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Resonant-frequency primitive waveforms and star waves in lattices

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Abstract

For square and triangular lattices we have found a new line-localized primitive waveform (LPW) existing at a resonant frequency. In two-dimensional (2D) case, the LPW represents a line of oscillating particles, while the lattice outside this line remains at rest. We show that: (a) A single LPW does not conduct energy; however, a band consisting of two or more neighboring LPWs is a conductor with the phase-shift-dependent energy flux velocity. (b) Any canonical sinusoidal wave consists of LPWs. In turn, the LPW can be represented by a superposition of the sinusoidal waves (these two types of waves are connected by the discrete Fourier transform). (c) There are two (three) LPW orientations for the square (triangular) lattice, and this is why the sinusoidal-wave group velocity orientation is piecewise constant at this frequency; it coincides with the nearest LPW orientation. (d) LPW can also exist at a lower frequency being localized at the lattice halfplane boundary. Further, for 3D lattices plane-localized waveforms are found to exist in a frequency region. Finally, for the point harmonic excitation of 2D lattices we show that starlike waves develop with the rays in the LPW directions.

1. Introduction

In the last decades escalating activity may be observed in the field of *Waves in lattices* or, more general, *Waves in structured media*. This rather old topic [1,2] has been developed in application to various structures. For example, one-dimensional (1D) periodic systems of roads [3] and periodically supported structures, a plate [4] and a beam [5], were considered. A comprehensive review of works where various other systems were considered can be found in Ref. [6]. The topic has got the second wind beginning from the late 1980s when artificial "crystals" where revealed as the band-gap materials allowing to control the propagation of waves of different nature: electronic and electromagnetic waves (electronic and photonic crystals [7–9]) and waves of sound and vibration (phononic crystals [10]). The number of publications related to this topic is growing exponentially [9,11].

Phononic crystals as two- (2D) and three-dimensional (3D) periodic structures of diverse materials and geometry were considered in many works. For example, metal–metal [12] and water–mercury [13,14] com-

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posites, honeycomb lattice [15], water with air bubbles [16,17], a polyethylene matrix with tungsten inclusions [18], an elastic body with voids [19], plastic–metal structures [20], different lattice structures [21–23]. In Ref. [24], a comparative analysis of the dispersion relations was carried out for the Cosserat continuum models and the corresponding discrete lattices. In addition to the translation of masses, the formulation in this paper also accounts their rotation. The similarity with the corresponding results in the present work indicates that the addition of the rotational inertia does not seem to affect the directional characteristics of the transversal LPW waves.

Also note that lattice models are used in fracture (analytical works in this field were summarized in Ref. [25]), mainly, with the aim to find the crack-speed-dependent dissipation and to establish connections between the far-field (macrolevel) and local (microlevel) energy release rates.

In the band gap materials, there exist resonant frequencies, such that there is no steady-state solution corresponding to an external non-selfequilibrated excitation. In the 1D case, resonant frequencies usually demarcate the pass and stop bands, and the group velocity $\mathrm{d}\omega/\mathrm{d}k = 0$ at this frequency. In this case, the energy flows from a source not as a wave but, roughly speaking, as heat (more precisely, the corresponding law depends on the order of the first non-zero derivative $\mathrm{d}^n\omega/\mathrm{d}k^n$ at this point [26]).

In 2D/3D cases, a resonant point can also exist in the interior of pass bands. In particular, it corresponds to the X-point in the Brillouin diagram for the lower branch of the dispersion curves (see e.g. Figs. 1 and 2 in Ref. [12]). Note that this branch for a continuous structured material is similar to the corresponding dispersion curves for the discrete square lattice considered in this paper (the diagram for the lattice is shown in Fig. 1). Such resonant points in 2D case also differ by the fact that the group velocity is at zero only for some special wave orientations. In the following, we will consider the lattice waves just for this resonant frequency. Excitation of continuous periodic structures and lattices at non-resonant frequencies are considered in many works: e.g. in Ref. [5] (periodically supported beam), [27–29] (2D lattices), [30] (bubbly liquids), [31–33] (elastic structures of diverse types), [34,35] (railway track structures).

