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ANALYSIS OF STRUCTURAL–ACOUSTIC COUPLING PROBLEMS BY A TWO-LEVEL BOUNDARY INTEGRAL METHOD, PART 1: A GENERAL FORMULATION AND TEST PROBLEMS

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A generalized formulation of a boundary integral equations method is presented for analysis of forced vibrations of a composite elastic structure immersed in compressible inviscid fluid. The structure is supposed to consist of parts which are membranes, plates, spherical, conical or cylindrical shells. Both the interaction between the acoustic medium and the composite structure as a whole, and the interactions between the parts of the structure, are described by boundary integral equations. These boundary integral equations are assembled in a two-level system. The first level boundary integral equations govern the dynamics of the above-mentioned "simple" parts of the structure. They contain unknown boundary displacements and forces, contact acoustic pressure and driving loads. The kernels of these equations are Green functions of "simple" unbounded structures vibrating *in vacuo*. These functions have explicit analytical forms. The boundary integral equation of the second level governs the interaction between the fluid and the structure as a whole. The classical boundary integral equation related to the contact acoustic pressure is modified by substitution of Somigliana-type formulae for normal displacments on each part of the structure. As a result the second level equation constitutes the connection between the contact acoustic pressure, driving loads and the boundary displacements and forces on the edges of the simple parts of the structure. The kernels of this boundary integral equation are convolutions of Green functions of simple unbounded structures and unbounded fluid. The validity of the method proposed is demonstrated for several simple test problems analyzed earlier by other authors. The aim of this part of the paper is to outline and evaluate the numerical procedure. A detailed analysis of the equations of vibrations of a composite thin-walled structure in an acoustic medium is presented in Part 2 of this paper.
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1. INTRODUCTION

Consider an arbitrary thin-walled structure, in contact with an ideal compressible fluid. A velocity potential φ satisfies the Helmholtz equation in the fluid:

$$\nabla^2\varphi + (\omega/c)^2\varphi = 0. \tag{1}$$

The dependence of all functions herein upon time is taken in the form $\exp(-\mathrm{i}\omega t)$ and this multiplier is henceforth omitted, ω is the circular frequency of the oscillations and c is the speed of sound in the fluid (a list of notation is given in Appendix B).

The acoustic pressure p is related to φ by

$$p = \mathrm{i}\rho\omega\varphi. \tag{2}$$

Here ρ is the ambient density of the fluid (assumed to be uniform and at rest apart from the acoustic disturbances).

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The vibrations of the structure are governed by the differential equations

$$[L]\{w\} = \{q\} + \delta_{j3}p. \tag{3}$$

$\{w\}$ is the displacements vector, the component of $\{w\}$ that is normal to the surface of the thin-walled structure is denoted by w_3, and $[L]$ is a tensor differential operator defined by the mathematical model chosen to describe the vibrations of the structure. The positive direction of the displacement w_3 corresponds to that of the outward unit normal to the fluid domain. $\{q\}$ is a vector of external driving loads applied to the structure. These driving loads are considered to be given and they are generated by some external sources of excitation of the vibrations. δ_{j3} is the Kronecker symbol. The coupling term enters only the third equation for the vibrations of the structure (that which corresponds to the normal displacement w_3).

Inasmuch as an inviscid acoustic fluid medium is considered, only the displacement components w_3 appears in the condition on the "wet" surface of the structure:

$$\partial\varphi/\partial v_+ = u = -\mathrm{i}\omega w_3. \tag{4}$$

u is the normal velocity of a point on a surface of the structure as well as the normal velocity of the corresponding particle of fluid; v_+ is distance along the outward unit normal to the fluid domain at the surface.

Exact analytical solutions for coupled problems of acousto- and hydroelasticity defined by equations (1)–(4) have been presented in references [1–6] for unbounded membranes, plates, cylindrical shells and full spherical shells. These model problems give physical insight into the interaction between an acoustic medium and an elastic body. Exact solutions are not available for problems of vibrations of composite thin-walled structures, but various numerical procedures may be selected for assessing the acoustic response of the structure. The most common approach consists of replacing problem (1)–(4) by its finite element (F.E.) approximation, with the conditions (4) satisfied at the interfacial nodes of the meshes for the fluid and for the structure [7–9]. To consider an unbounded acoustic medium, semi-infinite elements are used [7, 8] which satisfy the Sommerfeld conditions. In this formulation the system of linear algebraic equations contains all unknown amplitudes of the displacements over the surface of the structure and the amplitudes of the acoustic pressure in the fluid. This system has a high order and special attention has to be paid to the storage and handling of this information.

An alternative approach—a combination of finite elements for the structure and boundary elements for the acoustic medium—has appeared to be more efficient [10–12]. The boundary integral equation related to the acoustic pressure is replaced by a system of linear algebraic equations which connects the values of the acoustic pressure at the nodes of the boundary elements mesh with the normal velocities at the same points. Substitution of the solution for this system into the F.E. approximation of equation (3) gives the traditional F.E. relationship between the amplitudes of the displacements and the driving loads. The matrix of rigidity for the "dry" structure is then supplemented by the so-called added mass matrix, which is obtained by the inversion of the previously mentioned system of linear algebraic equations for the acoustic pressure. The valuable advantage of this method for numerical analysis is that it permits one to utilize all the well-known F.E. packages (NASTRAN, ANSYS, COSMOS, etc.) to solve the structural–acoustic coupling problem.

Another numerical approach to analyzing structural–acoustic coupling problems is a combination of the boundary integral equation method for the fluid and a modal analysis of vibrations for the structure [13–17]. Natural modes of vibrations of a "dry" structure provide an orthonormal basis which entirely describes the dynamic properties of the structure. This numerical procedure is very convenient and efficient when the natural modes of vibrations are simple (for example, hinged plates and shells). An efficient technique is described in reference [13] for analysis of vibrations of composite thin-walled shells of revolution. The pivotal condensation method is used to obtain the natural modes of vibrations for the whole structure. Then unknown displacements and the acoustic pressure on the surface of the structure are expanded into modal series. The Bubnov–Galerkin orthogonalization procedure results in a system of linear algebraic equations relating the amplitudes of the vibrations to the acoustic pressure for each mode.

It should be noted that all the numerical techniques mentioned are based on the same idea; to formulate two conjoint systems of linear algebraic equations (the first one related to the acoustic pressure at the nodes of a mesh on the surface of a structure and the second one related to displacements at the same nodes) and then to invert the matrix. These formulations may be used for analysis of vibrations for structures of complex geometries, but they yield little physical insight into the process of structural–acoustic interaction.

One more way in which to analyze structural–acoustic coupling problems is the Green functions method. The basic idea of this approach is to use Green functions of the structure to express the normal displacement w_3 in terms of the driving loads and acoustic pressure. Then there are two ways in which to assess the acoustic response of the structure. The first one is described in references [18–20]. It is based on the use of Green functions constructed for fluid-loaded unbounded structures. This numerical procedure has been used to analyze the problem of vibrations of a plate [18] and of a cylindrical shell of finite length in an infinitely long rigid baffle [19, 20]. The Green functions are constructed by the Fourier transform method. An alternative approach provides that Green functions constructed for a "dry" structure of finite length are incorporated into the numerical analysis. In reference [21], the finite element method is used to construct the Green function for the plane structure considered; in reference [22] the Green function for static loading is used in formulation of the boundary integral equations for the structure. Boundary integral equations are formulated for both the acoustic medium and the elastic structure, and this system of equations is solved in a manner which is typical of F.E.–B.E. coupling.

