Forerunning mode transition in a continuous waveguide

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ABSTRACT

We have discovered a forerunning mode transition as the periodic wave changing the state of a uniform continuous waveguide. The latter is represented by an elastic beam initially rested on an elastic foundation. Under the action of an incident sinusoidal wave, the separation from the foundation occurs propagating in the form of a *transition wave*. The critical displacement is the separation criterion. Under these conditions, the steady-state mode exists with the transition wave speed independent of the incident wave amplitude. We show that such a regime exists only in a bounded domain of the incident wave parameters. Outside this domain, for higher amplitudes, the steady-state mode is replaced by a set of local separation segments periodically emerging at a distance ahead of the main transition point. The crucial feature of this waveguide is that the incident wave group speed is greater than the phase speed. This allows the incident wave to deliver the energy required for the separation. The analytical solution allows us to show in detail how the steady-state mode transforms into the forerunning one. The latter studied numerically turns out to be periodic. As the incident wave amplitude grows the period decreases, while the transition wave speed averaged over the period increases to the group velocity of the wave. As an important part of the analysis, the complete set of solutions is presented for the waves excited by the oscillating or/and moving force acting on the free beam. In particular, an asymptotic solution is evaluated for the resonant wave corresponding to a certain relation between the load's speed and frequency.

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1. Introduction

An unusual *transition wave* considered in this paper belongs to the class of processes like phase transition or crack growth (which also can be considered as a phase transition (Truskinovsky, 1996)), or other similar events, where a change of the body structure or state spreads as a wave. The transition requires a certain energy, and such a wave can propagate being forced by the action of external loads or spontaneously drawing energy stored in the waveguide (Mishuris and Slepyan, 2014; Ayzenberg-Stepanenko et al., 2014). The transition wave in an elastic beam initially rested on a continuous or a discrete elastic foundation (Brun et al., 2013) can be considered as an example. In this work, a (negative) jump in the foundation stiffness propagates under gravity forces as a steady-state wave. (This was a theoretical base related to a bridge collapse, Brun et al., 2014.)

In this paper, the separation from the foundation under a sinusoidal incident wave is considered. Two ordered regimes of the transition wave are found: steady-state and forerunning. Below, in the Introduction, the main features of such processes

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are outlined and then the physical formulation of the considered problem is given.

The formulation of such a problem includes the dynamic equations for the waveguide in its initial and final states and, in addition, relations for the transition from the former to the latter. Note that the transition can occur instantaneously or during a period. The former mode of transition adopted in this paper is simpler for mathematical analysis. In this case, the waveguide is separated by a moving transition point (or an interface) into two parts, the intact part is placed in front of this point, while the modified part is placed behind this point. Note that generally, in the framework of a continuous waveguide, the formulation of the transition criterion is not trivial (Slepyan, 2002).

In analytical studies, transition waves are commonly examined under the steady-state formulation, assuming that the dynamic state depends on $\eta = x - Vt$ but not on x and t separately (x is the coordinate, t is time and V is the transition wave speed). The steady-state solution is based on the above-mentioned equations, the external load or/and conditions at infinity and some local conditions concerning the transition: a critical displacement, force or an energy release. An additional inequality, also local, can be needed in the case of a wave action. These equations and conditions can define the solution uniquely. Alternatively, a set of solutions is defined by such a formulation. In this latter case, some nonlocal conditions should be involved.

Mathematically, these considerations are sufficient. However, from the physical point of view it is not so. The main question is whether the steady-state regime can exist. This question arises due to the fact that the solution must satisfy certain additional, nonlocal conditions concerning the waveguide state at $\eta \neq 0$.

This may be a kinematic condition behind the transition point, $\eta < 0$, which can restrict the displacements, and a condition in front of the transition point, $\eta > 0$, which restricts a state parameter level to be subcritical (in accordance with the formulation, it becomes critical only at $\eta = 0$).

The latter condition is crucial for the problem under consideration. It is known as the *admissibility condition* (Marder and Gross, 1995), which states, in general, that the transition criterion should not be satisfied at $\eta > 0$, that is before the moment assumed in the problem formulation. Note that this condition was formulated for the lattice fracture; however, it is valid in a general case. This condition allows one to select a consistent solution from the set defined by the steady-state formulation. It also can essentially bound the region of existence of the steady-state regime.

As an example, the first analytical work on the lattice fracture (Slepyan, 1981a) can be mentioned. In this paper, the macrolevel energy release rate is obtained as a function of two parameters, the microlevel fracture energy and the crack speed. The corresponding micro-to-macro energy ratio is obtained as a non-monotonic function of the speed, such that there are some different crack speeds corresponding to a given macrolevel energy release rate. The Marder–Gross condition results in the conclusion that only the highest crack speed is admissible (if no fracture occurs outside of the prospective crack line).

In the context of the considered problem, another point is important. There is an essential difference in steady-state transition excited by a non-oscillating and oscillating incident waves. Under the action of a non-oscillating wave, the speed of the transition point increases approaching the incident wave group velocity as the action of the force increases. In contrast, in the steady-state transition under a sinusoidal incident wave, the speed of the transition point coincides with the phase speed of the incident wave regardless of the wave amplitude. In this case, as long as an energy release is required for the transition, the incident wave group speed must exceed the phase speed (since the energy flux velocity is equal to the group velocity). It can be seen below that under a sufficiently large wave amplitude, the transition point speed is below the group speed but faster than the phase speed of the incident wave. However, this appears in a different transition wave mode: it is not steady-state any longer.

The analytical solutions for fracture under a sinusoidal wave were presented by Slepyan (1981b, 2010) for a lattice and a continuous body, respectively. The papers most related to the considered issue, how the steady-state regime is replaced by a more complicated ordered mode, relate to the lattice fracture dynamics, Mishuris et al. (2009) and Slepyan et al. (2010). It was first found and discussed in these papers that there exist ordered clustering regimes changing as the incident wave amplitude grows. In a bounded wave amplitude region, the crack speed is equal to the incident wave phase speed, and the steady-state regime is valid. In this regime, the lattice bonds on the crack path break one after another at regular time intervals. Then, as the wave amplitude exceeds the critical level, the two-bond clustering occurs with two alternating values of the local crack speed. The crack speed averaged over the cluster again is constant but greater than the phase speed of the incident wave. In the further growth of the wave amplitude, the number of the bonds in the cluster increases, while the averaged crack speed is constant in each corresponding wave amplitude region. As the wave amplitude grows this averaged-over-the-cluster crack speed approaches the group speed of the incident wave. Such a clustering was also observed in the spontaneous crack propagation in a two-line chain with internal potential energy, Ayzenberg-Stepanenko et al. (2014). Note that the transition waves in lattices were considered in many works, see, e.g., Slepyan and Ayzenberg-Stepanenko (2004), Slepyan et al. (2005), Vainchtein (2010) and the references herein.