In the present work, 2D square and triangular lattices are considered at the resonant frequency. The paper is mainly aimed to show, for these and some other lattices, the existence of localized primitive waveforms (LPWs), and to discuss some consequences. The LPW is a 'selfequilibrated' standing wave strictly localized on a line of a certain orientation (there are two orientations of the LPW lines for the square lattice and three for the triangular lattice). While a single LPW line cannot conduct energy, a band consisting of two or more activated LPW lines is a conductor. Any sinusoidal wave, and hence any wave at this frequency, can be represented as a set of the LPWs, and it bears evidence of the features of the LPW. In particular, the sinusoidal-wave group velocity orientation coincides with the LPW orientation nearest to the wave propagation direction, and this is why it is piece-wise constant at this frequency. In this paper, analytical and numerical results are presented regarding the formation of starlike waves with the rays in LPW directions. These results illustrate the key role of the LPWs in the features of a general wave field. Note that the non-

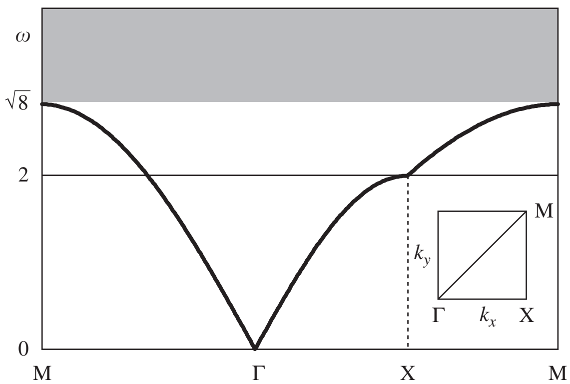

Fig. 1. Dispersion diagram for the square lattice. Frequency ω is plotted against the wave vector $\boldsymbol{k}$ (k_x, k_y) varying along the Brillouin contour shown at the right.

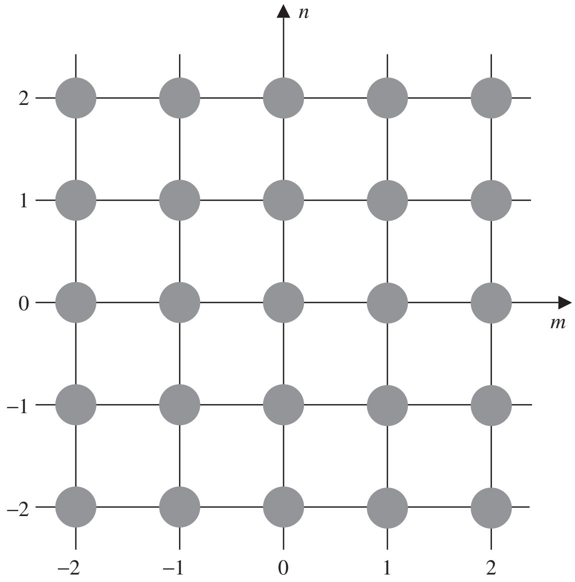

Fig. 2. The square lattice.

resonant steady-state response of periodic structures to the action of ≪a≫ local harmonic and impulsive sources with the wave beaming phenomenon was studied, in particular, in the above-discussed papers [27–32].

2. The LPW

Consider transversal oscillations of a 2D square lattice presented in Fig. 2. In mechanical terms, this structure represents a plane net of massless strings with point masses at the knots. The bond stiffness, μ, the lattice point mass, M, and the cell size, a, are assumed to be natural units. In these terms, the long wave speed $c = a\sqrt{\mu/M} = 1$ and the time unit is $a/c = \sqrt{M/\mu}$. In the linear approximation, the homogeneous dynamic equations are

$$\ddot{U}_{m,n}(t) + 4U_{m,n}(t) - U_{m+1,n}(t) - U_{m-1,n}(t) - U_{m,n+1}(t) - U_{m,n-1}(t) = 0. \tag{1}$$

