J. Mech. Phys. Solids Vol. 41, No. 6, pp. 1019–1033, 1993.
Printed in Great Britain.

0022–5096/93 $6.00+0.00

PRINCIPLE OF MAXIMUM ENERGY DISSIPATION RATE IN CRACK DYNAMICS

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(*Received* 16 *June* 1992; *in revised form* 17 *December* 1992)

ABSTRACT

THE PRINCIPLE of maximum energy dissipation rate is introduced as an energy criterion for crack dynamics. That allows us to explain observed limiting crack speeds in brittle materials, and to complete the crack dynamics formulation. The upper limits of the crack speed in perfectly elastic and elastic–plastic bodies are obtained. It is found that the theoretical maximum crack speed in an isotropic elastic body (in the first mode of crack propagation) is approximately equal to half the shear wave speed. In the case of an elastic–plastic body, the criterion is formulated as a maximum plastic strain work per unit time. The self-similar problem for the fracture mode III is solved (assuming the plastic zone to be narrow) and the crack speed limit is found as a function of the ratio of loading to yield limit: the plasticity decreases the crack speed limit and the latter tends to zero with the yield limit. The comparison of these theoretical results with some experimental data shows that under ordinary conditions crack propagation appears to conform to the "maximum dissipation rate" process.

1. INTRODUCTION

CONSIDER AN ELASTIC BODY under given external forces. A dynamic problem is completely defined if the boundary of the body, the boundary and initial conditions are given. However, in a fracture, the boundary is not known in advance: an additional part of the boundary is formed by the crack propagation, and this process is outside the elasticity framework. If we want to consider the problem on the macrolevel, we need a criterion to obtain the speed and the trajectory of the crack—to obtain the crack velocity as a vector.

The energy criterion for fracture consists of the comparison of two quantities: the macroscopic energy release G caused by a crack propagation (in elastostatics—the energy release under a crack position variation) with the surface energy (GRIFFITH, 1920) or an effective surface energy (IRWIN, 1948; OROWAN, 1955).

A dynamic problem of crack propagation along a given trajectory can be solved if the effective surface energy is constant or, in any case, is a function of the crack speed. In this case, the use of the energy criterion, in principle, allows the crack speed to be obtained as a function of time. However, the experimental data show that it is impossible to believe that the effective surface energy is constant during the entire process and, what is more, it is difficult to identify some kind of stable connection between the effective surface energy and the crack speed.

Really, the crack propagation process is usually characterized by two periods: in the first of them, the crack velocity increases and the energy release per unit area is almost constant; in the second period, the crack speed is constant but the energy release increases. It is clear that the classical energy criterion is not valid for the second period of the crack propagation.

Numerous experimental results which concern the crack speed limits in brittle materials are well known. A survey of brittle crack speed was presented by RAVI-CHANDAR and KNAUSS (1984). This survey is very important for us and it is shown in Table 1. Here ν is Poisson's ratio, c_2 is the shear wave speed and c_R is the Rayleigh wave speed. The references of the original papers can be found in the paper by RAVI-CHANDAR and KNAUSS (1984).

The results show that the crack speed (under ordinary conditions) does not achieve the Rayleigh wave speed c_R. The experimental values of the crack speed limits only equal about half this speed. These results contradict any model based on a speed-independent fracture energy [this case was considered by FREUND (1972a)], or any model with a bounded fracture energy. One can see also from the experimental results by RAVI-CHANDAR and KNAUSS (1984) that the crack speed after an acceleration stays constant under a strongly variable stress intensity factor. In this period the crack propagation looks like a process which is independent of the energy flux into the propagating crack tip.

Branching is often pointed out as the cause of the crack speed limitation in brittle materials. Some experimental data seem to contradict this as RAVI-CHANDAR and KNAUSS (1984) noticed. Indeed, under certain conditions during the constant speed propagation phase (not above $0.45c_R$) there is no evidence of crack branching. Taking this phenomenon into account it is possible to suppose that the branching is not a reason for the crack speed limitation but it is an effect of this limitation. Branching arises if the energy release is so high that not all this energy can be absorbed in a "fracture process zone".

Another result should be found in a "weakly bonded plane" in which the energy absorption is strongly limited. It was pointed out by RAVI-CHANDAR and KNAUSS

TABLE 1. *Survey of brittle crack speed* v *by* RAVI-CHANDAR *and* KNAUSS (1984)

Material	Author	ν	v/c_2	v/c_R
Glass	Bowden	0.22	0.42	0.51
	Edgerton	0.22	0.43	0.47
	Schardin	0.22	0.47	0.52
	Anthony	0.22	0.60	0.66
Plexiglas	Cotterell	0.35	0.54	0.58
	Paxson	0.35	0.58	0.62
	Dulaney	0.35	0.58	0.62
Homalite-100	Beebe	0.31	0.31	0.33
	Kobayashi	0.345	0.35	0.38
	Dally	0.31	0.35	0.38
	Smith	0.31	0.38	0.41

(1984) and investigated experimentally by LEE and KNAUSS (1989). In this case, the crack speed peaks at the Rayleigh wave speed.