In the present work, we consider a beam initially rested on an elastic foundation and subjected to the action of a sinusoidal wave. Under this action, the separation of the beam from the foundation propagates as a transition wave. The aim is to find, in such a continuous waveguide, the domain where the steady-state regime can exist and to study the nontrivial transition mode existing outside this domain. In this simple model, the transformation of the steady-state mode into more complicated forerunning mode can be observed in detail. The ordered forerunning mode manifests itself as a periodic process as in the case of the above-mentioned clustering mode in the lattice; however, the forerunning mode differs much from the latter. The main peculiarity is the periodic occurrence of detachment segments in front of the main transition wave

at a distance of it (see Fig. 5). As the incident wave amplitude grows the transition wave period decreases, while the separation wave speed averaged over the period increases to the group velocity of the incident wave. In contrast to the discrete structures, both these parameters continuously depend on the wave amplitude. Recall that the steady-state transition wave propagating in the same and some other structures under gravity forces was studied in Brun et al. (2013).

The paper is organized as follows. In the next section, we present the complete set of relations for the waves excited in the free beam by the moving-oscillating force. The relations between the wave parameters and the amplitude, speed and frequency of the force are shown including those for the resonant case corresponding to a certain relation between the load's speed and frequency. The results presented in this section provide a basis for the examination of the main problem considered in Sections 3.2 and 3.3.

Note that Sommerfeld (1904, 1905), first considered the moving source actions, not to mention the well-known Vavilov–Cherenkov effect as the electromagnetic radiation emitted when an electron passes through a dielectric medium (see Landau et al., 1984). Different problems related to the moving load are considered in the books by Slepyan (1972) and Friba (1999), see also Cai et al. (1988).

Next, in Section 3, the analytical solution for the steady-state transition under the sinusoidal incident wave is presented. The bounds of the domain in the wave speed-amplitude plane are determined where the steady-state mode can exist. It is shown how the forerunning mode arises when the wave amplitude reaches the upper bound of the domain.

Further, in Section 3.3, the forerunning mode is studied numerically in the framework of the transient problem. The transition wave was assumed to be excited by a moving non-oscillating force. The corresponding wave amplitude was determined based on the relations in Section 2. For the steady-state regime, a good agreement was found between the numerical and analytical results. The numerical simulations conducted for the higher wave amplitudes, outside the steady-state domain, revealed that the transition wave fast approaches the ordered forerunning mode. The latter was studied in detail.

Finally, discussions and conclusions are presented.

2. Flexural waves excited by a moving-oscillating force in a free beam

2.1. The equation, dispersion relations and waves

We based on the Bernoulli–Euler equation for an elastic beam under moving-oscillating external force

$$EI\frac{\partial^4 w(x, t)}{\partial x^4} + \varrho S\frac{\partial^2 w(x, t)}{\partial t^2} = P\delta(x - vt)e^{i\omega t}, \tag{1}$$

where E is the elastic modulus, I and S are the cross-section moment of inertia and the area, ϱ is the density of the beam material, w is the transverse displacement, x and t are the coordinate and time, P is the amplitude of the force moving along the beam with constant speed v and oscillating with frequency ω and δ is the Dirac delta function.

Note that everywhere in this paper, v and ω are the speed of the external force and its frequency, whereas V and Ω are the phase speed and frequency of the sinusoidal wave excited by the force. In the case of a non-oscillating force, $V = v$, and for an unmoving force $\Omega = \omega$. In the steady-state regime, the transition wave speed coincides with the incident wave phase speed. The phase and group speeds are defined as

$$V = \frac{\Omega}{k}, \quad V_g = \frac{d\Omega}{dk} \quad \left(k = \frac{2\pi}{\lambda}\right), \tag{2}$$

where k and λ are the wavenumber and wavelength, respectively. These speeds may differ in both magnitude and direction. The group speed coincides with the energy flux speed in the wave and hence represents the upper bound for the transition wave speed (if the incident sinusoidal wave is the only source of the energy delivering to the transition point).

We take $r = \sqrt{I/S}$ and $c = \sqrt{E/\varrho}$ as natural length and speed units (accordingly, r/c is the time unit), and ES as the force unit. In terms of the corresponding non-dimensional variables (we use the same notations), the equation becomes

$$\frac{\partial^4 w(x, t)}{\partial x^4} + \frac{\partial^2 w(x, t)}{\partial t^2} = P\delta(x - vt)e^{i\omega t}. \tag{3}$$

The force excites sinusoidal waves, whose frequencies, Ω, and wave numbers, k, are defined by the dispersion and Doppler relations

$$\Omega = \pm k^2, \quad \Omega = \omega + kv. \tag{4}$$

It follows from this that

$$k_{1,2} = -\frac{1}{2}\left(v \pm \sqrt{v^2 - 4\omega}\right)(v^2 \geq 4\omega), \quad k_{1,2} = -\frac{1}{2}\left(v \pm i\sqrt{4\omega - v^2}\right)(v^2 \leq 4\omega),$$

$$k_{3,4} = \frac{1}{2}\left(v \mp \sqrt{v^2 + 4\omega}\right),$$

(5)

where the nonzero real k corresponds to the sinusoidal waves with the phase and group speeds

$$V = \pm k, \quad V_g = 2V.$$

(6)

The wave propagates in front of the moving-oscillating force, at $\eta > 0$, if its group velocity exceeds v; otherwise, it is placed at $\eta < 0$.

The dispersion diagram and the Doppler rays (4) are shown in Fig. 1 for different speeds, v, for a value of $\omega > 0$ and for $\omega = 0, v > 0$.

2.2. Wave amplitudes

We consider the steady-state solution as the limit of the transient solution corresponding to zero initial conditions. We derive it directly starting from the Laplace and Fourier transforms on t and x respectively. It follows from (3) that

$$w^{LF}(k, s) = \frac{P}{[s - i(\omega + kv)](k^4 + s^2)},$$

$$w(x, t) = \frac{1}{4\pi^2 i}\int_{-i\infty+0}^{i\infty+0}\int_{-\infty}^{\infty}\frac{Pe^{-ikx+st}}{[s - i(\omega + kv)](k^4 + s^2)}.$$

(7)

Substituting $x = vt + \eta, s = s' + i(\omega + kv)$ we obtain

$$w^{LF_\eta}(k, s') = e^{i\omega t}W^{LF_\eta}(k, s'), \quad W^{LF_\eta}(k, s') = \frac{P}{s'[k^4 + (s' + i(\omega + kv))^2]},$$

(8)

where the superscript F_η denotes the Fourier transform on η. Assuming that the limit exists we find
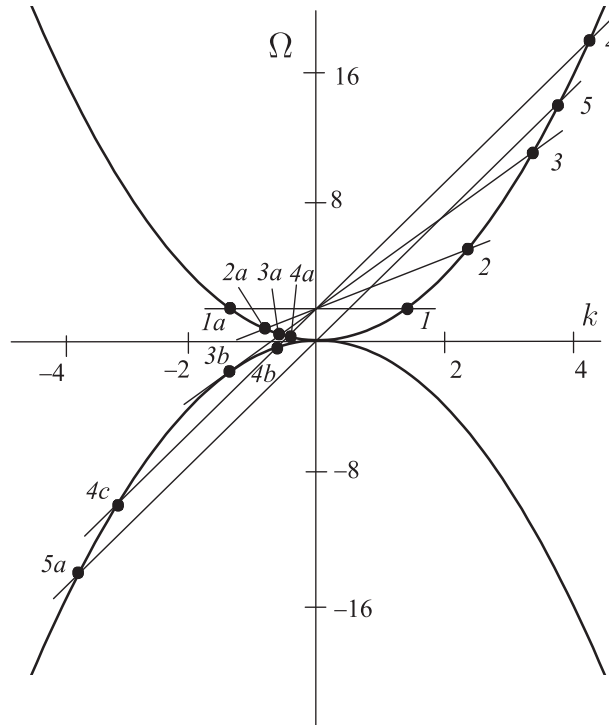
Fig. 1. Dispersion diagram for the free beam, $\Omega = \pm k^2$, and the Doppler rays, which intersections with the dispersion curves, marked by 1, 1a, ..., 5, 5a, correspond to the wave numbers, k, and frequencies, Ω, of the sinusoidal waves excited by the force moving with speed v and oscillating with frequency ω. The rays correspond to: an unmoving force, $\Omega = \omega > 0, v = 0$ (1–1a), to a subcritical speed, $\omega > 0, 0 < v < v_c$ (2–2a), to the resonant regime, $\omega > 0, v = v_c = 2\sqrt{\omega}$ (3–3a–3b), to a supercritical speed, $\omega > 0, v > v_c$ (4–4a–4b–4c) and to a non-oscillating moving force, $\omega = 0, v > 0$ (5–5a). The latter ray has also an intersection with both dispersion curves at the origin, $\Omega = k = 0$.