Below, along with the integer numbers m, n and p (the latter is used for a 3D lattice) we use continuous coordinates x, y and z, respectively. It is common knowledge that there exists an elementary solution as a sinusoidal wave

$$U_{m,n} = u_{m,n}(\omega)e^{i\omega t}, \quad u_{m,n}(\omega) = \exp[-i(k_x m + k_y n)], \quad -\pi < (k_x, k_y) \leqslant \pi, \tag{2}$$

with the dispersion relation

$$\omega = \sqrt{4 - 2\cos k_x - 2\cos k_y}. \tag{3}$$

In these relations, ω is the frequency and k_x, k_y are components of the wave vector $\boldsymbol{k}$. The latter defines phase velocity of the wave as $\boldsymbol{v} = \omega/k^2(k_x, k_y)$, while the group velocity of the wave is defined as $\boldsymbol{v}_g = (\partial\omega/\partial k_x, \partial\omega/\partial k_y)$. A general solution to Eq. (1) can be represented by superposition of these waves.

The dispersion relation (3) defines, in particular, two resonant frequencies $\omega = \omega_1 = 2$ $(k_y = \pm\pi \pm k_x)$ and $\omega = \omega_2 = \sqrt{8}$ $(k_x = k_y = \pi)$, where there is no steady-state solution in the case of a non-selfequilibrated harmonic excitation of the lattice. Below we consider the case $\omega = 2$ as the most interesting: the phenomena discussed in this paper appear only at this frequency. It follows from Eq. (3) that in this case the equifrequency contour is the square perimeter, Fig. 3(a). It should be noted that equifrequency contours were considered, e.g. in Refs. [24,27,36] for different frequencies; however, the resonant-frequency contour was missed (see e.g. Fig. 8 in Ref. [27]).

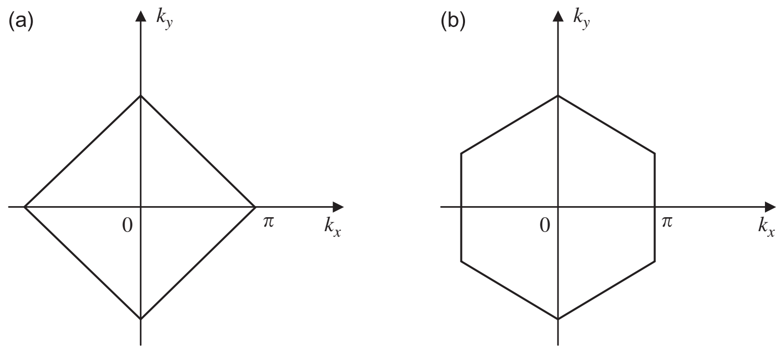

Fig. 3. Resonant frequency contours for (a) the square lattice, $\omega = 2$, and (b) the triangular lattice, $\omega = \sqrt{8}$.

The first purpose of this paper is to note that in the case $\omega = 2$ there exists another, 'most primitive' solution of Eq. (1),

$$U_{m,n} = \mathscr{L}_0^+ \mathrm{e}^{\mathrm{i}\omega t}, \quad \mathscr{L}_0^+ = (-1)^n \delta_{m,n} \; [\delta_{m,n} = 1 \; (m = n), \; \delta_{m,n} = 0 \; (m \neq n)], \tag{4}$$

representing antiphase oscillations of neighboring masses placed along a single right-inclined diagonal, $m = n$, with the other particles quiescent. Due to the symmetry and uniformity of the lattice such a LPW can also be associated with the left-inclined diagonal, $u_{m,n} = \mathscr{L}_0^- = (-1)^n \delta_{-m,n}$, and also with any other diagonal line $\mathscr{L}_\nu^\pm = (-1)^n \delta_{\pm m, n-\nu}$, where ν is any integer. The LPWs associated with different lines can be superimposed. In particular, any sinusoidal wave at this frequency can be represented by superposition of LPWs. Namely, the LPW and the sinusoidal wave are connected by the discrete Fourier transform

$$u_{m,n}(2) = S^\pm(k) = (-1)^n \mathrm{e}^{-\mathrm{i}k(m \mp n)} = \sum_{\nu=-\infty}^{\infty} \mathscr{L}_\nu^\pm \mathrm{e}^{\pm \mathrm{i}k\nu} \quad (k = k_x),$$
$$\mathscr{L}_\nu^\pm = \frac{1}{2\pi} \int_{-\pi}^{\pi} S^\pm(k) \mathrm{e}^{\mp \mathrm{i}k\nu} \mathrm{d}k. \tag{5}$$

The existence of the preferential directions associated with the LPW places the key role in the lattice properties at the resonant frequency.