A further development of numerical methods in structural–acoustic coupling problems is a formulation of the problem for composite thin-walled structures in a way which gives room for detailed asymptotic analysis of each particular problem. This aim may be achieved by the use of the advanced formation of the B.I.E. method suggested and developed in references [23, 24]. The basic idea of this method is the consistent use of boundary integral equations to describe both the interaction of the acoustic medium with the structure and the interactions of elementary parts of the structure with each other.

2. A SYSTEM OF BOUNDARY INTEGRAL EQUATIONS

The thin-walled structure considered consists of N parts. Surfaces corresponding to each of them are denoted $S^{(n)}$, $(n = 1, 2, \ldots, N)$, respectively. Let each element of a composite structure be a section of an unbounded plate, cylindrical or spherical shell. Only M $(M \leqslant N)$ parts of the composite structure are in contact with the acoustic medium. A harmonic driving load is arbitrarily distributed along the surface of the whole structure.

The boundary integral equation of motion of an acoustic medium is well-known [25, 26]:

$$(1 - C)p(\mathbf{X}) + \int_S [F_0(\mathbf{X}, \mathbf{Y})p(\mathbf{Y}) - \mathrm{i}\rho\omega G(|\mathbf{X} - \mathbf{Y}|)u(\mathbf{Y})]\,\mathrm{d}S_y = 0. \qquad (5)$$

Here $\mathbf{X}$ and $\mathbf{Y}$ are, respectively, the radius–vectors of the observation point and the source, S is the boundary of the acoustic medium, u is the normal velocity of the acoustic medium on the surface S (positive values correspond to the outward normal for the fluid domain), $4\pi C$ is the spatial angle unoccupied by the medium (if a tangential plane exists at the point $\mathbf{X}$, then $C = 1/2$).

The function $G(|\mathbf{X}-\mathbf{Y}|)$ is the fundamental solution of the inhomogeneous Helmholtz equation corresponding to a point source in an unbounded acoustic medium; $\mathrm{i}\rho\omega G(|\mathbf{X}-\mathbf{Y}|)$ is the acoustic pressure generated by this source. The function G depends only upon the distance between an observation point and a source point $|\mathbf{X}-\mathbf{Y}|$; $F_0(\mathbf{X},\mathbf{Y})$ is the normal derivative of $G(|\mathbf{X}-\mathbf{Y}|)$ on S at the point $\mathbf{Y}$: i.e.,

$$F_0(\mathbf{X},\mathbf{Y}) = \partial G(|\mathbf{X}-\mathbf{Y}|)/\partial \nu_+,$$

where ν_+ is distance along the unit outward normal to the fluid domain. This function may also be represented as

$$F_0(\mathbf{X},\mathbf{Y}) = \lambda F(|\mathbf{X}-\mathbf{Y}|),$$

$$F(|\mathbf{X}-\mathbf{Y}|) = \mathrm{d}G(|\mathbf{X}-\mathbf{Y}|)/\mathrm{d}(|\mathbf{X}-\mathbf{Y}|), \qquad \lambda = \boldsymbol{\nu}_+(\mathbf{Y}-\mathbf{X})/|\mathbf{X}-\mathbf{Y}|.$$

Here $F(|\mathbf{X}-\mathbf{Y}|)$ is a function depending only upon the distance $|\mathbf{X}-\mathbf{Y}|$, and λ is the scalar product of the two unit vectors $\boldsymbol{\nu}_+$ and $(\mathbf{Y}-\mathbf{X})/|\mathbf{X}-\mathbf{Y}|$.

Let the motions of each nth simple part of the thin-walled structure be described by equation (3):

$$\sum_{\alpha=1}^{K} L_{j\alpha}^{(n)} w_\alpha(\mathbf{X}) = q_j^{(n)}(\mathbf{X}) + \delta_{j3} p(\mathbf{X}) H(\mathbf{X}), \qquad \mathbf{X}\in S^{(n)}, \quad n = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, K. \tag{6}$$

If this part of the structure is in contact with an acoustic medium, then the acoustic pressure should be taken into account: $H(\mathbf{X}) = 0$, $n > M$; $H(\mathbf{X}) = 1$ and $n \leqslant M$. $w_\alpha^{(n)}$ are the components of the generalized displacements vector on the nth part of the thin-walled structure, incorporated in the differential equations of the vibrations. The number of these components K is determined by the mathematical model chosen. For instance, when a Mindlin–Timoshenko shell model is used, there are five components of this vector. The first three of them are the tangential (w_1 and w_2) and normal (w_3) displacements, and the other two components are the angles of rotation in the tangential directions w_4 and w_5. If the Novozhilov–Goldenveizer theory of thin shells is used, then there are only three components of $\{w\}$; w_1, w_2 and w_3. If a plate without shear deformation and rotary inertia is considered (Kirchhoff–Love theory) then there is the only one component of the vector; the normal displacement w_3. Relevant differential operators are well-known [2, 27] and therefore are not presented here.

Equation (6), defined for each elementary part of the structure, should be supplemented by conditions of continuity of the generalized displacements and equilibrium of the generalized forces on the edges for neighboring parts of the structure. It should be noted that for any Kirchhoff–Love type theory of thin-walled structures the full number $\bar{K}$ of components of the generalized displacements vector on an edge equals $K+1$. In this formulation the additional component is an angle of rotation, defined as the derivative of the normal displacement by the co-ordinate normal to the edge: $\partial w_3/\partial \nu_r$ (ν_r is the distance along the unit outward normal to the contour Γ in the plane tangential to $S^{(n)}$ at the point $\mathbf{Y}$). A vector of generalized forces on the edge of the nth part of the structure has the same number of components. The most general formulation of these conditions (when the interaction of the nth and the lth parts of the structure is considered) has the following form:

$$\sum_{\alpha=1}^{\bar{K}_n} \mu_{\alpha j}^{(n)} w_\alpha^{(n)} + \sum_{\alpha=1}^{\bar{K}_n} \varkappa_{\alpha j}^{(n)} Q_\alpha^{(n)} = \sum_{\beta=1}^{\bar{K}_l} \mu_{\beta j}^{(l)} w_\beta^{(l)} + \sum_{\beta=1}^{\bar{K}_l} \varkappa_{\beta j}^{(l)} Q_\beta^{(l)},$$

$$j = 1, 2, \ldots, \max(\bar{K}_n, \bar{K}_l). \tag{7}$$

Here $\bar{K}_n$ and $\bar{K}_l$ are the numbers of components of the vectors of the generalized forces and displacements on the interfacial edge for the nth and the lth parts of the structure, respectively; $\mu_{\alpha j}^{(n)}$, $\varkappa_{\alpha j}^{(n)}$, $\mu_{\beta j}^{(l)}$ and $\varkappa_{\beta j}^{(l)}$ are some numerical coefficients. If the coefficients $\mu_{\alpha j}^{(n)}$ and $\mu_{\beta j}^{(l)}$ are equal to zero, then one has some equilibrium conditions. If $\varkappa_{\alpha j}^{(n)}$ and $\varkappa_{\beta j}^{(l)}$ are equal to zero, then equation (7) gives the conditions of continuity for the displacements. The number of conditions (7) for each particular case of conjunction (Goldenvejzer–Novozhilov shell – Mindlin plate, Mindlin plate – Kirchhoff plate, etc.) should be equal to the maximum number of components of the generalized displacements vectors for the nth and the lth parts of the structure.