$$W^{F_\eta}(k) = \lim_{s' \to +0} s' W^{LF_\eta}(k, s') = \frac{P}{k^4 - (\omega + kv - i0)^2}. \tag{9}$$

A complete steady-state solution follows from (9) as a sum of four residues in the Fourier inverse transform. The waves propagating to the right (left) are defined by the real zeros of the denominator in (9) coming to the real axis from below, $k = k_i - i0$ (from above, $k = k_i + i0$). It follows that for $v < 2\sqrt{\omega}$ the waves are defined as

$$W(\eta) = -\frac{1}{4}\left(\frac{ie^{-ik_4\eta}}{\Omega_4\sqrt{\omega + v^2/4}} + \frac{e^{-ik_1\eta}}{\Omega_1\sqrt{\omega - v^2/4}}\right)PH(\eta),$$

$$W(\eta) = -\frac{1}{4}\left(\frac{ie^{-ik_3\eta}}{\Omega_3\sqrt{\omega + v^2/4}} + \frac{e^{-ik_2\eta}}{\Omega_2\sqrt{\omega - v^2/4}}\right)PH(-\eta), \tag{10}$$

where

$$\Omega_i = \omega + k_i v, \quad k_{1,2} = -\frac{1}{2}\left(v \pm i\sqrt{\omega - v^2/4}\right), \quad k_{3,4} = \frac{1}{2}\left(v \mp \sqrt{\omega + v^2/4}\right), \tag{11}$$

and $H(\eta)$ is the Heaviside step function, $H(\eta) = 1$ $(\eta > 0)$, $H(\eta) = 0$ $(\eta < 0)$. In these relations, only the first terms correspond to the sinusoidal waves (because $k_{1,2}$ are complex).

There is no steady-state mode at least in the resonant excitation, $v = 2\sqrt{\omega}$. The transient problems are considered below separately.

For $v > 2\sqrt{\omega}$ two waves propagate in each direction (in this case, all wavenumbers are real)

$$W(\eta) = -\frac{i}{4}\left(\frac{e^{-ik_4\eta}}{\Omega_4\sqrt{v^2/4 + \omega}} + \frac{e^{-ik_1\eta}}{\Omega_1\sqrt{v^2/4 - \omega}}\right)PH(\eta),$$

$$W(\eta) = -\frac{i}{4}\left(\frac{e^{-ik_3\eta}}{\Omega_3\sqrt{v^2/4 + \omega}} + \frac{e^{-ik_2\eta}}{\Omega_2\sqrt{v^2/4 - \omega}}\right)PH(-\eta), \tag{12}$$

where

$$\Omega_i = \omega + k_i v, \quad k_{1,2} = -\frac{v}{2} \mp \sqrt{v^2/4 - \omega}, \quad k_{3,4} = \frac{v}{2} \mp \sqrt{v^2/4 + \omega}. \tag{13}$$

In a special case on a non-oscillating moving force, $\omega = 0$, $v > 0$, referring to (9) we have

$$W^{F_\eta}(k) = \frac{P}{(k - v + i0)(k + v + i0)(k + i0)^2}, \tag{14}$$

and

$$W(\eta) = -\frac{P \sin v\eta}{v^3}H(\eta) - \frac{P\eta}{v^2}H(-\eta). \tag{15}$$

Note that this result also follows from (12) as the limit at $\omega = +0$.

In a particular case of an unmoving oscillation force

$$k_1 = -k_2 = i\sqrt{\omega}, \quad k_4 = -k_3 = \sqrt{\omega}, \quad \Omega_{3,4} = \omega, \tag{16}$$

and it follows from (10) that

$$W(\eta) = -\frac{P}{4\omega^{3/2}}\left(ie^{-i\sqrt{\omega}|\eta|} + e^{-\sqrt{\omega}|\eta|}\right). \tag{17}$$

Note that in the transition to dimensional quantities, one should make replacements in accordance with the above definitions, namely

$$W \to \frac{W}{r}, \quad (\Omega, \omega) \to (\Omega, \omega)\frac{r}{c}, \quad v \to \frac{v}{c}, \quad P \to \frac{P}{ES}. \tag{18}$$

The above relations represent the sinusoidal waves, except the following cases:

$$\text{(a) } \omega = 0, v > 0; \quad \text{(b) } v = 2\sqrt{\omega} (\omega > 0), \quad \text{(c) } \omega = v = 0. \tag{19}$$

In the case (a), $\Omega_2 = \Omega_3 = 0$ in the expression of W for $\eta < 0$ (15). This is, however, a removable singularity. In fact, the relation represents a steady-state wave but not a sinusoidal one

$$\lim_{\omega \to 0} W(\eta) = w(\eta) = -\frac{\eta}{v^2} \quad (\eta < 0).$$
(20)

Along with the sinusoidal waves, such a wave appears in the steady-state solution considered in Section 3.2, where the foundation action places the role of the non-oscillating moving force. The other singular points (in the cases (b) and (c) in (19)) correspond to transient waves which also arise in the case $v = 0$, $\omega > 0$. Such waves are considered below.

2.3. The transient regimes with no steady-state limit

We now consider transient regimes under zero initial conditions.

2.3.1. The resonant wave

In the case (b) with $v = 2\sqrt{\omega}$ (19), the steady-state expressions failed. Under this equality, the wave group velocity coincides with the load velocity, and the expression $k^4 - (\omega + kv)^2$ has a second-order zero: $k_1 = k_2 = -v/2$ (point 3b in Fig. 1). To proceed, we put $k = q - v/2$ in (8)

$$W^{LF_\eta}(s, k) = \frac{P}{s'[(q - v/2)^4 + (s' + i(vq - v^2/4))^2]} = \frac{P}{s'(s' + iq^2)[s' - i(q^2 - 2vq + v^2/2)]}.$$
(21)

Only the poles at $s = 0$ and $s' = -iq^2$ should be taken into account to obtain an asymptotic expression for the growing resonant wave. It is defined by the Fourier inverse transform over an arbitrary small (nonzero) vicinity of the point $q = 0$, which results in

$$W(x, t) \sim \frac{P}{\pi\sqrt{\omega}} \exp\left[i\frac{v}{2}\left(x - \frac{vt}{2}\right)\right] \int_0^\infty \frac{2\sin^2(q^2 t/2) + i\sin(q^2 t)}{q^2} \cos(q\eta)\, dq$$

$$= \frac{P\sqrt{t}}{\sqrt{\omega}} \exp\left[i\frac{v}{2}\left(x - \frac{vt}{2}\right)\right]\left[\frac{1}{\sqrt{\pi}} e^{i(\xi^2 + \pi/4)} + \xi\left(\mathrm{FresnelS}(\xi\sqrt{2/\pi}) + \mathrm{FresnelC}(\xi\sqrt{2/\pi}) - 1 + i(\mathrm{FresnelS}(\xi\sqrt{2/\pi}) - \mathrm{FresnelC}(\xi\sqrt{2/\pi}))\right)\right], \quad \xi = \frac{\eta}{2\sqrt{t}}.$$
(22)

These formulas show that the wave propagating together with the load grows as $\sqrt{t}$ and also expands in both directions as $\sqrt{t}$.