3. Energy flux

For the sinusoidal wave the group velocity vector is oriented as the external normal to the above-mentioned equifrequency contour (Fig. 3(a)). So the group velocity direction is independent of the wave orientation within each quadrant in the x, y-plane, and it coincides with the LPW orientation nearest to the wave vector. It also follows from Eq. (5) that for $\omega = 2$ the modulus, $|\boldsymbol{v}_g|$, is

$$v_g = \frac{\sqrt{2}}{2} \sin \frac{\pi}{1 + |\tan \alpha|}, \tag{6}$$

where α is the angle between the x-axis and the wave vector: $\tan \alpha = k_y / k_x$. So the group velocity is zero at those and only at those four points where its direction changes. In spite of the fact that the group velocity is non-zero almost everywhere, the resonance (in the above sense) does exist, although the oscillation amplitudes grow very slowly (see below).

Now consider a single LPW. Since there is no diagonal bond here, while the nearest particles connected with diagonal ones are at rest, there is no energy flux in this 'wave'. However, the energy flux becomes non-zero if two neighboring 'insulators' are activated with a phase displacement, say, $u_{m,n} = \mathscr{L}_0^+ + \mathrm{e}^{\mathrm{i}\phi} \mathscr{L}_1^+$. In this latter case, there exists a step curve comprising of the bonds connecting the oscillating particles; this step curve is a 'conductor'. A direct analysis evidences that the energy flux and the energy flux velocity are the same as those in such a step curve in the sinusoidal wave with the wave direction angle $\alpha = \arctan[(\pi - \phi)/\phi]$. This is an expected result, and it is still valid in the case of three- or more diagonal-line band activated. Indeed, expanding the activated band we construct the sinusoidal wave $S^{+}(k)$. Clearly, the energy flux is directed along such a band, and this is a structure-based explanation of the diagonal group velocity orientation.

4. Some other structures

The existence of the LPW follows directly from the lattice structure. Indeed, consider the right inclined diagonal and a neighboring particle, say, $m, m+1$ or $m+1, m$ (Fig. 4). It is connected with the diagonal particles, m, m and $m+1, m+1$. Since the diagonal particles are involved in antiphase oscillations, their actions on a near-diagonal particle are selfequilibrated. The particles outside the diagonal can thus be at rest. So the existence of the LPW is a consequence of a certain symmetry of the lattice structure.

4.1. Triangular lattice

These geometrical considerations are also applicable to a triangular lattice with the particles at

$$x = m + n/2, \quad y = \sqrt{3}n/2 \quad (m, n = 0, \pm 1, \ldots). \tag{7}$$

In this case, there exist LPWs associated with each of the three bond lines. It can be seen that, in spite of the direct connection of the oscillating particles, a single LPW does not conduct energy as in the case of the square lattice.

The non-dimensional homogeneous dynamic equation for the transversal motion of this lattice is

$$\ddot{U}_{m,n} = U_{m+1,n} + U_{m-1,n} + U_{m,n+1} + U_{m,n-1} + U_{m-1,n+1} + U_{m+1,n-1} - 6U_{m,n}. \tag{8}$$

It defines the dispersion relation as

$$\omega = \sqrt{8 - 4\cos(k/2)[\cos(k/2) + \cos(\sqrt{3}p/2)]}. \tag{9}$$

It follows that the LPW resonant frequency $\omega = \sqrt{8}$, while the $k_x - k_y$ plot is a regular hexagon perimeter (Fig. 3(b)). The group velocity is

$$v_g = \frac{1}{\sqrt{8}}\cos\left(\frac{\sqrt{3}\pi}{2}\tan\alpha\right), \quad \beta = 0 \quad (-\pi/6 < \alpha = \arctan k_y/k_x < \pi/6). \tag{10}$$