Taking into account these experimental results one can see that under ordinary conditions of crack propagation in a homogeneous material, the upper limit of energy absorption is high enough. In this case, the crack speed can be constant during a long period of crack propagation because the energy release—the energy absorption per unit area—can increase. In contrast with this, the case of a weakly bonded plane corresponds to a low limit under which the energy absorption cannot increase, and the crack speed is forced to tend to its theoretical limit.

So, the crack propagation process essentially depends on the distance between the upper limit of energy absorption and the lower limit—the surface energy. Use of the speed independent fracture energy is justifiable if this distance is small enough, and the classical energy criterion becomes indefinite if the distance is not small.

Interesting results have been obtained recently by FINEBERG *et al.* (1991, 1992). They discovered an almost regular roughness structure of the crack surfaces and high frequency oscillation of the crack speed in polymethylmethacrylate. These phenomena (the non-regular roughness was observed earlier repeatedly) arise when the crack speed is high enough, especially when the crack speed peaks at its limit. The energy release is almost constant when the crack speed increases, and it increases when the averaged crack speed is constant (when it is equal to the crack speed limit: about $0.5c_R$). The fact that the energy radiation is an effect of the crack speed oscillation was pointed out by RICE (1978) and SLEPYAN (1978). A periodical variability of the sizes of the fracture process zone as a result of crack speed oscillations was pointed out by BOTSIS and CHUDNOVSKY (1987).

Another reason for the effective surface energy increase is the influence of the structure of the medium. Recent developments in crack propagation is elastic periodically structured media such as chains, lattices, composite materials and rock joints by SLEPYAN (1981, 1990), KULACHMETOVA *et al.* (1984), MICHAILOV and SLEPYAN (1986) demonstrate a number of effects which cannot be discovered using the classical model of non-structured solids. The energy radiation from the front of the fracture is the most important phenomenon. It can be heat transfer, sound emission or high frequency seismic oscillations depending upon the scale of the structure. This energy outflow essentially depends on the crack speed and increases without bound if the crack speed tends to the critical value. Based on exact solutions these results are valid for a wide range of periodic structures. Numerical results were found by WEINER and PEAR (1975).

The experimental and theoretical results show that the effective surface energy is formed in a brittle material under influences of some "micro" factors such as the structure of the medium, the roughness of the crack surfaces and the crack velocity oscillation. All these factors cause a radiation—high frequency waves which carry energy from the crack. The roughness increases the crack surface area, and this phenomenon also increases the effective surface energy. The roughness and the crack velocity oscillation, in their turn, depend on the structure of the medium and on macrolevel factors such as the energy release (G) and the averaged smooth crack speed (v). Thus, we have here the coupled problem of the interaction of macro- and micro-processes.

From this point of view investigations of problems are important in which micromechanisms of energy absorption (during the macro–micro energy transfer) are taken into account. However, the classical energy criterion has a fundamental weakness. Firstly, the energy consumption is unstable under the above mentioned phenomena. Secondly, it is unstable under a small change of the material structure. [The simplest example of this kind of instability is given in the paper by SLEPYAN (1984). In that example, the process is considered in which a thread falls to a rigid plate. If the bending rigidity of the thread is equal to zero all kinetic energy flows into the moving contact point. However, if the thread has any positive value of the bending rigidity all kinetic energy is carried away by high frequency bending waves, and there is no energy flux into the contact point.] Lastly, the classical energy criterion is not sufficient for the crack dynamics formulation, as is discussed in the next section. To overcome these difficulties, the principle of maximum energy dissipation rate is introduced as an energy criterion for crack dynamics.

2. INDEFINITENESS OF THE CRACK VELOCITY UNDER THE CLASSICAL ENERGY CRITERION

There is an additional essential difficulty in the crack dynamics foundation. As a matter of fact, the classical energy criterion is not sufficient in principle. One can see that, taking into account the fact that crack velocity is a vector. This vector, being normal to the crack edge, has two components but at the same time, we only have the scalar criterion. Moreover, in the general case of a three-dimensional body we have a set of possible distributions of the velocity at the crack edge. Thus, there is a set of vectors which satisfies the energy criterion in a two-dimensional problem, and a set of distributions of the vectors—in a three-dimensional problem.