The substitution of equation (4) into equation (5) permits one to reformulate the latter as

$$(1 - C)p(\mathbf{X}) + \int_S [\lambda F(|\mathbf{X} - \mathbf{Y}|)p(\mathbf{Y}) - \rho\omega^2 G(|\mathbf{X} - \mathbf{Y}|)w_3(\mathbf{Y})]\, \mathrm{d}S_z = 0. \tag{8}$$

This integral equation contains two unknown functions; the amplitudes of the contact acoustic pressure p and of the normal displacement w_3. It should be emphasized that in equation (8) both the fundamental solution G and its derivative F depend only upon the distance between a point of observation and a source point.

To obtain the integral equation with the sole unknown function p the displacement w_3 should be expressed in terms of the driving load $\{q\}$ and the contact acoustic pressure p. This operation may be done in a way suggested in reference [23], by using an integral representation of w_3 in terms of the Green function of the whole structure:

$$w_3(\mathbf{X}) = \int_S \sum_{\alpha=1}^{K} q_\alpha(\mathbf{Z}) W_{3\alpha}^0(\mathbf{X}, \mathbf{Z})\, \mathrm{d}S_z + \int_S H(\mathbf{Z})p(\mathbf{Z})W_{33}^0(\mathbf{X}, \mathbf{Z})\, \mathrm{d}S_z. \tag{9}$$

$W_{3\alpha}^0$ is the Green function of the whole structure.

For any composite structure the Green function $W_{3\alpha}^0(\mathbf{X}, \mathbf{Z})$ depends upon two vectorial variables (the co-ordinates of a point of observation and a source separately) and these functions cannot be represented in a closed analytical form. In fact, to obtain them it is necessary to use some kind of numerical procedure. Then there are no advantages in the use of Green functions rather than just the simple F.E. procedure, because the latter may be incorporated directly into the numerical solution.

If a structure consists of fragments of ‘‘homogeneous’’ surfaces, plates, spheres or cylindrical shells, then an efficient numerical procedure may be used. This is based on the fact that the dynamics of these parts and their interaction may also be described by boundary integral equations. The kernels of the latter (Green functions of unbounded shells) may be represented in an analytical form and depend upon only one argument: the distance between an observation point and a source point.

The integral in the left side of equation (8) consists of the sum of integrals over the mentioned areas:

$$(1 - C)p(\mathbf{X}) + \sum_{n=1}^{M} \int_{S^{(n)}} [\lambda F(|\mathbf{X} - \mathbf{Y}|)p(\mathbf{Y}) - \rho\omega^2 G(|\mathbf{X} - \mathbf{Y}|)w_3^{(n)}(\mathbf{Y})]\, \mathrm{d}S_y^{(n)} = 0. \tag{10}$$

Each function $w_3^{(n)}$ is defined by the Somigliana-type formula (which may be easily derived from the reciprocity theorem)

$$w_3^{(n)}(\mathbf{Y}) = \int_{\Gamma^{(n)}} \sum_{\alpha=1}^{\bar{K}_n} [W_{3\alpha}^{(n)}(\mathbf{Y}-\mathbf{Z}_r)Q_\alpha^{(n)}(\mathbf{Z}_r) - \sum_{\beta=1}^{3} Q_{3\beta\alpha}^{(n)}(\mathbf{Y}-\mathbf{Z}_r)v_\beta(\mathbf{Z}_r)w_\alpha^{(n)}(\mathbf{Z}_r)]\,\mathrm{d}\Gamma_z^{(n)}$$
$$+ \int_{S^{(n)}} \left[\sum_{\alpha=1}^{K_n} q_\alpha^{(n)}(\mathbf{Z})W_{3\alpha}^{(n)}(\mathbf{Y}-\mathbf{Z}) + p(\mathbf{Z})W_{33}^{(n)}(\mathbf{Y}-\mathbf{Z})H(\mathbf{Z})\right]\mathrm{d}S_z^{(n)},$$
$$n = 1, 2, \ldots, N. \tag{11}$$

Here $w_\alpha^{(n)}(\mathbf{Z}_r)$ and $Q^{(n)}(\mathbf{Z}_r)$ are the generalized displacements and forces on a contour $\Gamma^{(n)}$; $W_{3\alpha}^{(n)}(\mathbf{Y}-\mathbf{Z})$ is the Green function of the relevant unbounded structure; $Q_{3\beta\alpha}^{(n)}(\mathbf{Y}-\mathbf{Z})$ is the tensor of forces corresponding to the fundamental solution $W_{3\alpha}^{(n)}(\mathbf{Y}-\mathbf{Z})$; $v_\beta(\mathbf{Z}_r)$ is the unit outward normal to the domain $S^{(n)}$ at the contour $\mathbf{\Gamma}^{(n)}$ in the plane tangential to $S^{(n)}$ at the point $\mathbf{Z}_r$.

The substitution of equation (11) into equation (10) gives the main integral equation for the interaction of the composite structure with the acoustic medium in the following form:

$$(1-C)p(\mathbf{X}) + \sum_{n=1}^{M} \left\{ \int_{S^{(n)}} [\lambda F(|\mathbf{X}-\mathbf{Y}|) - \rho\omega^2 \Phi_{33}^{(n)}(\mathbf{X},\mathbf{Y})]p(\mathbf{Y})\,\mathrm{d}S_y^{(n)} \right.$$
$$- \rho\omega^2 \int_{\Gamma^{(n)}} \sum_{\alpha=1}^{\bar{K}_n} [Q_\alpha^{(n)}(\mathbf{Z}_r)\Phi_{3\alpha}^{(n)}(\mathbf{X},\mathbf{Z}_r) + w_\alpha^{(n)}(\mathbf{Z}_r) \sum_{\beta=1}^{3} \Psi_{3\beta\alpha}^{(n)}(\mathbf{X},\mathbf{Z}_r)v_\beta(\mathbf{Z}_r)]\,\mathrm{d}\Gamma_z^{(n)}$$
$$= \rho\omega^2 \sum_{n=1}^{M} \int_{S^{(n)}} \sum_{\alpha=1}^{K_n} q_\alpha^{(n)}(\mathbf{Y})\Phi_{3\alpha}^{(n)}(\mathbf{X},\mathbf{Y})\,\mathrm{d}S_y^{(n)},$$
$$\Phi_{3\alpha}^{(n)}(\mathbf{X},\mathbf{Y}) = \int_{S^{(n)}} G(|\mathbf{X}-\mathbf{Z}|)W_{3\alpha}^{(n)}(\mathbf{Y}-\mathbf{Z})\,\mathrm{d}S_z^{(n)},$$
$$\Psi_{3\alpha\beta}^{(n)}(\mathbf{X},\mathbf{Y}_r) = \int_{S^{(n)}} G(|\mathbf{X}-\mathbf{Z}|)Q_{3\beta\alpha}^{(n)}(\mathbf{Z}-\mathbf{Y}_r)\,\mathrm{d}S_z^{(n)}. \tag{12}$$