2.3.2. A constant unmoving force

In the case (c) with $v = \omega = 0$ where the steady-state regime also does not exist, the first expression in (7) becomes

$$w^{LF}(k, s) = \frac{P}{s(k^4 + s^2)}.$$
(23)

It follows from this that

$$\frac{w(x, t)}{P} = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos(k^2 t)}{k^4} \cos(kx)\, dk$$

$$= \frac{2t^{3/2}}{3\sqrt{\pi}}[\sin(\xi^2 + \pi/4) - \xi^2 \cos(\xi^2 + \pi/4)] + \frac{xt}{2}[\mathrm{FrenelS}(\xi\sqrt{2/\pi}) \in -\mathrm{FrenelC}(\xi\sqrt{2/\pi})]$$

$$+ \frac{x^3}{12}[1 - \mathrm{FrenelS}(\xi\sqrt{2/\pi}) - \mathrm{FrenelC}(\xi\sqrt{2/\pi})], \quad \xi = \frac{x}{2\sqrt{t}}.$$
(24)

In this case, the displacements growing as $t^{3/2}$ also expand in both directions as $\sqrt{t}$.

2.3.3. An oscillating unmoving force

In this case, along with the steady-state sinusoidal waves defined in (10), there may exist a growing displacement. Indeed, the Fourier transform following from the first expression in (7) with $v = 0$ is

$$w(k, t) = \frac{P}{k^4 - \omega^2}\left(e^{i\omega t} - \cos(k^2 t) - \frac{i\omega \sin(k^2 t)}{k^2}\right).$$
(25)

The unboundedly growing wave is defined by the most singular last term. Asymptotically, the integration in an arbitrary small vicinity of point $k = 0$ is sufficient. Finally, we find the growing term, w_+, additional to the sinusoidal waves defined in (10)

$$w_+(x, t) = \frac{iP}{\pi} \int_0^\infty \frac{\sin(k^2 t)\cos(kx)}{\omega k^2}$$

$$= \frac{iP\sqrt{t}}{\omega}\left[\frac{1}{\sqrt{\pi}} \cos(\xi^2 - \pi/4) + \xi\mathrm{FresnelS}\left(\xi\sqrt{\frac{2}{\pi}}\right) - \xi\mathrm{FresnelC}\left(\xi\sqrt{\frac{2}{\pi}}\right)\right], \quad \xi = \frac{x}{2\sqrt{t}}.$$
(26)

This solution appears to be pure imaginary. This means that it corresponds to an action of the external force $P \sin(\omega t)H(t)$, that is to the imaginary part of the complex wave. So, the cosine force does excite only bounded waves. This seeming paradox relates to the motion of a free rigid mass under a suddenly applied harmonic force, where a similar effect can be easily seen.

3. Transition wave

3.1. Formulation

Bearing in mind the results presented in Section 2, let us consider the beam initially resting on a massless elastic foundation in the right region, $x > 0$. At $t=0$ the beam at the left, $x = - a < 0$, is subjected to an oscillating or/and moving force. Under the incident wave excited by the force, the connection to the foundation breaks at any point where the beam displacement reaches the critical value. So, the beam remains attached to the foundation while

$$w(x, t) < w_c, \quad t < t_*(x), \tag{27}$$

and separates from it at the moment, $t = t_*(x)$, when the displacement reaches the critical value. The corresponding equations are

$$EI\frac{\partial^4 w(x, t)}{\partial x^4} + \varrho S\frac{\partial^2 w(x, t)}{\partial t^2} = P\delta(x - vt + a)e^{i\omega t} \tag{28}$$

in the domain, where the beam is free, and

$$EI\frac{\partial^4 w(x, t)}{\partial x^4} + \varrho S\frac{\partial^2 w(x, t)}{\partial t^2} + \varkappa w(x, t) = 0 \tag{29}$$

in the domain, where the beam is attached to the foundation.

As can be seen in Fig. 1, the group speed of the wave, $d\omega/dk$, excited by this force is greater than the speed, v, of the latter (for positive ω and v). So, the excited wave propagates faster than the load. It is assumed that under the action of this wave, the separation spreads fast enough, such that the force remains in the free beam area. Note that this is the case for both steady-state and forerunning regimes considered below.

For greater generality, we use here a different normalization, namely, we consider the quantities

$$l = \left(\frac{r^2 ES}{\varkappa}\right)^{1/4}, \quad \tau = \sqrt{\frac{\varrho S}{\varkappa}}, \quad p = \varkappa l^2 \tag{30}$$

as the natural units of length, time and force, respectively. In these terms, Eqs. (28) and (29) become

$$\frac{\partial^4 w(x, t)}{\partial x^4} + \frac{\partial^2 w(x, t)}{\partial t^2} = P\delta(x - vt + a)e^{i\omega t} \quad \text{(the free beam region)} \tag{31}$$

and

$$\frac{\partial^4 w(x, t)}{\partial x^4} + \frac{\partial^2 w(x, t)}{\partial t^2} + w(x, t) = 0 \quad \text{(the intact region)}, \tag{32}$$

where we preserve the same notations for the non-dimensional values.

Note that the change in the normalization does not affect the results presented in Section 2. All the relations remain valid with respect to the differently normalized values since the Eqs. (3) and (31) coincide (except nonessential shift in the force position).

The corresponding dispersion relations become

$$\Omega = \pm k^2 \quad \text{(the free beam)}, \quad \Omega = \pm \sqrt{1 + k^4} \quad \text{(the supported beam)}, \tag{33}$$

as shown in Fig. 2.

In addition, we have five conditions at the separation point: four continuity conditions with respect to $w(x, t)$, $w'(x, t)$, $w''(x, t)$, $w'''(x, t)$, and the transition criterion, $w(x, t) = w_c$. The conditions at infinity are as follows. In the steady-state formulation, a sinusoidal wave of the amplitude A and frequency Ω propagates to the right along the free beam with the phase speed $V = \Omega/k$, and there is no other energy flux from plus/minus infinity. In the numerical study of the forerunning mode, an external force as the incident wave source is explicitly presented, and no energy flux from plus/minus infinity is assumed.