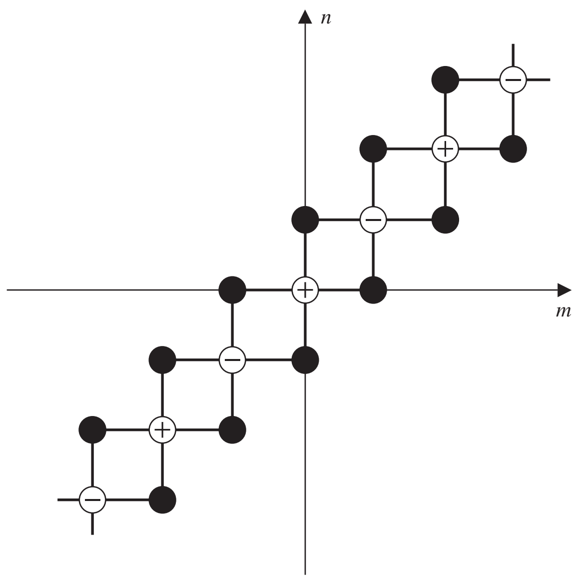

Fig. 4. The three-particle-width band. Particles involved in antiphase oscillations (white circles with ±) and immobile particles (black circles).

So, just as for the square lattice, the group velocity is directed as the nearest LPW orientation, and at the vertices, where it changes the direction, $v_g = 0$.

4.2. Cubic lattice

In scalar oscillations of a cubic lattice, there exists a plane LPW. We assume that the cubic lattice obeys the discrete analogue of the 3D wave equation, similar to Eq. (1) but with respect to the scalar function (say, the displacement along z-coordinate) of three discrete coordinates. It follows that for any k_z ($-\pi \leqslant k_z \leqslant \pi$), at the frequency $\omega = \sqrt{6 - 2\cos k_z}$, there exists a waveform localized at a diagonal plane $n = \nu \pm m$

$$u_{m,n,p} = (-1)^n \mathrm{e}^{-\mathrm{i}k_z p} \delta_{\pm m, n-\nu}. \tag{11}$$

Note that the plane LPW also exists in a 3D lattice where the 2D triangular lattice planes are connected just as the planes in the cubic lattice.

4.3. Material-bond lattice

It follows from the symmetry considerations that the LPW can also exist in such a lattice with non-zero bond density. (A non-zero-bond-density lattice as a network of strings without concentrated masses was considered, in particular, in Ref. [21], where vibrations of the strings, with no associated nodal displacements were mentioned. Fracture of a material-bond lattice was studied in Ref. [37].)

4.4. Bounded lattice

The LPW exists not only in infinite lattices or lattice strips. It also can exist in a finite structure with proper boundaries. For instance, if the boundaries, where the nodes are fixed, coincide with the left-inclined diagonals of a square lattice, they bound alternating longer and shorter right-inclined diagonals; each of the latter diagonals can support a finite LPW. The LPW can also exist in a closed properly structured ring. In particular, this ring can be made by means of a pure bend of an LPW-oriented lattice strip. In the latter case, the inner localized oscillations cannot be detected from the outside at all.

5. Transient problem

We now consider inhomogeneous equations (1) for the case where a single external force $P_{0,0} = P(t) = Q(t)\mathrm{e}^{2\mathrm{i}t}$ is applied at $t = 0$ to the particle $m = n = 0$. The Laplace transform on t and the discrete Fourier transforms on m with parameters s and k, respectively, lead to

$$u_n^{LF}(s - \mathrm{i}\omega, k) = \frac{Q^L(s - \mathrm{i}\omega)}{2\sqrt{Z^2 - 1}}(Z - \sqrt{Z^2 - 1})^{|n|}, \quad Z = 2 + \frac{s^2}{2} - \cos k. \tag{12}$$