The values for the boundary forces and displacements may be found from equation (11) by letting $\mathbf{Y}$ be a point on the contour $\Gamma^{(n)}$:

$$w_j^{(n)}(Y_r) = \int_{\Gamma^{(n)}} \sum_{\alpha=1}^{\bar{K}_n} [Q_\alpha^{(n)}(\mathbf{Z}_r)W_{\alpha j}^{(n)}(\mathbf{Y}-\mathbf{Z}_r) - \sum_{\beta=1}^{3} Q_{j\alpha\beta}^{(n)}(\mathbf{Y}_r-\mathbf{Z}_r)v_\beta(\mathbf{Z}_r)w_\alpha^{(n)}(\mathbf{Z}_r)]\,\mathrm{d}\Gamma_z^{(n)}$$
$$+ \int_{S^{(n)}} \left[\sum_{\alpha=1}^{K_n} q_\alpha^{(n)}(\mathbf{Z})W_{\alpha j}^{(n)}(\mathbf{Y}_r-\mathbf{Z}) + p(\mathbf{Z})W_{3j}^{(n)}(\mathbf{Y}_r-\mathbf{Z})H(\mathbf{Z})\right]\mathrm{d}S_z^{(n)},$$
$$j = 1, 2, \ldots, K_n + 1. \tag{13}$$

The system of boundary equations (11), (12) is supplemented by the boundary conditions (7) (conditions on interfacial regions).

It is well-known in acoustics that special attention in the analysis of a radiation field should be paid to the interfacial regions of a composite structure [19, 25]. The forces and displacements at these interfacial regions are the unknown functions in the two-level formulation, and their influence upon the contact acoustic pressure may be easily evaluated.

In the framework of the boundary integral equations method for structural–acoustic coupling problems one may use either the system of two equations (8) and (9), or the two-level system of equations (12), (13) and (7) . Despite the increase in the number of equations, the two-level formulation brings a significant reduction in the computing time required, because the kernels of equation (12)—convolutions of Green functions of the acoustic medium and the parts of the structure—have simple asymptotic expansions in some ranges of the parameters and therefore may be evaluated analytically. This permits one to elaborate some simplified models of interaction and to estimate their validity.

The two-level system of boundary integral equations formulated above is based on the use of fundamental solutions (Green functions) for each isolated component of the "acoustic medium – complex thin-walled structure" system. It is essential to compile a catalog of the fundamental solutions. The fundamental solutions used in analyses of particular test problems considered in section 3 are presented in Appendix A.

3. TEST PROBLEMS

The problem of the most interest in many technical applications in that of the vibrations of a cylindrical shell in an unbounded acoustic medium. The shell is of finite length, and it has plane end caps and, possibly, several intermediate plane bulkheads. The cyclo-symmetric formulation of the problem is possible when an arbitrary driving load can be expanded into a series of functions of an angular co-ordinate, and thus each circumferential mode of vibration of a thin-walled structure in an acoustic medium can be analyzed separately. Several test problems have been analyzed to check the validity and the efficiency of the numerical procedure based on the two-level B.I.E. method formulated in section 2. This choice of test problems is aimed at validating the general numerical procedure part by part. Then the numerical procedure may be applied to analyze vibrations of the particular composite thin-walled structure mentioned above. The data for this structure are presented in Part 2 of this paper, in which details of the numerical analysis for this structure are presented.

3.1. VIBRATIONS OF AN INFINITELY LONG CYLINDRICAL SHELL (PLANE PROBLEM)

Vibrations of a circular ring (an infinitely long cylindrical shell) of radius R and thickness h immersed in an unbounded acoustic medium were analyzed first. The driving load was a sinusoidal one with m circumferential waves: $q(\theta) = q_m \cos m\theta$. In this particular case there are simple analytical formulae for the amplitude of the radial displacement [2, 4, 6]:

$$w_3 = q_m(1-\nu^2)E^{-1}h^{-1}\left[\left\{\left(\frac{m}{R^2}+\frac{h^2m^3}{12R^4}\right)^2\Big/\left(\Omega^2-\frac{m^2}{R^2}\right)\right\}+\frac{h^2m^4}{12R^4}+\frac{1}{R^2}\right.$$
$$\left.-\,\Omega^2\left(1+\frac{\rho R\alpha}{\rho_0 h}\right)\right]^{-1}\cos m\theta. \qquad (14)$$

Here

$$\alpha = \left[m - \frac{\omega R}{c}\,\mathrm{H}^{(1)}_{m-1}\left(\frac{\omega R}{c}\right)\Big/\mathrm{H}^{(1)}_{m}\left(\frac{\omega R}{c}\right)\right]^{-1},$$

E and ρ_0 are the Young's modulus and the density of the material, respectively, and $\mathrm{H}^{(1)}_k$ is a Hankel function of the first kind of order k. A similar expression exists for the circumferential displacement.

The numerical analysis was performed for a full ring with the boundary conditions at the points $\theta = 0$ and $\theta = 2\pi$ corresponding to uniform attachment. Vibrations of a steel shell in a water were considered with $c/c_0 = 0{\cdot}307$ and $\rho/\rho_0 = 0{\cdot}128$.

The two-level system of boundary equations for this particular case consisted of six equations of the first level (three equations (13) for each edge $\theta = 0$ and $\theta = 2\pi$) and equation (12) of the second level. Fundamental solutions for the shell were taken in the form (A9); for an unbounded acoustic medium the fundamental solution was taken in the form (A2). Then, after the boundary conditions were utilized, there were six algebraic unknowns representing displacements and forces of the edges $\theta = 0$ and $\theta = 2\pi$ and the unknown function p—a contact acoustic pressure.

Several ranges of excitation were analyzed. The numerical solution was obtained by piecewise constant approximation of the acoustic pressure in equations (12) and (13). Then a system of linear algebraic equations (SLAE) was to be solved. Special attention was paid to the convergence analysis. In all the cases considered the differences between the values for p obtained via equation (14) and by numerical analysis did not exceed 3% if the number of elements on the ring N_E satisfied the inequality

$$N_E > 8\pi R\lambda^{-1}, \tag{15}$$

where λ is the length of the leaky wave (the one corresponding to the pure imaginary root of a characteristic polynomial) existing in the isolated structure. The same estimation was obtained earlier in reference [25].

In Figure 1 some results are presented for a ring with $h/R = 0{\cdot}01$ and $qR(1 - \nu^2)/Eh = 1$. Figure 1(a) corresponds to vibrations at the dimensionless frequency $\omega R/c = 1{\cdot}1$; while Figure 1(b) corresponds to vibrations at the dimensionless frequency $\omega R/c = 11$. The amplitudes of radial (w_3) and circumferential (w_2) displacements are presented versus the number of circumferential waves of the driving load. Points marked by circles correspond to the exact formula (14); points marked by crosses correspond to the numerical solution.