3.2. Steady-state regime and the domain of its existence

We now consider the steady-state regime existing only for intersonic speeds (as already was noted in Brun et al., 2013). Eqs. (28)
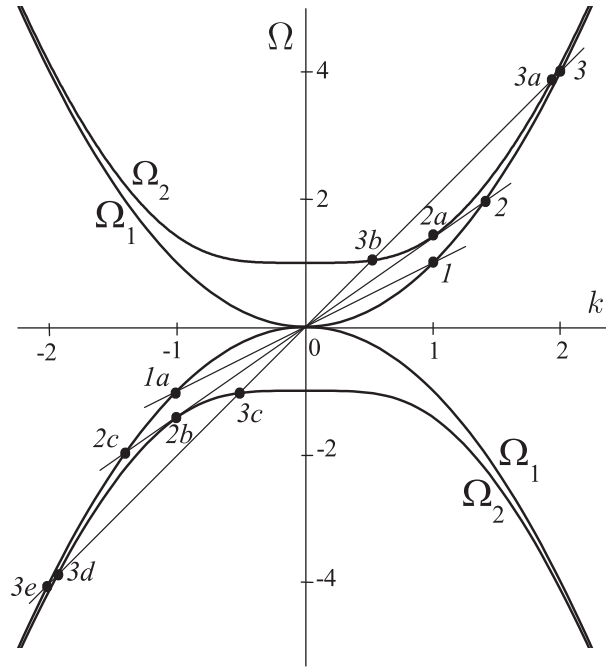
Fig. 2. Dispersion diagram for the free and supported beams, $\Omega_1 = \pm k^2$, and $\Omega_2 = \pm \sqrt{1 + k^4}$, respectively, and the rays corresponding to the intersonic (1–1a), supersonic (3–3a–3b–3c–3d–3e) wave phase speeds and the separating ray (2–2a–2b–2c) with $V = \sqrt{2}$. The intersections with the dispersion curves, marked by 1, 1a, …, 3d, 3e, correspond to the wave numbers, k, and frequencies, Ω, of the sinusoidal waves, which are radiated, in the steady-state regime, in front of the transition point, $\eta > 0$, for $\Omega = \Omega_2$ and $V_{g2} = d\Omega_2/dk > V_2 = \Omega_2/k$, that corresponds to the points 2a, 2b and 3a, 3d. It can be seen that no such waves are radiated behind the transition point, because the corresponding condition, $V_{g1} = d\Omega_1/dk < V_1 = \Omega_1/k$ does not exist for $\Omega = \Omega_1$. However, there exists a displacement linearly distributed at $\eta < 0$ corresponding to intersection point $\Omega_1 = k = 0$.

with $P=0$ and (32) are valid at $\eta < 0$ and $\eta > 0$, respectively, and $w = w(\eta)$, where $\eta = x - Vt$. Note that the steady state implies that the speed of the separation point coincides with the phase speed of the incident wave, V. The equations become

$$\frac{d^4 w(\eta)}{d\eta^4} + V^2 \frac{d^2 w(\eta)}{d\eta^2} + w(x, t) = 0 \quad (\eta > 0),$$

$$\frac{d^4 w(\eta)}{d\eta^4} + V^2 \frac{d^2 w(\eta)}{d\eta^2} = 0 \quad (\eta < 0). \tag{34}$$

In accordance with the dispersion diagram, Fig. 2, the general solution to these equations contains six unknown constants, two for waves at $\eta \geq 0$ exponentially decreasing or propagating to the right and four for the waves at $\eta \leq 0$ propagating in both directions. Recall that these constants are defined by the transition criterion and the continuity conditions

$$w(-0) = w(+0) = w_c, \quad w'(-0) = w'(+0), \quad w'(-0) = w''(+0), \quad w'''(-0) = w'''(+0), \tag{35}$$

and the relation

$$w_i(-0) + w_r(-0) = w(-0) \tag{36}$$

for the sum of the incident and reflected waves, respectively. In addition, we choose the steady-state solution satisfying a partial condition of admissibility

$$\frac{\partial w(\eta)}{\partial t} = -V \frac{dw(\eta)}{d\eta} > 0 \quad (\eta = 0). \tag{37}$$

This inequality prescribes that the displacement decreases in a right vicinity of the transition point in order to be in accordance with the Marder–Gross condition. These conditions allow us to construct a uniquely defined steady-state solution (presented below) formally valid for any large incident wave amplitude.

The main questions, however, remain: whether such solution really exists and what mode is formed otherwise. To answer the first question we have to check if the Marder–Gross admissibility condition is satisfied. Namely, if the critical displacement is not reached earlier than it is assumed in the problem formulation, that is $w(\eta) < w_c$ for any $\eta > 0$. Below we show that this condition is satisfied (and hence the solution is valid) only in a bounder domain in the incident wave phase speed – amplitude plane. For the incident wave parameters outside of this domain, the forerunning transition wave mode is disclosed analytically and described in detail numerically. Here, we call intersonic and supersonic regimes for $0 < V < \sqrt{2}$

and $V > \sqrt{2}$, respectively. Note that the limiting speed $V = \sqrt{2}$ coincides with the group speed, $d\Omega_2/dk$, of the wave in the intact area. Thus, as follows from the energy point of view, no sinusoidal wave can propagate to the right.

As for the supersonic incident wave, it can be concluded in advance that no steady-state solution exists in this domain. This follows directly from the fact that if the solution were exist, a sinusoidal wave would propagate at $\eta > 0$ with the amplitude $w_{max} \geq w_c$ ($w(+ 0) = w_c$) and thus would violate the Marder–Gross condition.

The solutions at the left and at the right, where some of the conditions in (35) and (36) are already taken into account, are

$$w(\eta) = A[\cos(V\eta + \phi) - \cos\phi] + w_c + C_1\eta \quad (\eta \leq 0). \tag{38}$$

and

$$w(\eta) = e^{-\alpha\eta}[w_c \cos(\beta\eta) + C_2 \sin(\beta\eta)] \quad (\eta \geq 0), \tag{39}$$

where $C_{1,2}$ are arbitrary constants and

$$\alpha = \frac{1}{2}\sqrt{2 - V^2}, \quad \beta = \frac{1}{2}\sqrt{2 + V^2}. \tag{40}$$

The remaining continuity conditions concern the derivatives of the displacement up to the third order at $\eta = 0$. Together with the partial admissibility condition (37), they define the constants $C_{1,2}$ and the phase shift, ϕ. We obtain

$$C_1 = -AV \sin\psi, \quad C_2 = -\frac{AV^2}{\beta}(V \sin\psi + \alpha \cos\psi),$$

$$\cos\phi = -V\sqrt{2 - V^2} \sin\psi - (1 - V^2)\cos\psi,$$

$$\sin\phi = V\sqrt{2 - V^2} \cos\psi - (1 - V^2)\sin\psi,$$

$$\psi = \arccos\frac{w_c}{AV^2}, \quad 0 \leq \psi < \frac{\pi}{2}. \tag{41}$$

It follows from this solution that the steady-state regime of the transition may exist only if the incident wave amplitude is large enough, while the speed is not too large, namely, if

$$A \geq \frac{w_c}{V^2}, \quad V < \sqrt{2}. \tag{42}$$

In addition to the above inequalities, the solution should satisfy a separation condition at $\eta < 0$ and the Marder–Gross admissibility condition at $\eta > 0$. In the formulation, we have assumed that the beam is free at $\eta < 0$, that is separated from the foundation, $w(\eta) \geq 0$. The numerical calculations conducted for a series of values of the (A, V)-couple evidence that the separation condition is satisfied. We now consider the admissibility condition, $w(\eta) < w_c (\eta > 0)$.