The energy release as a function of the instantaneous crack velocity vector, or as a functional of the instantaneous distribution of this vector, is unknown. Assume, only to demonstrate the above phenomenon, that in a two-dimensional problem on the microlevel, the energy release G, modulus of the instantaneous velocity $v_M = |\mathbf{v}_M|$ and its direction-angle θ (Fig. 1) are connected by the equation [we use the approximate relationship between the energy release and straight-line crack speed by FREUND (1990)]

$$G = f(t)\left(1 - \frac{v_M}{v^*}\right)\cos\theta = 2\gamma = \text{const} > 0, \tag{1}$$

where $f(t)$ is a function of time t and v^* is a critical speed. Thus, we assume that the

FIG. 1. Crack velocity as a vector.

classical criterion is valid on the microlevel. Let us assume that at a moment $t = t_0$, (1) is satisfied by values: $v_M = v_0(0 < v_0 < v^*)$, $\theta = 0$. Then, (1) is satisfied by a couple v_M, θ if this couple satisfies the relation

$$\left(1-\frac{v_M}{v^*}\right)\cos\theta = 1-\frac{v_0}{v^*},$$

from which one can see that

$$|\theta| \leqslant \arccos\left(1-\frac{v_0}{v^*}\right), \quad 0 \leqslant v_M \leqslant v_0. \tag{2}$$

This indefiniteness may be a reason for instability of straight-line crack propagation, a reason for roughness of crack surfaces, and, lastly, a reason for the experimental data disagreement with the theoretical predictions. In any case, the classical energy criterion gives us no possibility to define the crack velocity if the crack trajectory is unknown in advance.

From (1) it follows that

$$|\theta| \leqslant \theta_{max} = \arccos\frac{2\gamma}{f(t)}.$$

Let the angle θ be a variable value which satisfies this inequality:

$$\theta = \theta_{max}\cos\omega t,$$

where t is time and ω is frequency. The projection of the velocity v_M on the crack direction on the macrolevel (averaged direction $\theta = 0$) is

$$v_+(t) = v_M\cos\theta = v^*\left[\cos(\theta_{max}\cos\omega t) - \frac{2\gamma}{f(t)}\right].$$

Assuming the function $f(t)$ tends to infinity when t tends to infinity we have

$$v_+(t) \sim v^*\cos\left(\frac{\pi}{2}\cos\omega t\right).$$

The crack speed limit on the macrolevel is the averaged speed

$$v = v^*\frac{\omega}{\pi}\int_0^{\pi/\omega}\cos\left(\frac{\pi}{2}\cos\omega t\right)\mathrm{d}t \cong 0.472v^*.$$

This example shows that the crack velocity indefiniteness under the classical energy criterion on the microlevel may lead to the crack speed limitation on the macrolevel. At the same time, the energy release on the macrolevel which can be obtained by changing v_M to v and taking $\theta = 0$ in formula (1) increases unboundedly

$$G \sim f(t)\left(1-\frac{v}{v^*}\right) \cong 0.528f(t).$$

Under these conditions of indefiniteness it is natural to suppose that the crack velocity on the macrolevel—the averaged smooth velocity—obeys an extremal principle. It is possible to suppose the principle of maximum energy dissipation rate is in force—the principle of the maximum rate of macro–micro energy transfer. This principle is assumed, and its consequences are examined in relation to the experimental results.

3. FORMULATION OF THE CRITERION

Let G be the energy release per unit area of a dynamic crack, let N be the energy flux into the crack edge (also per unit area): $N = Gv$, and let M be the excess of the energy flux: $M = (G-2\gamma)v$, where γ is the effective surface energy for a quasistatic growth of the crack, $v = |\mathbf{v}|$, $\mathbf{v}$ is the crack velocity. Taking into account the above mentioned dynamic fracture phenomena assume that at each given moment the crack velocity as a vector distribution corresponds to the maximum energy dissipation rate—the maximum excess of the total energy flux into the propagating crack edge per unit time. We also use the Griffith's (or Irwin–Orowan's) criterion but only as the lower boundary of the energy release per unit area. So, we assume that the distribution of the crack velocity vector $\mathbf{v}$ is defined by the requirement

$$\int_\Gamma M(\mathbf{v})\,\mathrm{d}s = \int_\Gamma N(\mathbf{v})\,\mathrm{d}s - 2\gamma\int_\Gamma v\,\mathrm{d}s = \left(\int_\Gamma M(\mathbf{v})\,\mathrm{d}s\right)_{\max}, \quad v \geqslant 0, \quad M(\mathbf{v}) \geqslant 0, \quad (3)$$

where contour Γ is the crack edge, σ is the path along Γ. The second inequality corresponds to Griffith's criterion as a lower boundary of energy release.