For $\omega = 2$ we change $s = 2\mathrm{i} + s'$. In our case, asymptotes (or limiting values) of $u_{m,n}(t)$ ($t \to \infty$) are defined by asymptotes of $u_{m,n}^L(s')$ for $s' \to +0$. With this in mind we can put $s' = +0$ everywhere except the denominator in Eq. (12). As a result

$$u_{m,n}^L(s') \sim -\frac{\mathrm{i}Q}{s'}\frac{(-1)^n}{2\pi}\int_0^\pi \frac{\exp(\mathrm{i}k|n|)\cos km}{\sqrt{\sin^2 k + 4\mathrm{i}s'\cos k}}\mathrm{d}k. \tag{13}$$

It follows that for $m + n$ even

$$u^L_{0,0}(s)\sim -\frac{\mathrm{i}Q}{2\pi}\frac{\ln(4/s')}{s'}, \quad u_{0,0}(t)\sim -\frac{\mathrm{i}Q}{2\pi}[\ln(4t)+\gamma],$$
$$u_{m,m}(t)\approx -\mathrm{i}Q\frac{(-1)^m}{2\pi}\left(\ln\frac{4t}{|m|}+\gamma-2\right) \quad (m\neq 0),$$
$$u_{m,n}(t)\approx -\mathrm{i}Q\frac{(-1)^n}{2\pi}\left(\ln\frac{16t}{|m^2-n^2|}+\gamma-4\right) \quad (m\neq n), \tag{14}$$

where $\gamma\approx 0.577$ is the Euler constant. For $m+n$ odd we find that

$$\lim_{t\to\infty} u_{m,n}(t)=\frac{Q}{4}(-1)^v, \quad v=\max(|m|,|n|). \tag{15}$$

Let us fix a value of the amplitude, that is, fix the argument of logarithm in Eq. (14). It can be seen that this level of oscillations propagates along the diagonal, $n=m$, with a constant speed, while it spreads along a bond line as it were obey a parabolic equation: $m\sim\mathrm{const}\sqrt{t}$ $(n=0)$, $n\sim$ $\mathrm{const}\sqrt{t}$ $(m=0)$. This is in agreement with the above-discussed conclusions concerning the group velocity.

It follows that for the kinematic excitation, that is, for $u_{0,0}=u_{0,0}(t)=H(t)$

$$Q(t)\sim\int_0^\infty \frac{2\pi\mathrm{i}\,\exp(-xt)\,\mathrm{d}x}{x(\ln^2(x/4)+\pi^2)}\to 0. \tag{16}$$

For $m+n$ even

$$u_{m,m}\approx(-1)^m\left[1+\frac{\mathrm{i}Q(t)}{2\pi}(\ln|m|+2)\right] \quad (t\to\infty,\ m\neq 0),$$
$$u_{m,n}\approx(-1)^n\left[1+\frac{\mathrm{i}Q(t)}{2\pi}\left(\frac{1}{4}\ln|m^2-n^2|+4\right)\right] \quad (t\to\infty,\ m\neq n), \tag{17}$$

while for $m+n$ odd the amplitudes tend to zero.

The final pattern is presented in Fig. 5. However, for any finite region, it may take a long time for this limiting field to develop, since $Q(t)$ vanishes very slow: $\mathrm{i}Q(t)\sim\mathrm{const}/\ln t$ $(t\to\infty)$.

Numerical simulations show that the kinematic excitation of the square and triangular lattices lead to the star-wave dominant modes. In agreement with the analytical results, it appears that the 'star waves', Figs. 6