As can be seen in (39), the first maximum of $w(\eta)$ at $\eta > 0$ is the global maximum in the intact region, and the admissibility condition is satisfied if it is below w_c. The corresponding plots for $V = 0.5, 1, 1.4$ are presented in Fig. 3, where the results for different incident wave amplitudes are shown beginning from the lower bounds (42) and until the upper bounds, where the first maximum is equal to w_c.

The plots evidence that the steady-state regime exists in the domain between these bounds and does not exist outside it. The bounds plotted based on the analytical solution are shown in Fig. 4. In particular, the upper bound is plotted based on the implicit dependence A on V
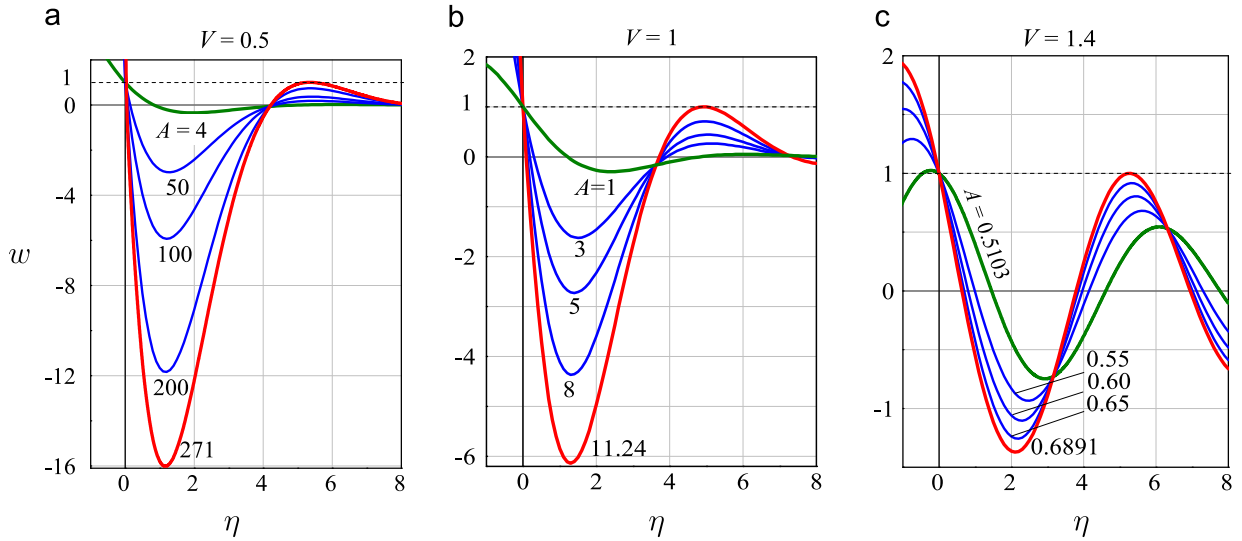
$$\max_\eta w(\eta) = \max_\eta e^{-\alpha\eta}[w_c \cos(\beta\eta) + C_2 \sin(\beta\eta)] = w_c \quad (Gn > 0), \tag{43}$$

If the incident wave amplitude is below the lower bound, the steady-state solution does not exist. Instead, some sporadic irregular developments of the separation were detected in the numerical simulations.

3.3. Forerunning mode transition

What happens if the incident wave amplitude appears on the upper boundary, when the displacement reaches the critical value at $\eta > 0$? The analytical solution evidences that the steady-state mode of the transition fails and shows the origination of the forerunner mode at a distance of the main transition wave in front of it.

The formation and evolution of the forerunning mode were studied numerically. The explicit finite difference scheme was used to solve Eq. (28) with $\omega = 0$ in the right side, which reflects a moving non-oscillating external force. Zero initial conditions are assumed. The transition criterion (27) was used. The beam was taken long enough, such that the reflections from its ends did not reach the considered area. The scheme satisfying the stability condition was tested by comparing the results with those found analytically for the steady-state regime. The numerical transient solution approaches the steady-state limit, where exists. This evidences that the numerical scheme is correct and the steady-state solution is stable (where exists). The corresponding incident wave amplitude was found from the relation (15) as $P = V^3A$. Recall that, in the case of the non-oscillating force, the incident wave phase speed is equal to the load speed, $V=v$.

Fig. 3. The steady-state regime. The displacements $w(\eta)$ (mainly at $\eta > 0$) corresponding to the incident wave phase speeds, $V = 0.5$, 1 and 1.4 (the figures (a), (b) and (c), respectively). The curves in each figure correspond to different incident wave amplitudes, A, beginning from the lower bound (42) (the green curves in the electronic version) and until the upper bound (the red curves in the electronic version). In the latter bound, the displacement reaches the critical level w_c (the dash line) at a distance of the separating point $\eta = 0$ in front of it. So, the upper bound of the steady-state mode is the lower bound of the forerunning mode. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)
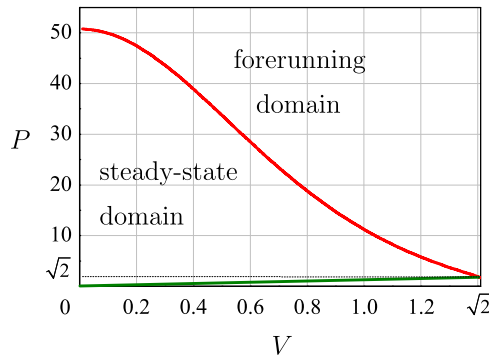
Fig. 4. The lower (the green curve in electronic version) and upper (the red curve in electronic version) bounds of the domain where the steady-state solution does exist ($P = V^3A$; A and V are the incident wave amplitude and phase speed, respectively). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

The established forerunning mode appears periodic as can be seen in Fig. 5. We present two graphical schemes of this nontrivial beam-foundation separation mode. In one of them, the separation path, $x(t)$, is a two-valued function of time, Fig. 5a, whereas the same process is represented by continuous/discontinuous separation lines at different time points, Fig. 5b.

The evolution of the forerunning wave obtained in the numerical simulations is illustrated in Fig. 6. In an initial stage, while the transition wave propagates, the evanescent wave penetrated into the intact area remains below the critical level. It increases gradually in time and, at a moment when it reaches the threshold, gives rise to the forerunner. This scenario is repeated periodically. In the representation of the numerical results, we normalize the displacement and the incident wave amplitude, $w(x, t)$ and A, taking w_c as the length unit. We use the same notations for the non-dimensional displacement and incident wave amplitude, $w(x, t)$, A and write 1 instead of w_c.

The development of the forerunning transition wave in the intersonic regime ($V < \sqrt{2}$) can be observed in Fig. 7 for some values of the incident wave amplitudes, A ($P = V^3A$). An initial stage is shown in Fig. 7a, and the established regime is demonstrated in Fig. 7b. The lower lines ($P = 11$) correspond to the steady-state regime. Recall that for $V = 1$ the latter fails at $A = 11.24$. From these plots, the local and averaged over the period speeds can be estimated.

The graphs of the averaged speed of the transition wave and the forerunning mode period as functions of the wave amplitude are presented in Fig. 8 for the intersonic and supersonic regimes, $V < \sqrt{2}$, Fig. 8a, and $V > \sqrt{2}$, Fig. 8b, respectively.

