3. WIENER PROCESS. STOCHASTIC ITÔ INTEGRAL

3.1. **Wiener process. Main properties.** A Wiener process (notation $W = (W_t)_{t\ge 0}$) is named in the honor of Prof. Norbert Wiener; other name is the Brownian motion (notation $B = (B_t)_{t\ge 0}$). Wiener process is Gaussian process. As any Gaussian process, Wiener process is completely described by its expectation and correlation functions.

We give below a description of main properties of $W = (W_t)_{t\ge 0}$:

1. $W_0 \equiv 0$;
2. paths (trajectories) of Wiener process are continuous functions of $t \in [0, \infty)$;
3. expectation $\boldsymbol{E} W_t \equiv 0$;
4. correlation function $\boldsymbol{E}(W_t W_s) = t \wedge s$, $(a \wedge b = \min(a, b))$;
5. for any $t_1, ..., t_n$ the random vector $(W_{t_1}, ..., W_{t_n})$ is Gaussian;
6. For any s, t

$$\begin{aligned} \boldsymbol{E} W_t^2 &\equiv t \\ \boldsymbol{E}[W_t - W_s] &\equiv 0, \\ \boldsymbol{E}[W_t - W_s]^2 &= |t - s|; \end{aligned}$$

7. Increments of Wiener process on non overlapping intervals are independent, i.e. for $(s_1, t_1) \cap (s_2, t_2) = \varnothing$ the random variables $W_{t_2} - W_{s_2}$, $W_{t_1} - W_{s_1}$ are independent;
8. paths of Wiener process are not differentiable functions;
9. martingale property (notation $W_0^s = \{W_u, 0 \le u \le s\}$)

$$\begin{aligned} &\boldsymbol{E}(W_t | W_0^s) = W_s \\ &\boldsymbol{E}\{(W_t - W_s)^2 | W_0^s\} = t - s. \end{aligned}$$

Proofs. **1-5** is nothing but the definition of Wiener process. **6.** is implied by **3.** and **4.**

4. provides the orthogonality of increments for non overlapping intervals, that is for $s_1 < s_2 < s_3 < s_4$

$$\boldsymbol{E}\big(W_{s_4} - W_{s_3}\big)\big(W_{s_2} - W_{s_1}\big) = (s_2 - s_1) - (s_2 - s_1) = 0.$$

The required independence property for these random variables follows from well known fact:

orthogonal Gaussian random variables are independent.

To verify the validity of **8.**, with $h > 0$ let define $\Delta(h) = \frac{W_{s+h} - W_s}{h}$ and show that

$$\lim_{h \to 0} \Delta(h) \text{ “does not exists”}.$$

Assume that this limit exists. Then the limit for the Fourier transform (here $\boldsymbol{i} = \sqrt{-1}$)

$$\lim_{h\to 0} \boldsymbol{E} e^{\boldsymbol{i}\lambda\Delta(h)} \text{ “exists and is a continuous function of } \lambda\text{”}.$$

Hence, since the random variable $\Delta(h)$ is zero mean Gaussian with the variance $\boldsymbol{E}\frac{(W_{s+h}-W_s)^2}{h^2} = \frac{1}{h}$, we find

$$\boldsymbol{E} e^{\boldsymbol{i}\lambda\Delta(h)} = e^{-\frac{\lambda^2}{2h}} \xrightarrow[h\to 0]{} \left\{ \begin{array}{ll} 1 & \lambda = 0, \\ 0 & \lambda \neq 0 \end{array} \right\} := U(\lambda).$$

Since $U(\lambda)$ is discontinuous function the assumed differentiability is not valid.

9 . Both follow from the property for the increments of Wiener process to be independent for non overlapping intervals:

$$\begin{aligned} \boldsymbol{E}(W_t|W_0^s) &= \boldsymbol{E}(W_t - W_s + W_s|W_0^s) \\ &= W_s + \boldsymbol{E}(W_t - W_s|W_0^s) \\ &= W_s \end{aligned}$$

and

$$\boldsymbol{E}\big((W_t - W_s)^2|W_0^s\big) = \boldsymbol{E}(W_t - W_s)^2 = t - s.$$

□

There exists the alternative definition of Wiener process based on the martingale property. We formulate this result without of proof.