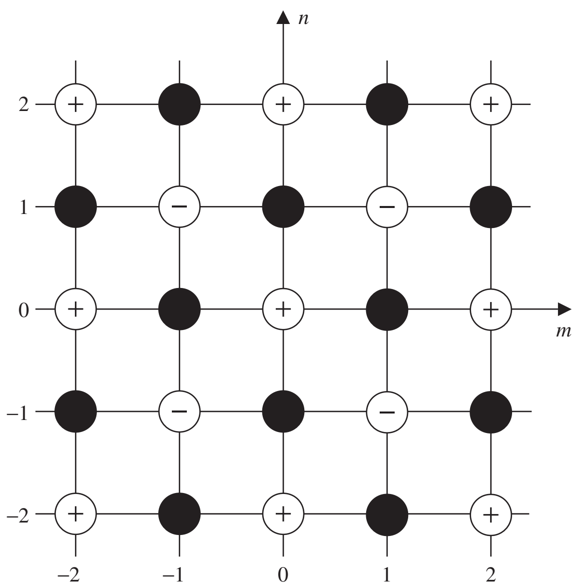

Fig. 5. The final state of the square lattice under the point kinematic excitation. Particles involved in antiphase oscillations (white circles with ±) and immobile particles (black circles).

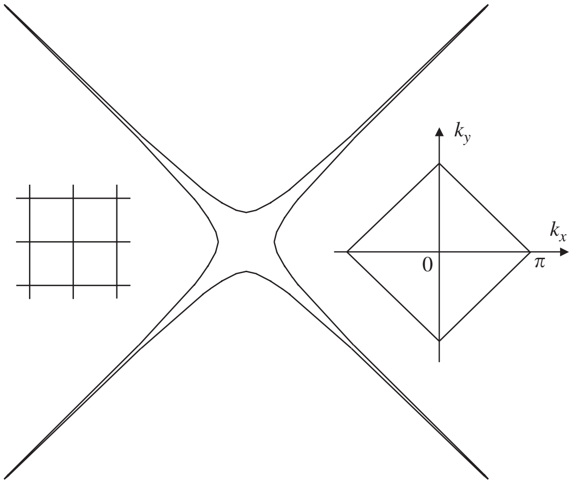

Fig. 6. The star wave contour for the kinematically excited square lattice as the dominant mode of oscillations at $\omega = 2$. Non-dimensional time $t = 500$ and the ray length $= 211$.

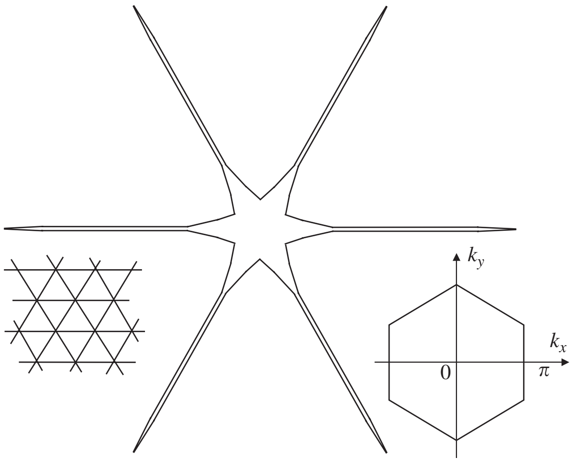

Fig. 7. The star wave contour for the kinematically excited triangular lattice as the dominant mode of oscillations at $\omega = \sqrt{6}$. Non-dimensional time $t = 500$ and the ray length $= 141$.

and 7, develop in accordance with the LPW orientations. The star wave contours are plotted in such a way that the oscillation amplitudes in the outer area $|u_{m,n}| < 0.1$ (recall that $u_{0,0} = 1$), while the inner area is minimal (the star cannot be narrowed without violation of the inequality). The plots correspond to (dimensional) time $t = 500\sqrt{M/\mu}$ when the ray lengths are equal to $211a$ and $141a$ for the square and triangular lattices, respectively. Recall that M, μ and a are the node mass, the bond stiffness and the bond length, respectively. In addition to the star and the lattice geometry, the corresponding $k_x - k_y$ contour is shown in each figure.

In conclusion, we note that the LPW can exist as the lattice surface waveform. Consider for example a square lattice half-plane (or a lattice strip) bounded by a diagonal line. In this case, each boundary particle is connected with only two others (see Fig. 4 assuming the right black particles removed). The single LPW still be in existence, now at the boundary; however, at a lower frequency: $\omega = \sqrt{2}$ (any particle with two bonds now represents an oscillator with the unit mass and the spring stiffness equal to 2). Similar conclusion is valid for the triangular lattice with the boundary waveform frequency $\omega = \sqrt{6}$.

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