One more test was performed for the same problem when the driving load considered was a point (i.e., line) radial force applied at the section $\theta = \pi$. The distribution of the contact acoustic pressure was compared with the one obtained in reference [21] by another

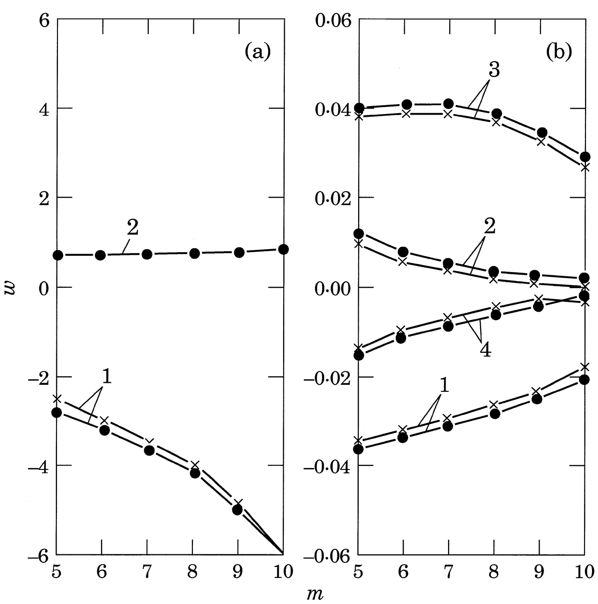

Figure 1. Amplitudes of displacements versus the number of circumferential waves. ×, Numerical solution; ●, exact solution (14). The curves on Figure 1(a) are plotted for $\omega R/c = 1{\cdot}1$; the curves on Figure 1(b) are plotted for $\omega R/c = 11$. The curves 1 and 2 represent the real parts for amplitudes of the radial (w_3) and tangential (w_2) displacements; the curves 3 and 4 on Figure 1(b) represent their imaginary parts.

numerical procedure. In reference [21], the concentrated force was simulated by the expression

$$q(\theta) = q_a[1 - (\theta/\theta_0)^2], \qquad -\theta_0 < \theta < \theta_0, \quad \theta_0 = \pi/100,$$

for a dimensionless frequency of $\omega R/c = 15$.

The distribution of the acoustic pressure appeared to be almost sinusoidal, with amplitudes slightly varying from one wave to another. The average value for the amplitude of the acoustic pressure was $p/q_0 = 0{\cdot}15$. The number of circumferential waves ($m = 36$) obtained in the numerical analysis performed by the B.I.E. method when $N_E = 150$ (150 elements around along the circumference of the ring) coincided with that obtained in reference [21].

3.2. EVALUATION OF THE GLOBAL IMPEDANCE OF A CYLINDRICAL SHELL OF FINITE LENGTH

A pure acoustical problem was analyzed: an evaluation of the contact acoustic pressure and the global impedance of a cylindrical shell of finite length l when the velocities on the surface of the shell are uniformly distributed. Detailed results for this problem were obtained in reference [28] and repeated in reference [24]. As in references [28] and [29] the velocities of the plane end caps were put to zero and the ones in the lateral surface of the cylinder were u_0. Then the boundary integral equation has the following form:

$$\tfrac{1}{2}p(x, r) + \int_0^1 F(x, r, 0, \eta)p(0, \eta)\eta \, \mathrm{d}\eta + \int_0^l F(x, 1, \xi, 1)p(\xi, 1) \, \mathrm{d}\xi$$
$$+ \int_0^1 F(x, r, l, \eta)p(l, \eta)\eta \, \mathrm{d}\eta + \mathrm{i}\rho\omega \int_0^l u_0 G(x, 1, \xi, 1) \, \mathrm{d}\xi = 0. \qquad (16)$$

Here $G(x, r, \xi, \eta)$ is the Green function (A3) for $m = 0$; (x, r) and (ξ, η) are the co-ordinates of an observation point and a source, respectively. The global impedance is determined by the formula [25, 28]:

$$Z = \frac{1}{\rho c |u_0|^2} \int_0^l p(\xi)\bar{u}_0(\xi) \, \mathrm{d}\xi, \qquad (17)$$

where $\bar{u}_0$ is the complex conjugate of the function u_0.

This test problem validated the efficiency of the numerical solution of the integral equation of the second level (12). Some values for the global impedance were obtained for $l = 6$ when $N_E = 48$ (N_E is the number of ring elements along the axis of the shell). These values are presented in Table 1.

3.3. VIBRATIONS OF A CIRCULAR PLATE IN AN UNBOUNDED RIGID BAFFLE

One more test problem readily available in the literature is that of vibrations with m circumferential waves (cyclosymmetric vibrations) of a plate in an acoustic medium. In

TABLE 1

Global impedance of a cylindrical shell of finite length

$\omega R/c$	[28, 29]	B.I.E.M.
0·2	0·104–0·3061	0·108–0·3121
0·4	0·331–0·4681	0·336–0·4741
0·6	0·517–0·4391	0·525–0·4481
1·0	0·712–0·3301	0·723–0·3391
1·5	0·802–0·2691	0·817–0·2811

particular, in reference [30] the analysis of the vibrations of a thin circular plate in an infinite rigid baffle has been performed.

The two-level system of boundary integral equations consisted of the equation of the second level (12) and two equations (13) of the first level:

$$\eta\left[-M_p(\eta)\frac{\partial W^{(p)}(r,\eta)}{\partial\eta}+Q_p(\eta)W^{(p)}(r,\eta)\right]\Bigg|_{\eta=R}+\int_0^R[q(\eta)+p(\eta)]W^{(p)}(r,\eta)\eta\,\mathrm{d}\eta=rw_3(r)$$
$$+\eta[-M_p^0(\eta,r)w_3'(\eta)+Q_p^0(\eta,r)w_3(\eta)]|_{\eta=R},\tag{18a}$$

$$\eta\left[-M_p(\eta)\frac{\partial^2 W^{(p)}(r,\eta)}{\partial\eta\,\partial r}+Q_p(\eta)\frac{\partial W^{(p)}(r,\eta)}{\partial r}\right]\Bigg|_{\eta=R}+\int_0^R[q(\eta)+p(\eta)]\frac{\partial W^{(p)}(r,\eta)}{\partial r}\eta\,\mathrm{d}\eta$$
$$=w_3(r)+rw_3'(r)+\eta\left[-\frac{\partial M_p^0(r,\eta)}{\partial r}w_3'(\eta)+\frac{\partial Q_p^0(r,\eta)}{\partial r}w_3(\eta)\right]\Bigg|_{\eta=R}.\tag{18b}$$

Here

$$M_p(\eta)=-D\left[\frac{\mathrm{d}^2w_3(\eta)}{\mathrm{d}\eta^2}+\frac{\nu}{\eta}\frac{\mathrm{d}w_3(\eta)}{\mathrm{d}\eta}-\frac{\nu m^2}{\eta^2}w_3(\eta)\right],$$

$$Q_p(\eta)=D\left[\frac{\mathrm{d}^3w_3(\eta)}{\mathrm{d}\eta^3}-\frac{1}{\eta}\frac{\mathrm{d}^2w_3(\eta)}{\mathrm{d}\eta^2}+\frac{1+m^2(2-\nu)}{\eta^2}\frac{\mathrm{d}w_3(\eta)}{\mathrm{d}\eta}-\frac{m^2(3-\nu)}{\eta^3}w_3(\eta)\right].$$

M_p^0 and Q_p^0 are represented similarly to M_p and Q_p via $W^{(p)}(r,\eta)$.