Thus we have the variational problem for the instantaneous crack velocity distribution. Of course, it is possible, in principle, to find this distribution only if the energy release as a functional of prior crack motion, and as a function of instantaneous crack velocity distribution, is known.

Consider a two-dimensional problem. In this case we may omit the integration symbols in relations (3). Now the Griffith (or Irwin–Orowan) criterion follows from (3) as the lower boundary of energy release per unit area. Really, $M(\mathbf{v}) < 0$ if $G < 2\gamma$, $v > 0$. In this case $M(0) = M_{\max}$, and the crack does not propagate. In the opposite case when $v > 0$ the principle gives us the equations

$$\frac{\partial N}{\partial v} = 2\gamma, \quad \frac{\partial N}{\partial \theta} = 0. \qquad (4)$$

The energy release G, and hence energy flux N, are functionals of $\mathbf{v}$ as was noted above, but they also depend on the instantaneous velocity $\mathbf{v}$, and just this velocity is meant in the equations (4).

Consider the equations

$$\frac{\partial M}{\partial v} = 0, \quad \left(\frac{\partial N}{\partial v} = 2\gamma\right) \qquad (5)$$

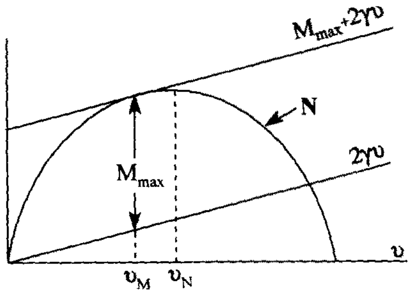

FIG. 2. The extremal crack speeds: (5) is satisfied by $v = v_M$ and (6) is satisfied by $v = v_N$.

and

$$\frac{\partial N}{\partial v} = 0. \tag{6}$$

Equation (5) is satisfied by $v = v_M$, and (6) is satisfied by $v = v_N$. Assume the energy flux N is a convex smooth function of the instantaneous speed v [that is what it is for a straight crack under ordinary conditions, as in the book by SLEPYAN (1990)]. Then, as one can see in Fig. 2, $v_M < v_N$. However, v_M tends to v_N if the energy flux N increases. Really, the energy flux N can be represented (see Section 4) in the form $N = f(v)g(t)$, and (6) follows from criterion (5) if $g(t) \to \infty$. Thus, we see that v_N is the crack speed limit, and to obtain this limit it is possible to use (6) for the total energy flux into the propagating crack tip. [Using this equation as well as the second equation (4) for example (1) (change v_M to v) we obtain the speed limit $v = 0.5v^*$, and the direction on the macrolevel $\theta = 0$.]

Here we call "dissipation" the excess of the energy transfer from the macro to the microlevel: the energy flux from the elastic body to the crack (assuming it is smooth) with the exception of the surface energy flux $2\gamma v$. All factors such as the structure of the medium, the roughness and the crack velocity oscillations are assumed to belong to the microlevel as well as the energy radiation from the crack under the influence of these factors. However, it is possible to obtain a "longwave" radiation on the macrolevel in a transient problem. This "macroscopic" radiation in contrast to the microscopic one is automatically described by the solution to the macroproblem.

At the same time it is impossible to identify an exact boundary to separate the macro and microlevels but it is in the usual run of difficulties in similar foundations.

The extremal dissipation principle was used earlier by NIKOLAEVSKIJ (1987) for the investigation of some aspects of crack growth in a visco-elastic material but without any variation of the crack velocity. A somewhat different formulation of the above principle for the two-dimensional case was given by SLEPYAN (1992).

We also consider crack propagation in an elastic–plastic body. In this case the rate of the plastic strain energy is required to be maximum.

Two-dimensional problems for straight cracks are considered below, to obtain the crack speed limits.

4. THE CRACK SPEED LIMIT IN AN ELASTIC BODY

Consider the generalized plane problem for a semi-infinite straight running crack in an unbounded elastic body (the initial conditions are at zero). As shown by KOSTROV *et al.* (1969) the energy flux into the crack tip is

$$N = \frac{v}{2\mu}\left[\frac{v^2}{c_2^2 R(v)}(\alpha K_{\mathrm{I}}^2 + \beta K_{\mathrm{II}}^2) + \frac{1}{\beta}K_{\mathrm{III}}^2\right],$$

$$\alpha = \left(1 - \frac{v^2}{c_1^2}\right)^{1/2}, \quad \beta = \left(1 - \frac{v^2}{c_2^2}\right)^{1/2}, \tag{7}$$

where μ is the shear modulus; $0 < v < c_2$; c_1 and c_2 are the dilatational and shear wave speeds; K_{I}, K_{II} and K_{III} are the stress intensity factors for the first, second and third fracture modes, respectively, $R(v) = 4\alpha\beta - (1+\beta^2)^2$. Expression (7) applies for a variable crack speed as well as a constant speed.