In the supersonic regime, the calculations were performed for $V = 2$. Established periodic transition modes were found in a rather narrow vicinity of $P = 0.8$, Fig. 9, and for $P \geq 3.2$, Fig. 10. The former is similar to a bridged crack, where the forerunners are not merged, whereas in the latter range, the forerunning mode appears similar to that detected in the intersonic regime.
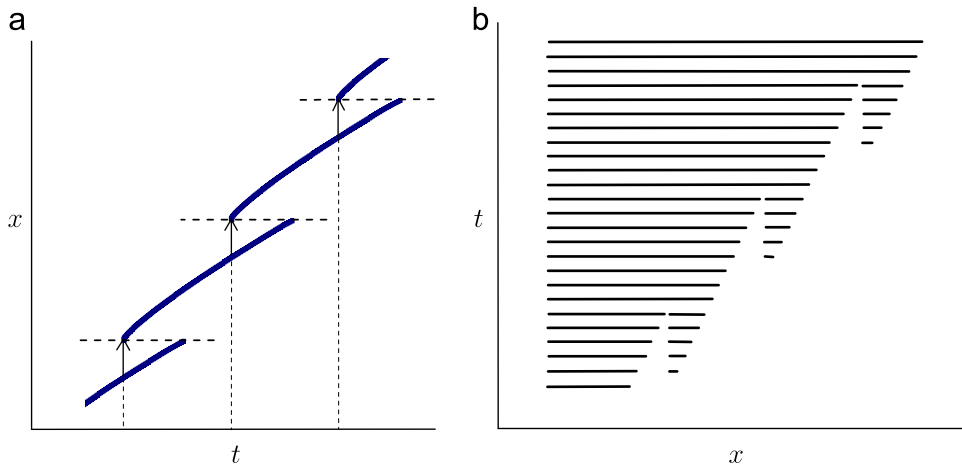
Fig. 5. Two different schemes of the same forerunning mode transition. At the left (a): A piecewise continuous curve is plotted corresponding to the moving separation point as a two-valued function $x(t)$. While a continuous separation segment spreads along the x-axis, another separation segment starts at a distance in front of it (see vertical dash lines with arrows), and during a time-period there exist two growing segments. Then the first segment reaches the second one (see horizontal dash lines), and these two segments are merged into one. Then a new segment arises, etc. At the right (b): The same process represented by continuous/discontinuous separation lines at different time points.
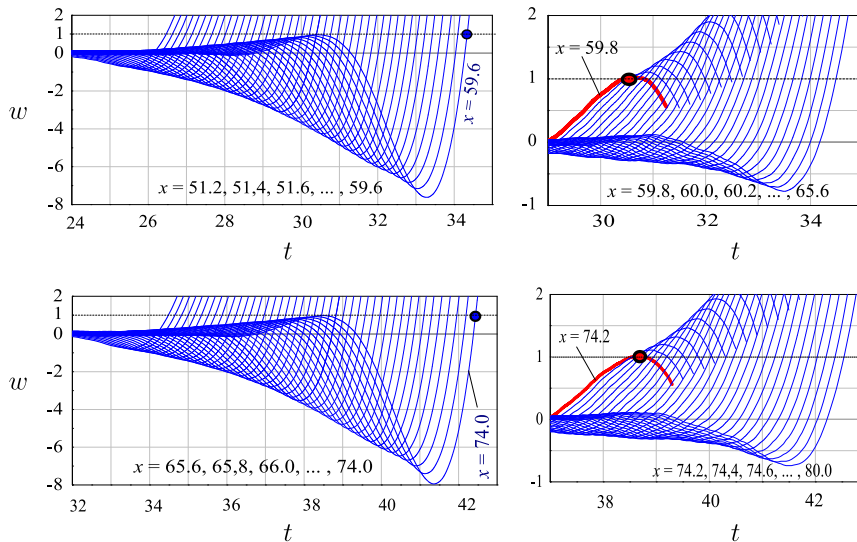

Fig. 6. The forerunning mode transition. The function, $w(x, t)$, for a set of x-points is plotted for $V=1$, $P=30$. In an initial stage shown in the left plots, a regular transition wave propagates, and the evanescent wave penetrated into the intact area remains below the critical level. However, it increases, at a moment it reaches the threshold (the bold curve in the right plots (red in the electronic version)), and the forerunner transition arises. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

4. Discussions and conclusions

A new transition wave has been found propagating in the continuous waveguide under the action of an incident sinusoidal wave. The transition wave manifests itself as transition segments periodically arising at a distance of the main wave front ahead of it. Such a wave mode replaces the steady-state one as the amplitude of the incident sinusoidal wave exceeds the critical level. The latter is determined analytically as a function of the incident wave speed.

This forerunning mode was described in detail. The mechanisms of the transformation of the steady-state mode transition into the forerunning one and of the development of the latter are elucidated. The dependencies of the forerunning mode parameters, the period T and T-averaged speed $\langle v \rangle$, on the incident wave amplitude A and speed V are determined. It was found that the period continuously decreases, while the transition speed increases to the group speed of the incident wave as the intensity of the latter increases. This is in contrast to the steady-state regime, where the transition and incident wave speeds coincide regardless of the intensity of the incident wave. This also differs from the clustering wave mode in the lattice, where the corresponding dependencies are piece-constant, and where both the period and the speed increase with the incident wave amplitude.
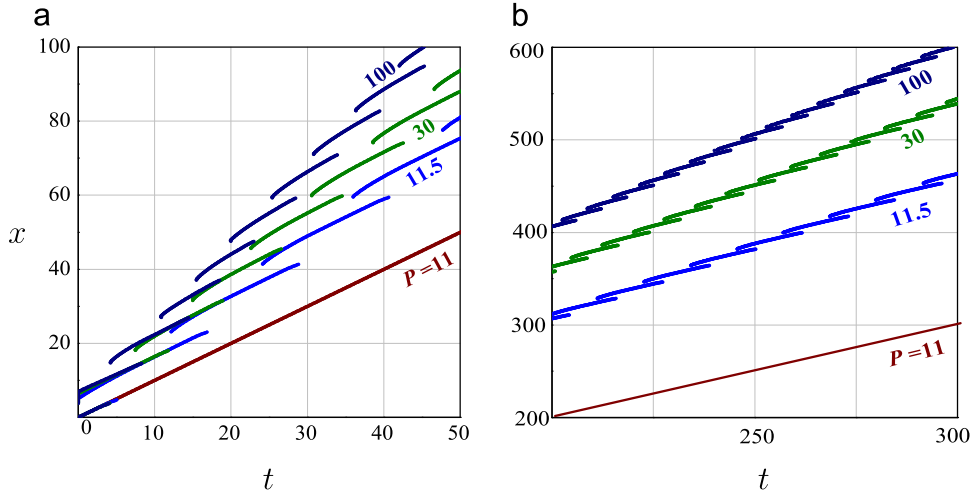
Fig. 7. The development of the forerunning transition wave in the intersonic regime for $V=1$. An initial stage is shown in Fig. 7a, and the established regime is demonstrated in Fig. 7b. The lower lines correspond to the steady-state regime. Recall that for $V=1$ the latter fails at $P = A = 11.24$. In these plots, the local and averaged over the period speeds can be estimated.
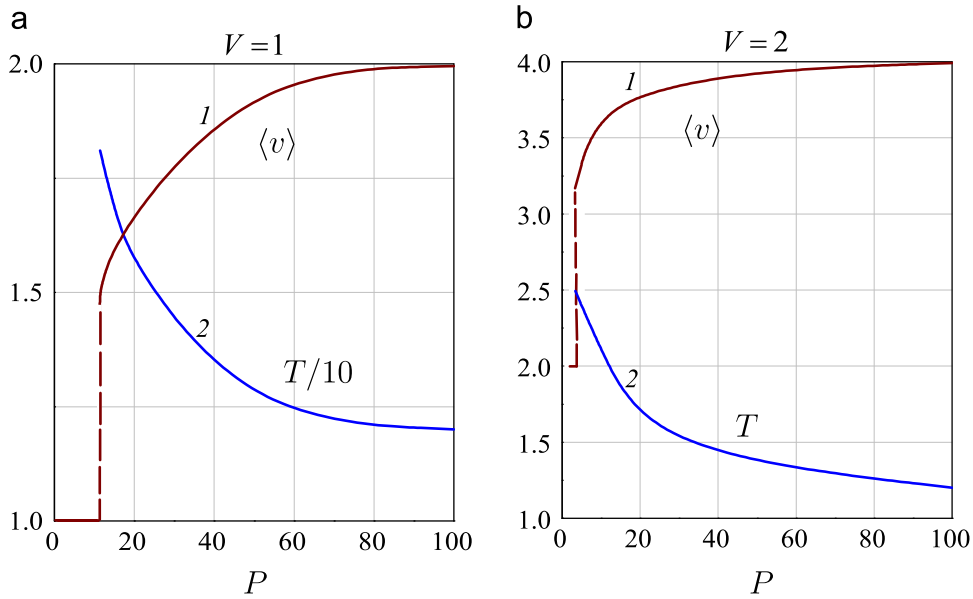