Levy's Theorem: *The random process $(W_t)_{t\geq 0}$ is Wiener process if $W_0 = 0$, the trajectories of W_t are continuous and the martingale property hold.*

3.2. **One more property of Wiener process.**

If ξ is zero mean Gaussian random variable with the variance $\sigma^2 = \boldsymbol{E}\xi^2$, then $\boldsymbol{E}|\xi| = \sqrt{\frac{\pi}{2}}\sigma$. Therefore, we have

$$\boldsymbol{E}|W_{t_{j+1}} - W_{t_j}| = \sqrt{\frac{\pi}{2}}\sqrt{t_{j+1} - t_j}$$

and thus the series $\sum_j \boldsymbol{E}|W_{t^n_{j+1}} - W_{t^n_j}|$ diverges with $t^n_{j+1} - t^n_j \to 0$, where $0 < t^n_1 < t^n_2 < \dots t^n_n \equiv t$.

However the increments of Wiener process obey very important property exposed in

Lemma 3.1. *Let $0 \equiv t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)}$ be the subdivision of the interval $[0,t]$ with $\max_j[t^{(n)}_{j+1} - t^{(n)}_j] \to 0, n \to \infty$. Then (here l.i.m. denotes the limit in L^2 sense)*

$$\underset{n\to\infty}{l.i.m.} \sum_{j=0}^{n-1} [W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^2 = t.$$

Proof: Note that $\boldsymbol{E}[W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^2 = [t_{j+1} - t_j]$ that is $\boldsymbol{E}\sum_{j=0}^{n-1}[W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^2 = t$. Consequently, it is sufficient to show only that $\lim_n \boldsymbol{E}\Big(\sum_{j=0}^{n-1}[W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^2 - t\Big)^2 = 0$. The latter holds since

$$\sum_{j=0}^{n-1}[W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^2 - t = \sum_{j=0}^{n-1}\Big([W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^2 - [t_{j+1} - t_j]\Big)$$

and so

$$\begin{aligned}
\lim_n \boldsymbol{E}\Big(\sum_{j=0}^{n-1}[W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^2 - t\Big)^2 &= \sum_{j=0}^{n-1}\boldsymbol{E}\Big([W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^2 - [t_{j+1} - t_j]\Big)^2 \\
&= \sum_{j=0}^{n-1}\Big(\boldsymbol{E}[W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^4 + [t_{j+1} - t_j]^2 \\
&\quad -2[t_{j+1} - t_j]\boldsymbol{E}[W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^2\Big) \\
&= \sum_{j=0}^{n-1}\Big(\Big\{3\boldsymbol{E}[W_{t^{(n)}_{j+1}} - W_{t^{(n)}_j}]^2\Big\}^2 - [t^{(n)}_{j+1} - t^{(n)}_j]^2\Big) \\
&= \sum_{j=0}^{n-1}\Big(3[t^{(n)}_{j+1} - t^{(n)}_j]^2 - [t^{(n)}_{j+1} - t^{(n)}_j]^2\Big)\Big)^2 \\
&= \sum_{j=0}^{n-1} 2[t^{(n)}_{j+1} - t^{(n)}_j]^2 \\
&\le 2\max_j[t^{(n)}_{j+1} - t^{(n)}_j]\sum_{j=0}^{n-1}[t^{(n)}_{j+1} - t^{(n)}_j] \\
&= 2t\max_j[t^{(n)}_{j+1} - t^{(n)}_j] \\
&\to 0,\ n \to \infty.
\end{aligned}$$

□

3.3. **The Itô Integral.**

For a pair $(W_t, f(t))$ of a Wiener process W_t a random process $f(t)$, we define the Itô integral

$$I(f) = \int_0^\infty f(t)dW_t.$$

Since paths of W_t are not differentiable and sums $\sum \boldsymbol{E}|W_{t_{j+1}} - W_{t_j}|$ diverge, the Itô integral is not "classical" integral.

We give below conditions under with the Itô integral might be defined.

(A.1.) $\boldsymbol{E}\int_0^\infty f^2(t)dt < \infty$;

(A.2.) for every fixed time t and any $h > 0$ the random variables $f(s), s \le t$ and increments $W_{t+h} - W_t$ are independent.

We start with the considerations of the particular case. Assume

$$f(t) = \sum_k \alpha_k I(t_k \le t < t_{k+1}), \tag{3.1}$$

where $0 = t_0 < t_1 < t_2, ..., < t_n, ...$ is deterministic sequences of time values and $\alpha_k, k = 0, 1, ...$ are random variable such that for fixed k and $h > 0$

$$\{\alpha_0, ..., \alpha_k\} \text{ and } W_{t_k+h} - W_{t_k} \text{ are independent.}$$

Due the assumption **(A.1)** $\sum_k \boldsymbol{E}\alpha_k^2[t_{k+1} - t_k] < \infty$.