To define the stress intensity factors as functions of the instantaneous crack speed we have to use the wave solution for non-uniform crack speed.

Some historical data concerning this topic are briefly presented. The first solution to the non-uniform crack speed problem ($v(t) < c_2$) was obtained by KOSTROV (1966) for the third fracture mode. The general solution for plane problem ($v(t) < c_{\mathrm{R}}$) was obtained by FREUND (1972b) assuming external loading is independent of time. The solution for loading with general dependence on time was published by KOSTROV (1974) for $v(t) < c_{\mathrm{R}}$, and by KOSTROV (1976) for $c_{\mathrm{R}} < v(t) < c_2$. Using the improved method SLEPYAN (1974) obtained the general solution for $v(t) < c_{\mathrm{R}}$ in a more simple and compact form (8). The same method was used by SLEPYAN and FISHKOV (1980) for the more general case in which the crack speed may cross the critical values: c_{R}, c_2, c_1 (the general problem for non-uniform speed of the separation point of boundary conditions was considered).

The general solution by SLEPYAN (1974) [see also SLEPYAN (1990)] for the crack opening displacement $2u_-(x,t)$ and for the traction distribution $\sigma_+(x,t)$ on the line of crack propagation in front of the crack ($x > l(t)$) is as follows

$$u_- = S_- ** [(S_+ ** \sigma_-)H(l(t)-x)],$$

$$\sigma_+ = -P_+ ** [(S_+ ** \sigma_-)H(x-l(t))], \tag{8}$$

where $S_\pm$ are functions (distributions) which arise as a result of factorization, $P_+ ** S_+ = \delta(x)\delta(t)$, δ is Dirac delta-function, H is unit step function. Stresses σ_- correspond to surface crack loading, and $l(t)$ is the crack tip coordinate; t is time, and x is the axis of crack propagation. The double convolutions such as $S_+ ** \sigma_-$ are the double integrals:

$$S_+ ** \sigma_- \equiv \int_{-\infty}^{\infty}\int_{-0}^{t+0} S_+(x-\xi, t-\tau)\sigma_-(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi.$$

The expression for stress intensity factors follows from the second formula (8) (see SLEPYAN, 1990, p. 212)

$$K = -\sqrt{2}f_+(v)[S_+(t,x) ** \sigma_-(t,x)], \tag{9}$$

where the function f_+ becomes definite for a definite fracture mode ($K = K_{\rm I}$, $K_{\rm II}$ or $K_{\rm III}$). The double convolution $S_+ ** \sigma_-$ is a functional of prior crack motion, but this convolution is independent of the instantaneous crack speed. Therefore we have no special interest in it if our aim is to obtain the crack speed limit v. One can find expressions for the function $f_+(v)$ in the same book by SLEPYAN (1990), pp. 212–215. Here multipliers which are independent of the crack speed are omitted. We have with this accuracy

$$K_{\rm I} = \left(1-\frac{v}{c_1}\right)^{-1/2}\left(1-\frac{v}{c_{\rm R}}\right)\psi, \quad K_{\rm II} = \left(1-\frac{v}{c_2}\right)^{-1/2}\left(1-\frac{v}{c_{\rm R}}\right)\psi,$$

$$\psi = \exp\left(\frac{v}{\pi}\int_{c_2/c_1}^{1}\frac{\phi(x)\,{\rm d}x}{c_2 - vx}\right),$$

$$0 \leqslant \phi(x) = \arctan\left(4x^2(1-x^2)^{1/2}\frac{(x^2-c_2^2/c_1^2)^{1/2}}{(2x^2-1)^2}\right) \leqslant \frac{\pi}{2}, \tag{10}$$

where $c_{\rm R}$ is the Rayleigh surface wave speed. The complete expression for $K_{\rm III}$ is simple enough to give it here [the solution was obtained by KOSTROV (1966)]

$$K_{\rm III} = -\left[2\left(1-\frac{v}{c_2}\right)\right]^{1/2}\int_{l(t)-c_2t}^{l(t)}\sigma\left(t-\frac{l(t)-\xi}{c_2},\xi\right)(l(t)-\xi)^{-1/2}\,{\rm d}\xi, \tag{11}$$

where $l(t)$ is the crack tip coordinate.