Equation (18b) is related to the angle of rotation in the radial direction. This equation is obtained by the differentiation of equation (18a) in respect to the co-ordinate of an observation point.

The boundary conditions were those of a plate with the edge clamped:

$$w_3(R)=w_3'(R)=0.\tag{19}$$

The main unknowns here are the acoustic pressure p and the amplitudes of the bending moment $M_p(R)$ and shear force $Q_p(R)$ on the circular edge of the plate. The Green function for the plate was taken in the form (A5), and the fundamental solution for the unbounded acoustic medium in the form (A3). The numerical analysis of forced vibrations excited by a driving load

$$q(r,\theta)=q_0\cos\theta\tag{20}$$

was performed with $N_E=25$ and $h=0{\cdot}01$. In Figure 2 the dependence of the amplitudes of the displacement at the point ($r=0{\cdot}5$, $\theta=0$) upon the frequency parameter $\omega R/c$ is presented. It is seen that the phase shift of the real part of the displacement occurs at $\omega R/c=0{\cdot}0753$ (see curve Re w in Figure 2). At this frequency there is also the maximum of the imaginary part of the amplitude of the displacement (see curve Im w in Figure 2). It can easily be shown that there is maximum energy flux from the vibrating plate at this frequency and therefore this is a case of resonance. The same dimensionless resonance frequency was obtained in reference [30] by another numerical procedure. In reference [30] the resonance frequency was $\omega=190$ rad/s. The radius of the plate was $R=0{\cdot}6$ m and the speed of sound in the acoustic medium was $c=1500$ m/s. The density ratio was $\rho/\rho_0=0{\cdot}128$ and the ratio of the speeds of sound was $c/c_0=0{\cdot}307$. Thus the dimensionless resonance frequency was $\omega R/c=0{\cdot}076$.

3.4. BEAM-TYPE VIBRATIONS OF A CIRCULAR CYLINDRICAL SHELL OF FINITE LENGTH

The next test problem is that of beam-type vibrations of a circular cylindrical shell of finite length l in an unbounded acoustic medium (the number of circumferential waves of

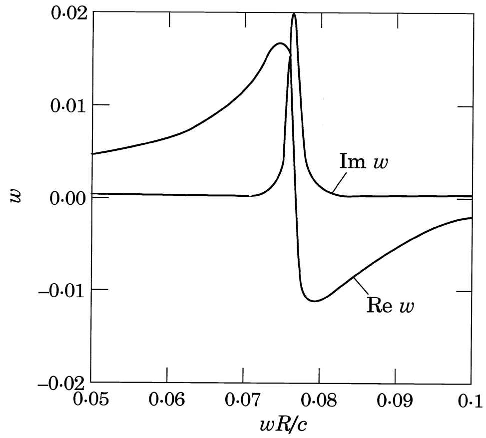

Figure 2. The real (Re w) and imaginary (Im w) parts of the amplitudes of the displacements for a circular plate in an unbounded acoustic medium versus the dimensionless frequency of excitation $\omega R/c$.

the vibrational mode considered is one; $m = 1$). This problem, under various assumptions, has been analyzed by many authors; in particular, see reference [31]). In this paper, the vibrations were considered under the assumption that the influence of the end caps is negligibly small and it is possible to ignore the interaction of the acoustic medium and the structure on these surfaces. This assumption may be easily introduced into a numerical solution: in fact, the area for integration in equation (12) becomes simply the lateral surface of the cylindrical shell. Boundary conditions of simply supported edges on both ends of the cylindrical shell were taken as in reference (31). There were then eight boundary equations of the first level (four equations (13) for each edge of the shell):

$$w_k(x) = \sum_{j=1}^{3} \int_0^l [q_j(\xi) + \delta_{j3} p(\xi)] W_{kj}(x, \xi)\, \mathrm{d}\xi + \sum_{\alpha=1}^{4} [Q_\alpha(\xi) W_{k\alpha}(X, \xi) - Q^0_{k\alpha}(X, \xi) w_\alpha(\xi)]|^{\xi=l}_{\xi=0},$$

$$k = 1, 2, 3, 4, \quad w_4(x) \equiv w_3'(x), \quad x = 0, \quad x = l \tag{21}$$

(the generalized forces $Q_\alpha(\xi)$ and $Q^0_{k\alpha}(x, \xi)$ are represented by formulae (A7)), and one equation of the second level. The latter was obtained by inserting equation (21) for $k = 3$ into equation (12). In this system the Green functions for the cylindrical shell have the form (A8) and the fundamental solution of the Helmholtz equation is of the form (A3).

One more assumption introduced in reference [31] was that it is possible to discard all the tangential displacements for the beam-type vibrations and to describe the vibrations of the shell by the single equation for the normal displacement w_3. In reference [31] a discretization procedure was used and all the derivatives in the equation of the beam vibrations were replaced by their finite difference approximations. The driving load was

$$q(x, \theta) = \sin(n\pi x/l) \cos\theta. \tag{22}$$

The numerical results were obtained, in particular, for the following values of the parameters: $n = 1$, $R/l = 0{\cdot}04$ and

$$\rho R^2/\rho_0 A_0 = 0{\cdot}414. \tag{23}$$

A_0 is the area of the cross-section of the beam.

The calculations by the B.I.E. method were performed for the ratios of the speeds of sound and the densities in the acoustic medium and in the structure material of $c/c_0 = 0{\cdot}307$ and $\rho/\rho_0 = 0{\cdot}128$. Then, for the equality (23) to hold, the ratio of the shell thickness to its radius was taken as $h/R = 0{\cdot}494$. The number of boundary elements was $N_E = 100$, and the collocation points for equation (12) were taken in the middle of each element. The whole SLAE then contained 108 equations.

In Figure 3 numerical results obtained by the B.I.E. method (curve 1) are compared with the ones obtained by the finite difference method [31] (curve 2). Curve 3 is also taken from reference [31], but this curve represents the amplitudes of the beam displacement versus the forcing frequency obtained in reference [32] for a model of a beam in an infinitely long rigid baffle. The frequency parameter is $\Omega_1 = \omega l/c_0$. The agreement of the results obtained by these three approaches appears to be good.

The first resonance frequency obtained in the framework of thin shell theory is less than those from references [31, 32] because the model of a shell with tangential displacements taken into account is less rigid than a model of a beam when tangential displacements are excluded. Even in this particular case of the very thick shell considered (in fact, the thickness of the shell is above the upper limit of validity of shell theory) the effect of the tangential displacements being taken into account is notable. With a decrease of h/R the error introduced by the use of a beam model should grow rapidly.