Set

$$I(f) := \sum_k \alpha_k [W_{t_{k+1}} - W_{t_k}]. \tag{3.2}$$

The sum in the right side of (3.2) converges in the mean square sense. In fact, $\alpha_k[W_{t_{k+1}} - W_{t_k}], k \ge 1$ forms the sequence of orthogonal zero mean random variables (in the third line of (3.3) $k > \ell$):

$$\begin{aligned}
&\boldsymbol{E}\alpha_k[W_{t_{k+1}} - W_{t_k}] = \boldsymbol{E}\alpha_k\boldsymbol{E}\Big([W_{t_{k+1}} - W_{t_k}]\Big) = 0 \\
&\boldsymbol{E}\Big(\alpha_k[W_{t_{k+1}} - W_{t_k}]\Big)^2 = \boldsymbol{E}\alpha_k^2\boldsymbol{E}\Big([W_{t_{k+1}} - W_{t_k}]^2\Big) = \boldsymbol{E}\alpha_k^2(t_{k+1} - t_k) \\
&\boldsymbol{E}\alpha_k[W_{t_{k+1}} - W_{t_k}]\alpha_\ell[W_{t_{\ell+1}} - W_{tell}] = \boldsymbol{E}\Big\{\alpha_k\alpha_\ell[W_{t_{\ell+1}} - W_{t_\ell}]\boldsymbol{E}\Big([W_{t_{k+1}} - W_{t_k}]\Big)\Big\} = 0.
\end{aligned} \tag{3.3}$$

Particularly, (3.3) provides

$$\begin{aligned}
\boldsymbol{E}I^2(f) &= \boldsymbol{E}\Big(\sum_k \alpha_k[W_{t_{k+1}} - W_{t_k}]\Big)^2 \\
&= \sum_k \boldsymbol{E}\alpha_k^2[W_{t_{k+1}} - W_{t_k}]^2 \\
&= \sum_k \boldsymbol{E}\alpha_k^2[t_{k+1} - t_k] \\
&= \int_0^\infty \boldsymbol{E}f^2(t)dt.
\end{aligned}$$

So, the integral $I(f)$ is a linear function in f, i.e.

$$I(c_1 f_1 + c_2 f_2) = c_1 I(f_1) + c_2 I(f_2) \tag{3.4}$$

for any constants c_1, c_2 and random processes $f_1(t)$, $f2(t)$ of (3.1) type.

To define now the Itô integral for a random process $f(t)$ satisfying only **(A.1)** and **(A.2)** we will some additional fact given below without proof.

Lemma 3.2. *Let the random process $f(t), t \geq 0$ be satisfied* **(A.1.)**, **(A.2)**. *Then there exists a sequence $f_n(t), t \geq 0, n \geq 1$ of piece-wise constant random processes*

$$f_n(t) = \sum_k \alpha_k^n I(t_k^n \leq t < t_{k+1}^n),$$

where $t_k^n, k = 0, 1, 2...$ is a condensing sequence of deterministic time values and for every k the random variables $\{\alpha_1, \dots \alpha_k\}$ are independent of $W_{t_k^n + h} - W_{t_k^n}, h > 0$, moreover for every n, f_n satisfies **(A.1.)**, **(A.2.)** *and*

$$\lim_{n \to \infty} \int_0^\infty \boldsymbol{E}\big(f(t) - f_n(t)\big)^2 dt = 0.$$

3.3.1. **Proof of existence $I(f)$.**

For fixed n, $I(f_n)$ is well defined. By linear property of $I(f_n)$ we have

$$I(f_n) - I(f_m) = I(f_n - f_m).$$

Hence

$$\begin{aligned}
\boldsymbol{E}\Big(I(f_n) - I(f_m)\Big)^2 &= \boldsymbol{E} I^2(f_n - f_m) \\
&= \int_0^\infty \boldsymbol{E}\big(f_n(t) - f_m(t)\big)^2 dt \\
&\leq 2\int_0^\infty \boldsymbol{E}\big(f(t) - f_n(t)\big)^2 dt + 2\int_0^\infty \boldsymbol{E}\big(f(t) - f_m(t)\big)^2 dt \\
&\to 0, \ n, m \to \infty.
\end{aligned}$$