The speed limit v is obtained as a value which satisfies the equation

$$\frac{{\rm d}N}{{\rm d}v} = 0, \quad 0 < v < c_2. \tag{12}$$

The results of calculations of the ratio of crack speed limit to shear wave speed for the fracture modes I and II are represented in Table 2.

The crack velocity limit for fracture Mode III is obtained directly from the equation

$$\frac{\rm d}{{\rm d}v}\left[v\left(\frac{1-v/c_2}{1+v/c_2}\right)^{1/2}\right] = 0$$

which follows from (7), (11) and (12). We obtain

TABLE 2. *The crack speed limit*

Poisson's ratio	0	0.1	0.2	0.3	0.4	0.5
$c_{\rm R}/c_2$	0.874	0.893	0.911	0.927	0.942	0.955
v/c_2 (fracture mode I)	0.476	0.492	0.507	0.517	0.520	0.482
v/c_2 (fracture mode II)	0.539	0.568	0.601	0.638	0.674	0.711

$$\frac{v}{c_2} = \frac{\sqrt{5}-1}{2} \cong 0.618.$$

The theoretical results for fracture mode I are close to the experimental results (see Table 1): the theoretical ratio for glass ($\nu = 0.22$): $v/c_2 = 0.51$, the averaged experimental ratio $v/c_2 = 0.48$. The same ratios for plexiglass ($\nu = 0.35$) are 0.52 and 0.57 respectively. The experimental results for Homalite-100 differ from the theoretical ones somewhat more (0.52 and 0.35 respectively). Perhaps the decrease of the speed limit is the effect of the viscosity and plasticity influence which is not taken into account in the theoretical consideration (the plasticity influence is estimated below). Note here that a small change of the crack speed (in comparison with the speed limit) leads to a smaller (second order) change of the energy dissipation rate, because the limit corresponds to maximum rate.

The theoretical results for fracture modes II and III predict higher values of the limit crack speed and shear cracks are expected to be faster. This result is confirmed, to some extent, indirectly from data for the shear crack speed. Some results are pointed out by BEROZA and SPUDICH (1988) using two methods of the natural data estimation. It turns out that the shear crack speeds $v = 0.7c_2$ from one method, and $v = 0.8c_2$ from another, are the most suitable for the natural data description (application to the 1984 Morgan Hill, California, earthquake), see also HEATON (1990). The theoretical results for shear cracks (see Table 2) correspond to somewhat lower speeds: $0.6 \leqslant v/c_2 \leqslant 0.7$.

To compare the crack motion under speed-independent fracture energy with the motion of the same crack under the principle of maximum energy dissipation rate, consider once again, the unbounded elastic body containing a half plane crack. Suppose that material is initially at rest, and that time-independent uniformly distributed normal loading σ_- is applied to the crack surfaces at the moment $t = 0$. At a moment $t = t_0$ the crack starts, and the loading propagates on the new surfaces of the crack. Thus

$$\sigma_- = AH(t)H(l(t)-x), \quad A = \text{const.}$$

In general outline, the cracks motions are as follows. In the case of constant fracture energy, the crack speed increases ($t \geqslant t_0$) and tends to the Rayleigh surface wave speed (curve 1 in Fig. 3). The crack speed under the discussed principle also increases at $t > t_0$ (curve 2 in Fig. 3) but in contrast with the previous case the crack speed tends to the above obtained limit v_N (line 3 in Fig. 3).

It should be noted that the above obtained crack speed limits for the infinite body are also valid for any homogeneous elastic body. To obtain a proof of this statement, it is necessary to take into account the fact that a boundary influence reaches the crack tip with some delay, and hence the boundary influence at any given moment does not depend on the crack speed at the same moment. Therefore, these boundary conditions are not essential if the instantaneous speed variation is used.

However, the crack speed limit depends on the body model. Consider the solution to the dynamic problem for a long rectangular plate being split in half by a wedge (FREUND, 1990). The principle of maximum energy dissipation rate leads (assuming cross-sections obey the beam theory) to extremal speed $v = c_0/\sqrt{3}$, $c_0 = \sqrt{E/\rho}$

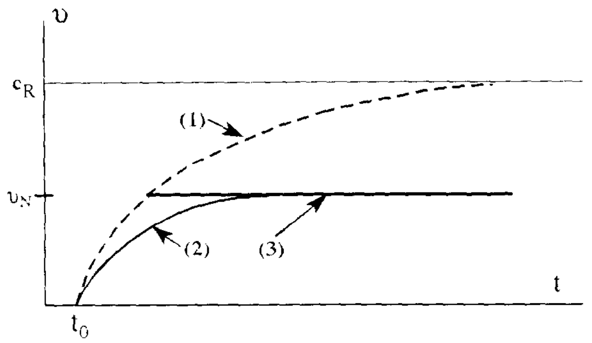

FIG. 3. Crack speed based on Griffith's criterion (1), crack speed based on criterion of maximum energy dissipation rate (2), and the crack speed limit (3).

where E is the Young's elastic modulus, and ρ is material mass density ($N =$ const $(1 - v^2 c_0^{-2})v$).