3.5. VIBRATIONS OF A STIFFENED CYLINDRICAL SHELL OF FINITE LENGTH IN AN INFINITELY LONG RIGID BAFFLE

The last test problem is that of the vibrations of a cylindrical shell in an infinite rigid baffle. There are many papers dealing with the influence of stiffeners upon vibrations and sound radiation from the shell, but in most of them are for high frequency excitation and attention is focused on the directivity of the sound radiation. In a low frequency range the vibrations of a shell may be adequately described by the structural–orthotropic shell model. There are various formulations of the averaging procedure, but the comparison of these techniques is beyond the scope of this paper. We used a technique presented in a detailed form in reference [33] to analyze vibrations and sound radiation of the shell considered in reference [34]. This shell is made of aluminium and submerged in water. The radius of the shell is $R = 0{\cdot}254$ m, its thickness is $h = 0{\cdot}008$ m and length is $l = 2{\cdot}54$ m. The stiffeners are rings having a rectangular cross-section 0·127 m high and 0·024 m wide. They are uniformly distributed along the surface; the distance from each other is 0·127 m. The boundary conditions are of clamped edges. A radial point force is located at the middle section of a shell. This force has been expanded in a trigonometric series in the angular

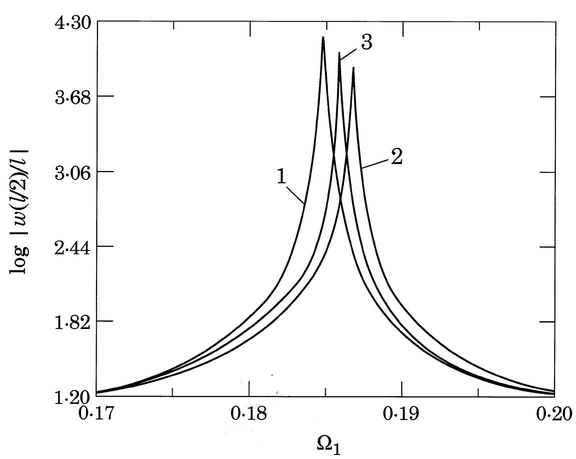

Figure 3. Amplitudes of displacements at the middle cross-section of a cylindrical shell ($w(l/2)$) versus the dimensionless frequency of excitation $\Omega_1 = wl/c_0$. The solution by the B.I.E. method is curve 1; the solution presented in reference]31] is curve 2; the solution presented in reference [32] is curve 3.

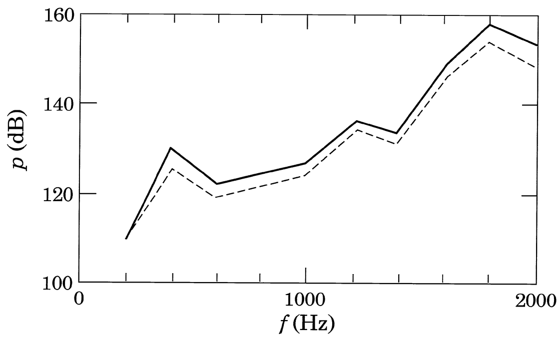

Figure 4. Pressure attenuation p (dB) versus frequency of excitation f (Hz). The solid line corresponds to solution by the boundary integral equations method; the dashed line corresponds to the solution presented in reference [34].

co-ordinate and each circumferential mode has been analyzed separately. The total response of the structure was obtained as the sum of the partial responses. The average pressure radiation [34] is shown in Figure 4. The solid line corresponds to the B.I.E. solution while the dashed line represents the solution of reference [34]. The number of boundary elements used is $N_E = 100$. The whole SLAE contained 108 equations for each circumferential mode. The agreement of the results obtained by these two approaches appears to be good.

4. CONCLUSIONS

A two-level system of boundary integral equations has been formulated for analysis of vibrations of composite thin-walled structures in an acoustic medium. The solution involves a combination of analytical and numerical methods. Several test problems have been considered to validate the efficiency and accuracy of the method proposed. The analysis of a two-level system of boundary integral equations for vibrations of a cylindrical shell having end caps and four intermediate bulkheads is performed in Part 2 of this paper.

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APPENDIX A: A CATALOG OF FUNDAMENTAL SOLUTIONS

The kernel $G(|\mathbf{X}-\mathbf{Y}|)$ of equation (8) is a fundamental solution of the Helmholtz equation (3). In a general three-dimensional case it corresponds to the field of a unit strength point source in an unbounded medium:

$$G(|\mathbf{X}-\mathbf{Y}|)=\frac{1}{4\pi}\frac{\exp(\mathrm{i}\omega A/c)}{A}, \qquad A=|\mathbf{X}-\mathbf{Y}|, \quad \mathrm{i}=\sqrt{-1}. \tag{A1}$$

When a two-dimensional problem is considered, the corresponding fundamental solution of the Helmholtz equation is

$$G(|\mathbf{X}-\mathbf{Y}|)=\frac{\mathrm{i}}{4}\,\mathrm{H}_0^{(1)}\left(\frac{\omega A}{c}\right), \tag{A2}$$

where $\mathrm{H}_0^{(1)}$ is the zero order Hankel function of the first kind.

One more fundamental solution of equation (3) is required to analyze vibrations of axisymmetrical thin-walled structures. It represents the potential of velocities induced by a ring source of radius R_0 in a three-dimensional unbounded acoustic medium. The distribution of intensities of point sources on this sources is sinusoidal with m circumferential waves in respect to the angular co-ordinate. Then the fundamental solution of equation (3) is

$$G_m(|\mathbf{X}-\mathbf{Y}|)=\frac{R_0}{4\pi}\int_0^{2\pi}\frac{\exp(\mathrm{i}\omega A/c)}{A}\cos m\theta\,\mathrm{d}\theta, \tag{A3}$$

$$\mathbf{X}(x, r, \theta_1), \quad \mathbf{Y}(\xi, R_0, \theta_2), \quad A=\sqrt{(x-\xi)^2+R_0^2+r^2-2R_0 r\cos\theta}, \qquad \theta=\theta_1-\theta_2.$$

These are the fundamental solutions for the Helmholtz equation to be used in analysis of a wide range of structural–acoustic coupling problems.

The fundamental solution (Green function) for the three-dimensional problem of vibrations of a thin Kirchhoff plate under point loading is

$$W^{(p)}(\mathbf{X},\mathbf{Y})=\frac{1}{8D_pS_p^2}\left[\mathbf{Y}_0(S_pA)+\frac{2}{\pi}K_0(S_pA)-\mathrm{i}\mathrm{J}_0(S_pA)\right], \tag{A4}$$

$$A=|\mathbf{X}-\mathbf{Y}|=\sqrt{r^2+R^2-2rR\cos\theta}, \qquad S_p^4=\rho_p h_p\omega^2 D_p^{-1}.$$

D_p, ρ_p and h_p are the bending rigidity, density and thickness of the plate, respectively, $\mathrm{Y}_0(\ )$, $\mathrm{K}_0(\ )$ and $\mathrm{J}_0(\ )$ are Bessel functions.

