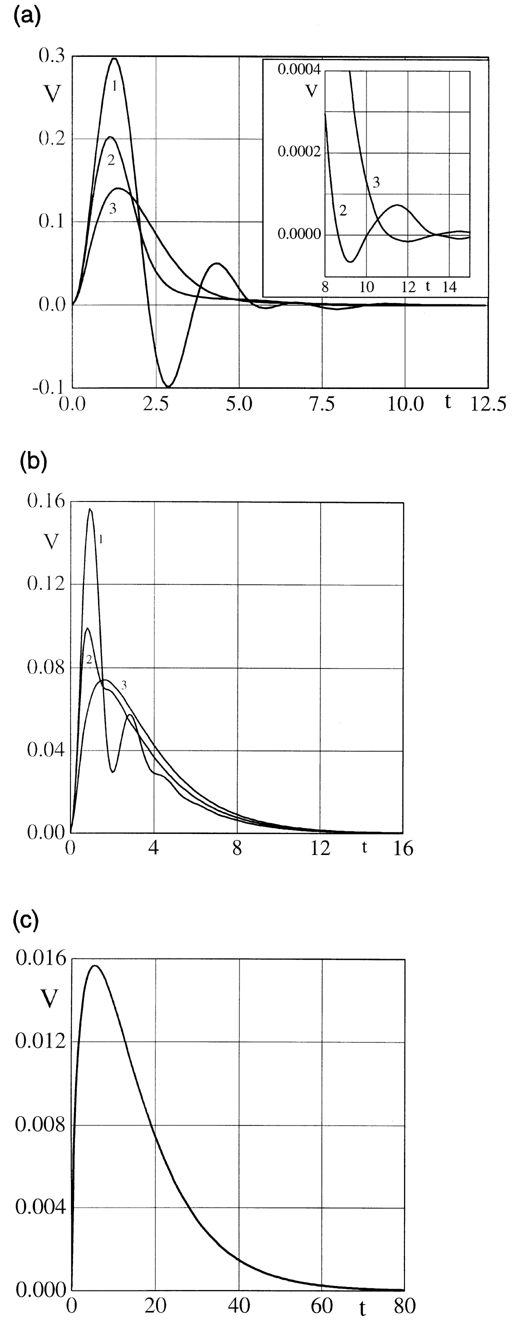

Fig. 9. The response of the two-phase viscoelastic chain ($\gamma = 0.25$). (a) The DOR domain: 1. $\alpha = 1$, $\beta = 0.5$; 2. $\alpha = 1$, $\beta = 0.25$; 3. $\alpha = 1$, $\beta = 0$. (b) The SAR domain: 1. $\alpha = 2$, $\beta = 0.5$; 2. $\alpha = 2$, $\beta = 0.25$; 3. $\alpha = 2$, $\beta = 0$. (c) The SAR domain: $\alpha = 10$, $\beta = 0$.

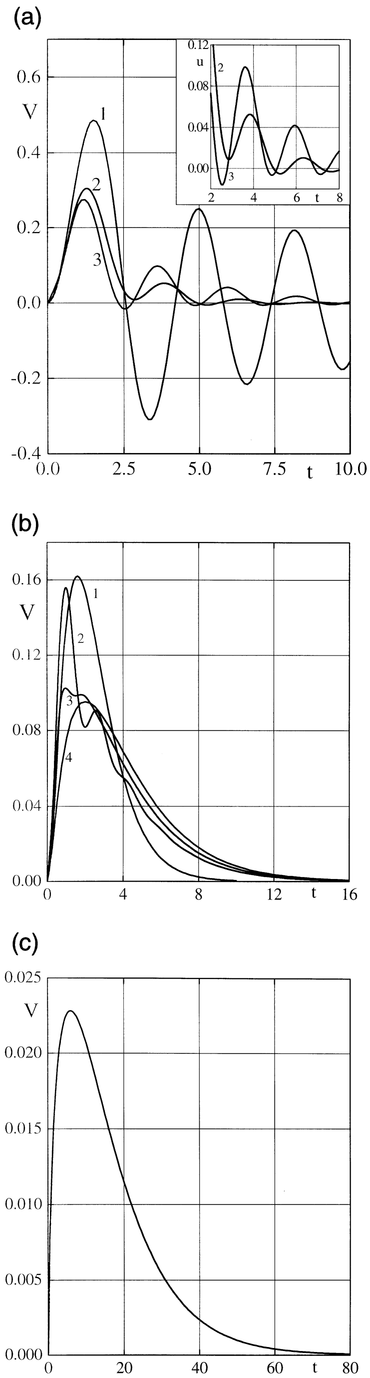

Fig. 10. The response of the two-phase viscoelastic chain ($\gamma = 1$). (a) The DOR domain: 1. the elastic chain, $\alpha = \beta$; 2. $\alpha = 1$, $\beta = 0.5$; 3. $\alpha = 2$, $\beta = 1$. (b) The SAR domain: 1. $\alpha = 1$, $\beta = 0$; 2. $\alpha = 2$, $\beta = 0.5$; 3. $\alpha = 2$, $\beta = 0.25$; 4. $\alpha = 2$, $\beta = 0$. (c) The SAR domain: $\alpha = 10$, $\beta = 0$.