5. ELASTIC–PLASTIC BODY

Consider an elastic–plastic problem for fracture mode III. We assume for simplicity that the plastic region is a narrow zone in front of the crack—we use the well-known foundation by BARENBLATT (1959), DUGDALE (1960), LEONOV (1961) and PANASYUK (1968). We consider the self-similar problem for a semi-infinite straight crack in the unbounded elastic body (the plasticity appears only in boundary conditions). We have the boundary conditions ($y = 0$) for the half-plane x, y ($-\infty < x < \infty$, $y \geqslant 0$)

$$\sigma_{yz} = \sigma_- = [-pH(vt - x) + kH(x - vt)H(wt - x)]H(t) \quad (x < wt),$$
$$u_z = u = 0 \quad (x > wt), \tag{13}$$

where z is the third axis, $-p = \text{const} < 0$ is a shear stress which acts on the crack surfaces ($x < vt$), $k = \text{const} > 0$ is the same stress but in plastic zone, w is the velocity of the plastic zone front, and v is the velocity of its internal boundary—the crack velocity (Fig. 4). The initial conditions are at zero. We also have an additional condition: there is no energy flux into the moving point $x = wt$.

The displacement of the crack surface—half of the crack opening displacement is possible to obtain using the first formula (8) for fracture mode III.

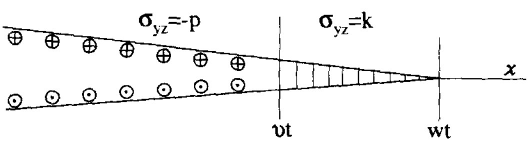

FIG. 4. Elastic–plastic problem for fracture mode III.

$$u = S_- ** [(S_+ ** \sigma_-)H(wt-x)],$$

$$S_+ = \sqrt{\frac{c}{\pi}}\, t_+^{-1/2}\delta(ct-x), \quad S_- = -\frac{1}{\mu}\sqrt{\frac{c}{\pi}}\, t_+^{-1/2}\delta(ct+x), \tag{14}$$

where $c = c_2$, symbol $**$ means the double convolution, the same as above and δ is the Dirac delta function, $t_+^{-1/2} = 1/\sqrt{t}\,(t > 0)$, $t_+^{-1/2} = 0$ $(t < 0)$.

The representation (14), and the condition that there is no energy flux into the point $x = wt$ lead to the equality

$$w = v + (c-v)\lambda^{-2}, \quad \lambda = \frac{k+p}{p} \tag{15}$$

and to the following expression for the crack surface speed

$$\frac{\partial u}{\partial t} = \frac{cp}{\pi\mu}\left\{\frac{2w}{c+w}\left(\frac{t-\tau}{\tau}\right)^{1/2} + \arccos\left(1-\frac{2\tau}{t}\right)\right.$$

$$\left. -\lambda\left[\frac{2w}{c+w}\left(\frac{x-vt+(c+v)\tau}{(c-v)\tau}\right)^{1/2} - \frac{v}{\sqrt{c^2-v^2}}\operatorname{arccosh}\left(1+2\tau\frac{c+v}{x-vt}\right)\right]\right\} \tag{16}$$

$$0 < \tau = \frac{wt-x}{c+w} < t \quad (0 \leqslant v < w < c, vt \leqslant x \leqslant wt).$$

In this case the dissipation rate is the plastic strain work per unit time

$$N = 2k\int_{vt}^{wt}\frac{\partial u}{\partial t}\,\mathrm{d}x. \tag{17}$$

Using (16) we obtain

$$N = \frac{cpk}{\pi\mu}\,t\left[\frac{2}{\lambda}(c^2-v^2)^{1/2} - (c-w)\operatorname{arccosh}\left(l-2\frac{w-v}{c+w}\right)\right]. \tag{18}$$

In the self-similar problem framework we can use the principle of maximum N only for the averaged speed. The extremal v-value which gives us the maximum work up to the moment: $\max \int_0^t N\,\mathrm{d}t$ since a variation of v in (18) at the given moment $t = t_1$ means the same variations at all times $0 < t < t_1$. In contrast to variation at the moment—the "local variation"—this is a "global variation". The result below must be considered just in this sense.