A Green function for the cyclosymmetric problem of vibrations of a Kirchhoff plate is the response to its line lateral loading on a circle of radius R (the intensity of loading is sinusoidal in respect to the angular co-ordinate θ):

$$W_m^{(p)}(r,R)=\frac{1}{8D_pS_p^2}\int_0^{2\pi}\left[\mathrm{Y}_0(S_pA)+\frac{2}{\pi}K_0(S_pA)-\mathrm{i}\mathrm{J}_0(S_pA)\right]\cos m\theta\,\mathrm{d}\theta. \tag{A5}$$

If vibrations of a cylindrical shell in the mode with m circumferential waves are analyzed and Goldenveizer–Novozhilov theory [27] is used, then the differential equations (3) have the following dimensionless forms:

$$w_1''-\frac{1-v}{2}m^2w_1+\Omega^2w_1+\frac{1+v}{2}mw_2'-vw_3'=\frac{q_1(1-v^2)}{Eh}, \qquad -\frac{1+v}{2}mw_1'+\frac{1-v}{2}w_2''$$

$$-m^2w_2+\Omega^2w_2-\frac{2-v}{12}h^2mw_3''+\left(m+\frac{m^3h^2}{12}\right)w_3=\frac{q_2(1-v^2)}{Eh},$$

$$-\nu w_1' + \frac{(2-\nu)}{12} h^2 m w_2'' - \left(m + \frac{m^3 h^2}{12}\right) w_2 + \frac{h^2}{12} w_3'''' - \frac{h^2}{12} 2m^2 w_3''$$

$$+ \left(1 + \frac{h^2}{12} m^4 - \Omega^2\right) w_3 = \frac{q_3(1-\nu^2)}{Eh}. \quad \text{(A6)}$$

Here $\Omega^2 = (1-\nu^2)\rho\omega^2 R^2 E^{-1}$ and $(\ )' = \mathrm{d}(\ )/\mathrm{d}x$. A set of Green functions for an infinitely long cylindrical shell in this case corresponds to four loadings at the section with axial co-ordinate $x = 0$, each of which is of unit strength in the longitudinal direction and sinusoidal with m waves in the circumferential direction. The four loadings are as follows:

(i) longitudinal force,

$$Q_{11}^0 = \frac{Eh}{1-\nu^2}[(W_{11})' + \nu m W_{12} + \nu W_{13}] = \tfrac{1}{2}\,\mathrm{sign}\,(\xi),$$

$$W_{12} = W_{13} = Q_{14}^0 = 0; \quad \text{(A7a)}$$

(ii) circumferential force,

$$Q_{22}^0 = \frac{Eh}{2(1+\nu)}[-mW_{21} + (W_{22})'] = \tfrac{1}{2}\,\mathrm{sign}\,(\xi),$$

$$W_{21} = Q_{23}^0 = (W_{23})' = 0; \quad \text{(A7b)}$$

(iii) shear force,

$$Q_{33}^0 = \frac{Eh^3}{12(1-\nu^2)}[(W_{33})''' + (1-2\nu)m^2(W_{33})' + (1-2\nu)m(W_{32})'] = \tfrac{1}{2}\,\mathrm{sign}\,(\xi),$$

$$W_{32} = Q_{32}^0 = (W_{33})' = 0; \quad \text{(A7c)}$$

(iv) longitudinal bending moment,

$$Q_{44}^0 = \frac{Eh^3}{12(1-\nu^2)}[-(W_{43})'' + \nu m^2 W_{43} + mW_{42}] = \tfrac{1}{2}\,\mathrm{sign}\,(\xi),$$

$$Q_{41}^0 = W_{42} = W_{43} = 0. \quad \text{(A7d)}$$

The first index corresponds to the loading number; the second one is the number of the component of the generalized displacements or generalized forces vector.

These sets of Green functions have the following form:

$$W_{j1}(x,\xi) = \sum_{\alpha=1}^{4} c_{j\alpha} a_{j\alpha} \exp S_\alpha |x-\xi|,$$

$$W_{j2}(x,\xi) = \sum_{\alpha=1}^{4} c_{j\alpha} b_{j\alpha} \exp S_\alpha |x-\xi|,$$

$$W_{j3}(x,\xi) = \sum_{\alpha=1}^{4} c_{j\alpha} \exp S_\alpha |x-\xi|, \qquad j = 1, 2, 3, 4. \quad \text{(A8)}$$

S_α are the roots of a characteristic equation (dispersion polynomial) such that Re $S_\alpha < 0$; if Re $S_\alpha = 0$ then Im $S_\alpha > 0$. $a_{j\alpha}$, $b_{j\alpha}$ and $c_{j\alpha}$ are the coefficients of the normal waves defined by the loading of the shell.

The subscript m and the superscript (c), indicating that a cylindrical shell is considered, are omitted in expressions (A7) and (A8).

For a plane structural–acoustic coupling problem the fundamental solution for vibrations of a circular arc may be easily obtained from equation (A6) by omitting the terms containing the longitudinal displacement and the derivatives with respect to the axial co-ordinate. Then the Green functions corresponding to (i) circumferential force, (ii) shear force and (iii) circumferential bending moment have the same form as in equations (A8):

$$W_{j2}(\theta,\psi)=\sum_{\alpha=1}^{3} b_{j\alpha}c_{j\alpha}\exp S_\alpha|\theta-\psi|, \qquad W_{j3}(\theta,\psi)=\sum_{\alpha=1}^{3} c_{j\alpha}\exp S_\alpha|\theta-\psi|. \tag{A9}$$

θ is the angular co-ordinate of an observation point and ψ is the angular co-ordinate of a source.

If a structure of finite length is considered it is possible to retain only the real parts of the Green functions for the parts of the structure, because the imaginary parts satisfy homogeneous equations and enable the radiation conditions to be satisfied (the group velocity of waves existing in the structure should be directed from a source to infinity). In the case of a finite structure, the radiation conditions are not essential. This aspect of the boundary integral equations method is discussed in detailed form in reference [24]. In respect to the efficiency of the calculations it is more convenient to use real functions than complex ones. If the effects of internal damping in the material of a structure are taken into account then, of course, complex Green functions for the structure should be used.

APPENDIX B: LIST OF NOTATION

φ	velocity potential
ω	circular frequency of oscillations
c	speed of sound in the acoustic medium
ρ	density of the acoustic medium
$\mathbf{v}_+$	outward unit normal to the fluid domain at the surface of a structure
$\{w\}$	displacements vector
w_3	component of $\{w\}$ which is normal to the surface of the thin-walled structure; the positive direction of w_3 corresponds to that of the outward normal $\mathbf{v}_+$ to the fluid domain
$[L]$	tensor differential operator defined by the mathematical model chosen to describe vibrations of the structure
$\{q\}$	vector of external driving loads
u	velocity of a point on a surface of the structure as well as a velocity of the corresponding particle of the acoustic medium
$\mathbf{X}$, $\mathbf{Y}$	radius-vectors of the observation point and source;
$4\pi(1-C)$	solid angle occupied by the acoustic medium
S	boundary of the acoustic medium
$S^{(n)}$	part of the boundary S
G	fundamental solution of the Helmholtz equation
$w_\alpha^{(n)}$	components of the generalized displacements vector on the nth part of a thin-walled structure
$Q_\alpha^{(n)}$	generalized forces on a contour $\mathbf{\Gamma}^{(n)}$
$W_{j\alpha}^{(n)}$	Green function of unbounded shell corresponding to the nth part of the structure
$W_{j\alpha}^{0}$	Green function of the whole structure
$Q_{j\alpha}^{(n)}$	tensor of forces corresponding to the fundamental solution $W_{j\alpha}^{(n)}$
$\mathbf{\Gamma}^{(n)}$	contour of nth part of the thin-walled structure;
$\mathbf{v}_\beta$	unit outward normal to the domain $S^{(n)}$ at the contour $\mathbf{\Gamma}^{(n)}$ in the tangential to $S^{(n)}$ at the point $\mathbf{Y}$
c_0	speed of sound in the material of the shell
v	Poisson ratio for the material of the shell
D_p	bending rigidity of the plate
ρ_p	density of the plate
h_p	thickness of the plate