$$\alpha = \frac{1 - 2\beta/\alpha}{\sqrt{2\gamma(1 - 3\beta/\alpha)}}. \tag{114}$$

This dependence is valid for all over the range $0 \leq \beta/\alpha < 1/3$ because the 4γ-MR domain [the MR domain corresponding to the second polynomial in (111)] is contained in the SAR domain lying to the right of the boundary (114). Indeed, the 4γ-MR domain boundary is defined by the relation

$$\beta^2 = \frac{1}{54\gamma}[18\gamma\alpha\beta - 1 \pm (1 - 12\gamma\alpha\beta)^{3/2}] \tag{115}$$

which can be obtained from the corresponding dependence for the oscillator (49) [also see (97)] by a linear transformation $\alpha \to \sqrt{\gamma}\alpha$, $\beta \to \sqrt{\gamma}\beta$. This dependence is characterized by the following limiting points:

$$\alpha = \frac{1}{\sqrt{\gamma}} \quad (\beta = 0), \quad \alpha = \alpha_{\min} = \frac{1}{2}\sqrt{\frac{3}{\gamma}} \quad \left(\beta = \beta_{\max} = \frac{1}{6}\frac{1}{\sqrt{3\gamma}}\right) \tag{116}$$

and these points are placed to the right from the boundary (114). The SAR, DOR and MR domains for a set of γ are shown in Fig. 8. Note that the corresponding result for $\gamma = 1$ is the same as for the lattice (Fig. 5). Some results obtained by the numerical inversion of the expression (110) are presented in Figs. 9 and 10 where $V = (u_0 - u_1)/p$.

For a small jump of the modulus, $\gamma \to 1$, a limit of the ratio, $V = V_1$, exists which can be used for the SAR/DOR interface asymptotic determination. This limit is [see (105)]

$$V_1 = \frac{4}{\sqrt{s^2 + 4E}(\sqrt{s^2 + 4E} + s)^2}. \tag{117}$$

For the elastic chain ($E = 1$) this involves

$$V_1 = J_2(2t) \tag{118}$$

where J_2 is the Bessel function. This function becomes negative at a finite time [$t \approx 2.6$; see Fig. 10(a), curve 1], and hence, contrary to the SAR domain, a slow wave cannot exist.

5. Conclusions

1. In this paper, a phase-transition wave and crack propagation are considered for some standard-viscoelastic-material systems, and the existence of the static-amplitude-response (SAR) and dynamic-overshoot-response (DOR) domains are shown. If the viscosity parameters belong to the latter, in particular, in the elastic case, the dynamic overshoot phenomenon leads to a fast wave or a fast

crack propagation. In a sense, such a process is similar to detonation. Indeed, an excess in the energy release under the phase transformation (or fracture) arises during a time-interval inherent for the element — as well as in detonation. This results in the activation of a neighboring element, that is, in the fast spread of the transformation. In particular, this phenomenon can manifest itself in a positive difference between the initiation and dynamic stress intensity factors in fracture. In contrast, in the case of the SAR domain where there is not any dynamic overshoot, the considered wave or a crack can propagate slowly, because there is no excess in the energy release. In this case, especially for high viscosity, in the heart of the SAR domain, the element activation or fracture takes a long time: the strain increases slowly and reaches the critical value with large delay. Thus, the SAR domain more likely corresponds to slow combustion or melting rather than detonation. In this sense, one can conclude that brittle materials behave as if they are in the DOR regime, while ductile materials correspond to the SAR domain. Note that the asymptote of the DOR/SAR interface, $\beta/\alpha \sim 1/3$ ($\alpha \to \infty$), is the same for the viscoelastic oscillator, lattice and two-phase chain considered in this paper.

2. For an elastic body, some possibilities to avoid oscillations under an impact and to eliminate the dynamic overshoot under a suddenly applied load are shown. This can be achieved by a proper control of the dynamic load in an initial portion of the loading period using a specifically designed shock absorber. Such a possibility can be important in the application to the design of a shock-proof structure and in some other applications, for example, in small-scale experiments where a short-term high-level acceleration is required to model the influence of gravity. At the same time, such a control can be applicable to the structure design to obtain a structure which can belong to the SAR (or DOR) domain with respect to fracture or phase transition. This could be achieved, in principle, by the creation of a proper post-peak tensile softening of the structure element. This goal can also be achieved by a combined influence of the stress/strain law control and viscosity.
3. There exists a structure-associated size effect in the SAR/DOR domains separation. Indeed, consider two samples with the same density $\rho = M/a^2$, modulus κ and viscosity times α and $\beta < \alpha$, but with different sizes of the lattice cell. They must show the same properties in macro-level dynamic deformation, but not in fracture or phase transition where only the nondimensional values, $\alpha\omega$ and $\beta\omega$, are important. The frequency $\omega = \sqrt{\kappa/M} = c/a$, where the shear wave velocity, $c = \sqrt{\kappa/\rho}$, is the same for both samples. Thus, the nondimensional relaxation/creep times are as large as the structure size, a, is small. Consequently, the sample with a smaller structure size can belong to the SAR domain, while another sample can belong to the DOR domain. (Under the same conditions a coarse-grained material appears to be more brittle as it should.)
4. The manifestation of the dynamic factor in fracture was considered by means of an example as the fracture Mode III in a standard-material viscoelastic square-cell lattice. At the same time, it is clear that the phenomenon exists in a

general case, although the SAR/DOR domains interface depends, of course, on the material structure and the fracture mode. In this connection, note that the transient-problem solutions (51), (88) and (110) are valid for any type of viscosity, that is, for any expression for the complex modulus, $E = E(s)$ ($E(0) = 1$).