There are two limit cases which show a basic tendency. Let $\lambda \to 1$ $(k \ll p)$. In this case

$$w \to c, \quad N \to 2\frac{cpkt}{\pi\mu}\sqrt{c^2-v^2} \tag{19}$$

and the dissipation rate is maximum if $v = 0$. Maximum energy dissipation rate is achieved only by the maximum rate of the length of the plastic zone: $(w-v)\,t \to ct$. Thus the crack speed tends to zero with the yield limit.

The second case corresponds with $\lambda \to \infty$ $(p \ll k)$. In this case

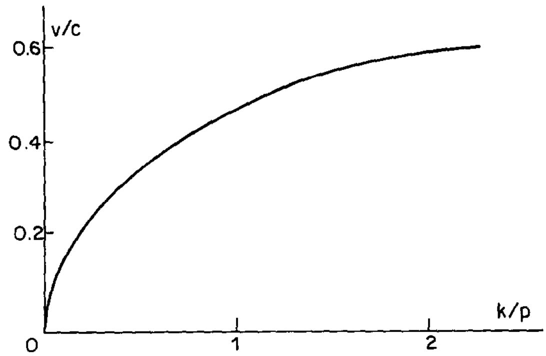

FIG. 5. Crack velocity limit in an elastic–plastic body.

$$w \to v, \quad N \to \frac{4cp^2}{\pi\mu} tv\sqrt{\frac{c-v}{c+v}}. \tag{20}$$

This result is the same as above for a perfectly elastic body. Really, if $\lambda \to \infty$ the plastic zone appears as a point and the influence of plasticity disappears. Furthermore the global and the local variations are the same in self-similar elastic problems. The results of calculations of v as a function of λ using (15), (18) are represented in Fig. 5.

6. CONCLUDING REMARKS

Two different kinds of variation are used above for the elastic and elastic–plastic problems: the local variation—the variation of the instantaneous speed and the global variation—assuming the crack speed is independent of time. These variations lead to the same results in the considered general and self-similar ($k \to \infty$) elastic problems. In other cases, however, results can be different. Consider for example the propagation of a semi-infinite crack in a homogeneous initially stressed elastic strip (Fig. 6). In a steady-state problem total strain energy flows (as a result of unloading of the strip by the crack) to the propagating crack tip if the crack speed is lower than the Rayleigh wave speed. [This phenomenon was noted by RICE (1968) for some equilibrium situations and by FREUND (1990) for the steady-state dynamic problem.] In this case

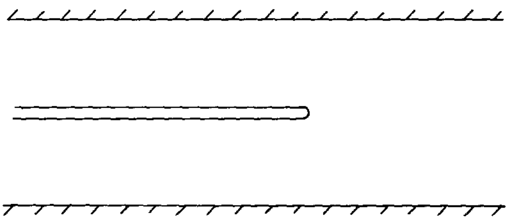

FIG. 6. Crack propagation in an initial stressed elastic strip.

the global variation leads to extremal speed which is as high as possible (but lower than the Rayleigh wave speed because the energy release G is a constant, and $M = (G-2\gamma)v$ is maximal if $v = v_{\max}$. However, if we use the local variation we obtain the extremal speed the same as above—about half the Rayleigh wave speed (see the last part of Section 4).

The "correct" approach is local but only on the macrolevel, and it seems that the difference with the global variation results may somehow influence the process. It may lead to speed oscillation. Likely problems including the above elastic–plastic problem are required to be examined as non-uniform crack speed problems.

The above experimental results show that under ordinary conditions the dynamic crack propagation looks like any energy absorption process, and obeys the extremal principle. It is necessary, however, to stress that the value of the extremal crack speed can be changed in the presence of another influence, such as the above mentioned weak bond. In such cases the variational principle should be used taking into account these additional conditions or restrictions, as was discussd in the Introduction. Such kinds of restrictions have to be included in the variational formulation of the criterion.

ACKNOWLEDGEMENTS

The author appreciates Professor J. R. Rice's interest in this work, his critical remarks and helpful advice. The author would also like to express his gratitude to Professor A. V. Cherkaev and A. I. Chudnovsky for helpful discussions, to Professors J. D. Achenbach, L. B. Freund, I. A. Kunin, K. A. Lurie, R. Parnes, K. Ravi-Chandar, G. Rodin and D. I. Shilkrut for their seminar invitiations and for their attention to this work. Lastly, the support provided by Professor J. P. Dempsey at Clarkson University through his U.S. Office of Naval Research Grant No. N00014-90-J-1360 is greatly appreciated.

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