Consequently, $I(f_n), n \geq 1$ is the fundamental sequence and so that by the Cauchy criteria this sequence converges the mean square sense to some limit, which we denote by $I(f)$. In other words we get

$$\lim_n \boldsymbol{E}\big(I(f) - I(f_n)\big)^2 = 0.$$

The random variable $I(f)$ is unique in the following sense. If $\widetilde{f}_n(t), n \geq 1$ is another approximating sequence with a limit $\widetilde{I}(f)$, then

$$\boldsymbol{E}\Big(I(f) - \widetilde{I}(f)\Big)^2 = 0.$$

In fact

$$\boldsymbol{E}\big(I(f)-\widetilde{I}(f)\big)^2=\boldsymbol{E}\big(I(f)-I(f_n)+I(f_n)-I(\widetilde{f}_n)+I(\widetilde{f}_n)-\widetilde{I}(f)\big)^2$$
$$\le 3\Big\{\boldsymbol{E}\Big(I(f)-I(f_n)\Big)^2+\boldsymbol{E}I^2(f_n)-\widetilde{f}_n)+\boldsymbol{E}\Big(I(\widetilde{f}_n)-\widetilde{I}(g)\Big)^2\}$$
$$=3\Bigg(\boldsymbol{E}\Big(I(f)-I(f_n)\Big)^2+\boldsymbol{E}\big(\widetilde{I}(f_n)-\widetilde{I}(f)\big)^2$$
$$+\int_0^\infty \boldsymbol{E}\big(f_n(t)-\widetilde{f}_n(t)\big)^2dt\Bigg)$$

and the first and second terms in the right side of this inequality tend to zero by the definition while the third

$$\int_0^\infty \boldsymbol{E}\big(f_n(t)-\widetilde{f}_n(t)\big)^2dt$$
$$\le 2\Bigg(\int_0^\infty \boldsymbol{E}\big((t)-f_n(t)\big)^2dt+\int_0^\infty \boldsymbol{E}\big(f(t)-\widetilde{f}_n(t)\big)^2dt\Bigg)\to 0,\ \to\infty.$$

The random variable $I(f)$ is named the Itô integral.

3.3.2. **Properties of $\boldsymbol{I(f)}$.**

(P.1.) For $f_i(t), i=1,2$, satisfying **(A.1.)** and **(A.2.)**, and any constants $c_i, i=1,2$

$$I(c_1f_1+c_2f_2)=c_1I(f_1)+c_2I(f_2).$$

(P.2.) $\boldsymbol{E}I^2(f)=\int_0^\infty \boldsymbol{E}f^2(t)dt.$

Proofs: For piece-wise constant processes **(P.1)**, **(P.2.)** are obviously valid. They remain valid under passing to limit in the mean square sense (see Home work 5.). □

Remark 1. Instead of **(A.1)** assume:

(A'.1.) $\boldsymbol{E}\int_0^T f^2(t)dt<\infty,\ T>0$. Then the Itô integral $I_T(f)=\int_0^T f(t)dW_t$ is defined as well by setting $I_T(f)=I(f_T)$, with $f_T(t)=f(t)I(T>t)$.

(A".1.) $\boldsymbol{P}(\int_0^T f^2(t)dt<\infty)=1$. then $I_T(f)$ is well defined as well.

Sketch Proof: Set $\tau_n=\min\{t\le T:\int_0^t f^2(s)ds\ge n\},\ n\ge 1$ and put $f_n(t)=f(t)I(\tau_n\ge t)$. Then $I_T(f_n)$ is well defined (see Problem 4 from Home work). Set

$$I_T(f)=I_T(f_1)+\sum_{n=1}^\infty[I_T(f_{n+1})-I_T(f_n)].$$

□

1. Home work: Wiener Process and Itô integral

1. Prove the following statement: let ξ, η be Gaussian vector with zero mean and orthogonal components, i.e. $\boldsymbol{E}\xi = 0$, $\boldsymbol{E}\eta = 0$ and

$$\boldsymbol{E}\xi\eta = 0.$$

Show that ξ and η are independent random variables.

2. Let $\xi_1, \dots, \xi_n, \dots$ be a sequence of Gaussian random variables. Assume that for any $z \in \mathbb{R}$ and $\boldsymbol{i} = \sqrt{-1}$

$$\lim_{k\to\infty} \boldsymbol{E} e^{\boldsymbol{i} z \xi_k} \text{ (exists).}$$

Show that

$$\left.\begin{array}{l} \lim\limits_{k\to\infty} \boldsymbol{E}\xi_k \\ \lim\limits_{k\to\infty} \boldsymbol{E}\xi_k^2 \end{array}\right\} \text{ (exist).}$$

3. Prove **(P.1.)** and **(P.2)**.

4. With $f(t)$, satisfying **(A.1)** and **(A.2)**, to prove that $f_n(t) = f(t)I(\tau_n \geq t)$, where $\tau_n = \inf\{t : \int_0^t f^2(s)ds \geq n]\}$, satisfies **(A.1)** and **(A.2)** as well.