Fig. 8. Dependencies of the transition wave averaged speed, $\langle v \rangle$ (curves 1), and the forerunning mode period, T (curves 2), on the wave amplitude, A ($P = V^3 A$), for the intersonic regime (a) and supersonic regime (b).

There is one more difference in the transition waves in continuous and discrete waveguides. If an energy release is required for the transition, as is usually the case, the steady-state transition wave can exist in a continuous waveguide only in the case where the group speed of the incident wave exceeds the phase speed. Such a transition wave cannot exist otherwise. In contrast, in a discrete waveguide, there is infinite set of the phase speeds corresponding to a given frequency, and a sufficiently low phase speed can always be found (see Mishuris et al., 2009).

Our considerations were based on Euler–Bernoulli equation and Winkler foundation model. However, the phenomenon discovered in this paper has a more general nature. The following conditions may be necessary for the existence of the steady-state regime transferring into the forerunning one. One concerns the phase and group velocities of the incident wave: the latter must exceed the former. The other conditions relate to the waves in the intact region. Not only sinusoidal waves must present there. Otherwise, the steady-state mode contradicts the Marder–Gross condition and hence cannot exist. The forerunning mode can substitute the steady-state one if the evanescent wave oscillate.

Finally, we note that Euler–Bernoulli equation, in its application to an elastic beam, is valid if the wave length, $2\pi/k$, is much greater than the beam cross-section height. In terms of the dimensional wavenumber k_d and the radius of inertia of the cross-section r, with refer to (30) this condition reads
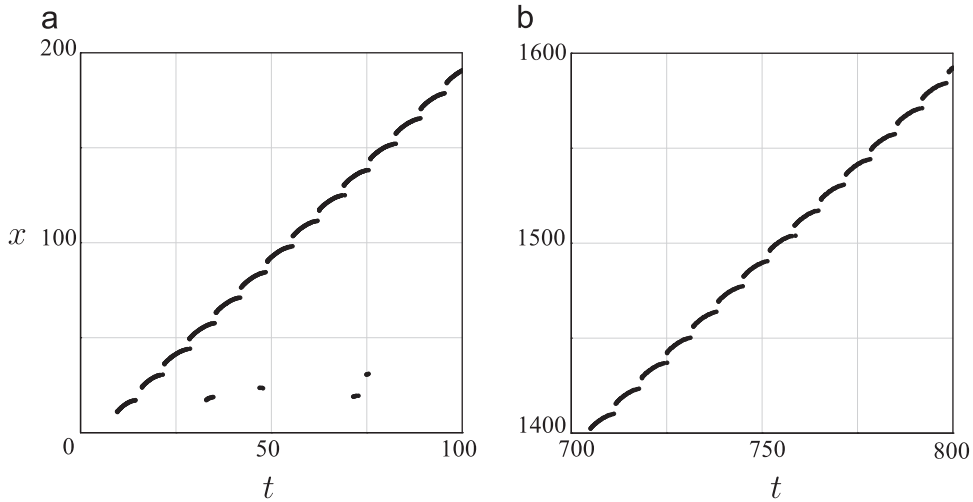
Fig. 9. The 'bridged-crack' periodic transition mode in the supersonic regime for different time intervals, $V=2$, $P = V^3A = 0.8$. Separate points shown in Fig. 9a correspond to local damages of some bridges detected only in the initial region of the transition. The other bridges remain intact in the calculation period, $0 < t < 800$.
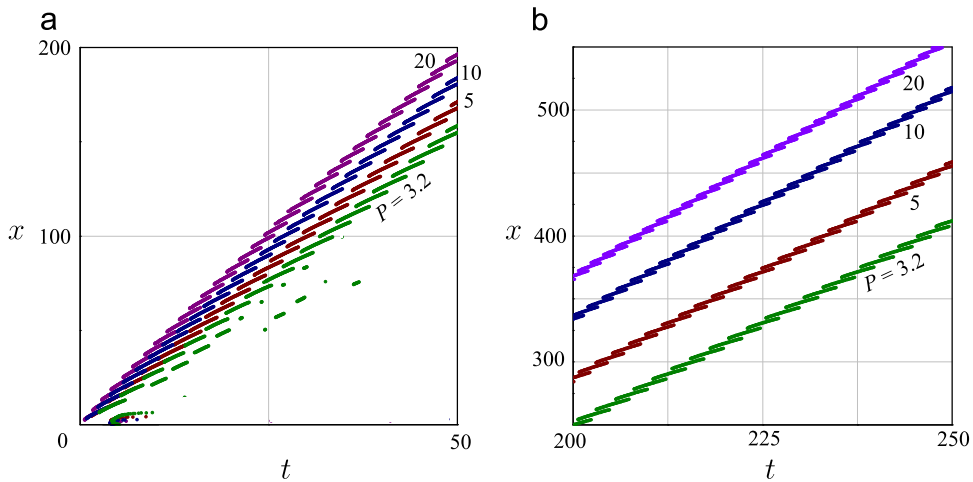
Fig. 10. The development of the forerunning transition wave in the supersonic regime for $V=2$, $P = V^3A = 3.2, 5, 10, 20$ shown in an initial time interval (a) and for the established regime (b).

$$|k_d|r = |k|\left(\frac{\varkappa r^2}{ES}\right)^{1/4} \ll 1. \tag{44}$$

Recall that for the intersonic regime the non-dimensional wavenumber modulus satisfies the inequality $|k| < \sqrt{2}$.

Acknowledgments

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