5. In this work, possibilities of slow propagation are examined, and a straight crack, as the bond breakage between two neighboring lines of particles, is considered. In contrast, in the case of the DOR domain, when only a fast crack can exist, the excess of the energy release can lead to the breakage of other bonds near the mentioned crack line which results in roughness of the crack surfaces, oscillations in the crack velocity and an increase of the resistance to the crack propagation.
6. It should be noted that the phenomena considered in this paper cannot manifest themselves, at least, cannot be visible, in the case where no inherent size unit exists, as for example, in a non-structured homogeneous viscoelastic plane with a semi-infinite crack.
7. In the determination of the SAR/DOR interface, it was seen that an 'oscillation conservation law' is valid, namely, if the overshoot exists the corresponding oscillations do not disappear in time although their amplitude can decrease exponentially under the influence of viscosity. This fact allowed one to derive an analytical description for the interface in each problem considered based on an asymptotic analysis.

Acknowledgements

This research was supported by Grant No. 9673-3-98 from the Ministry of Science, Israel.

References

Eatwell, G.P., Willis, J.R., 1982. The excitation of a fluid-loaded plate stiffened by a semi-infinite array of beams. IMA J. Appl. Mech. 29 (3), 247–270.

Fineberg, J., Gross, S.P., Marder, M., Swinney, H.L., 1991. Instability in dynamic fracture. Phys. Rev. Lett. 67 (4), 457–460.

Fineberg, J., Gross, S.P., Marder, M., Swinney, H.L., 1992. Instability in the propagation of fast cracks. Phys. Rev. B45 (10), 5146–5154.

Kulakhmetova, ShA., Saraikin, V.A., Slepyan, L.I., 1984. Plane problem of a crack in a lattice. Mechanics of Solids 19 (3), 101–108.

Marder, M., 1991. New dynamical equation for cracks. Phys. Rev. Lett. 66, 2484–2487.

Marder, M., Gross, S., 1995. Origin of crack tip instabilities. J. Mech. Phys. Solids 43, 1–48.

Marder, M., Xiangmin, Liu, 1993. Instability in lattice fracture. Phys. Rev. Lett. 71 (15), 2417–2420.

Slepyan, L.I., 1981a. Dynamics of a crack in a lattice. Sov. Phys. Dokl. 26, 538–540.

Slepyan, L.I., 1981b. Crack propagation in high-frequency lattice vibrations. Sov. Phys. Dokl. 26 (9), 900–902.

Slepyan, L.I., 1982a. Antiplane problem of a crack in a lattice. Mechanics of Solids 17 (5), 101–114.

Slepyan, L.I., 1982b. The relation between the solutions of mixed dynamic problems for a continuous elastic medium and a lattice. Sov. Phys. Dokl. 27 (9), 771–772.

Slepyan, L.I., 1984. Dynamics of brittle fracture in media with a structure. Mechanics of Solids 19 (6), 114–122.

Slepyan, L.I., 1986. Dynamics of brittle fracture in media with a structure — nonhomogeneous problems. In: Mathematical Methods of Mechanics of Solids. Nauka, Moscow, pp. 143–149 (in Russian).

Slepyan, L.I., 1990. Mechanics of Cracks, 2nd ed. Sudostroenie, Leningrad 1st ed.: 1981 — in Russian.

Slepyan, L.I., 1993. Role of microstructure in fracture and phase transition dynamics. In: Proceedings of the International Seminar 'Mecamat'93: Micromechanics of Materials', Eyrolles, Paris, pp. 246–257.

Slepyan, L.I., 1998. Some basic aspects of crack dynamics. In: Cherepanov, G. (Ed.), Fracture: A Topical Encyclopedia of Current Knowledge Dedicated to Alan Arnold Griffith. Krieger, Melbourne, USA, pp. 620–661.

Slepyan, L.I., Troyankina, L.V., 1984. Fracture wave in a chain structure. J. Appl. Mech. Techn. Phys. 2596, 921–927.

Slepyan, L.I., Troyankina, L.V., 1988. Shock waves in nonlinear chain. In: Strength and Visco-plasticity. Nauka, pp. 301–305 (in Russian).

Slepyan, L.I., Ayzenberg-Stepanenko, M.V., Dempsey, J.P., 1999. A lattice model for viscoelastic fracture. Mechanics of Time-Dependent Materials, in press.