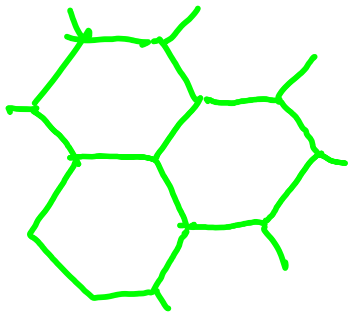
Can Structure Beat Shannon?

Lattice Codes Tell Their Story

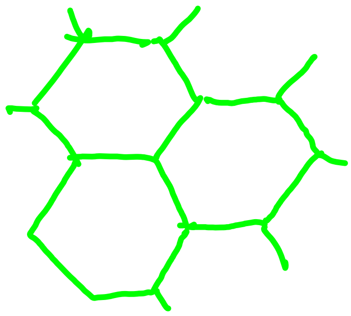


Rami Zamir

Plenary Talk ISIT 2010
Austin, Texas

Can Structure Beat Random?

Lattice Codes Tell Their Story



Rami Zamir

Plenary Talk ISIT 2010
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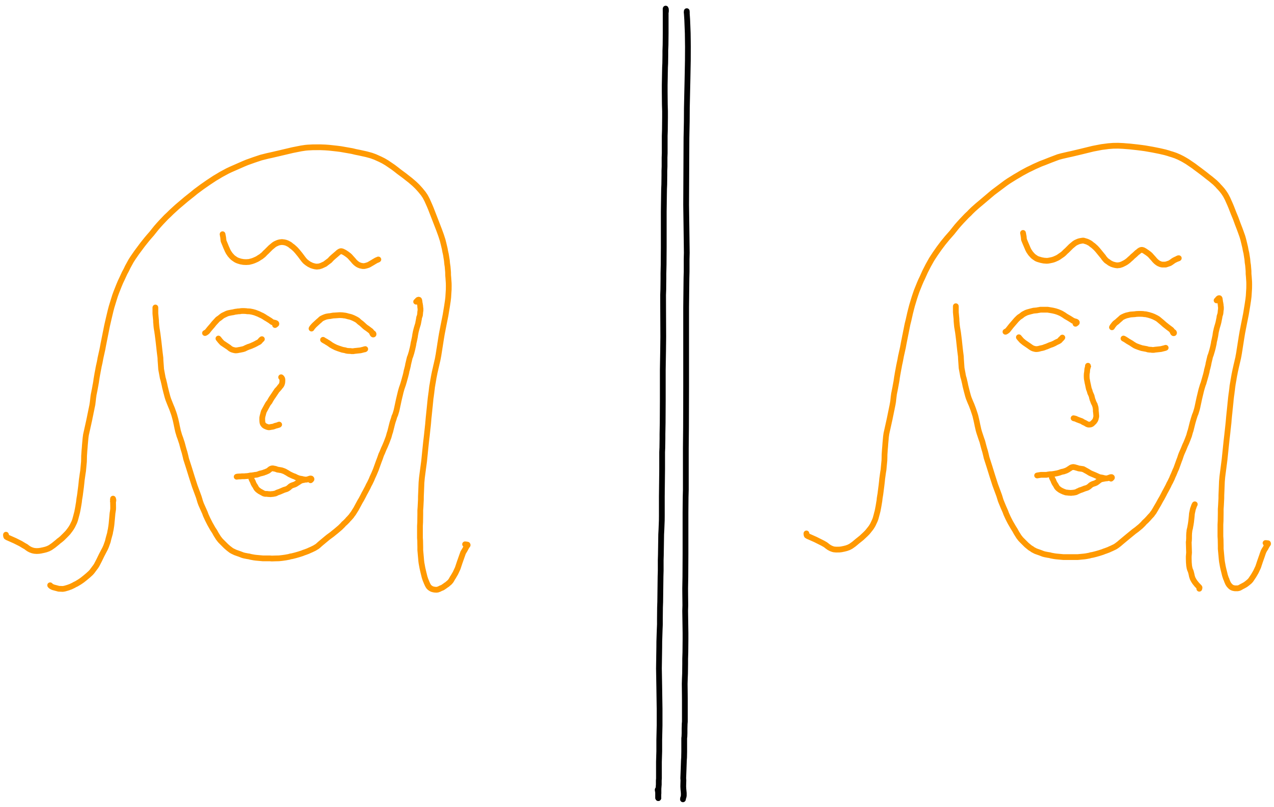
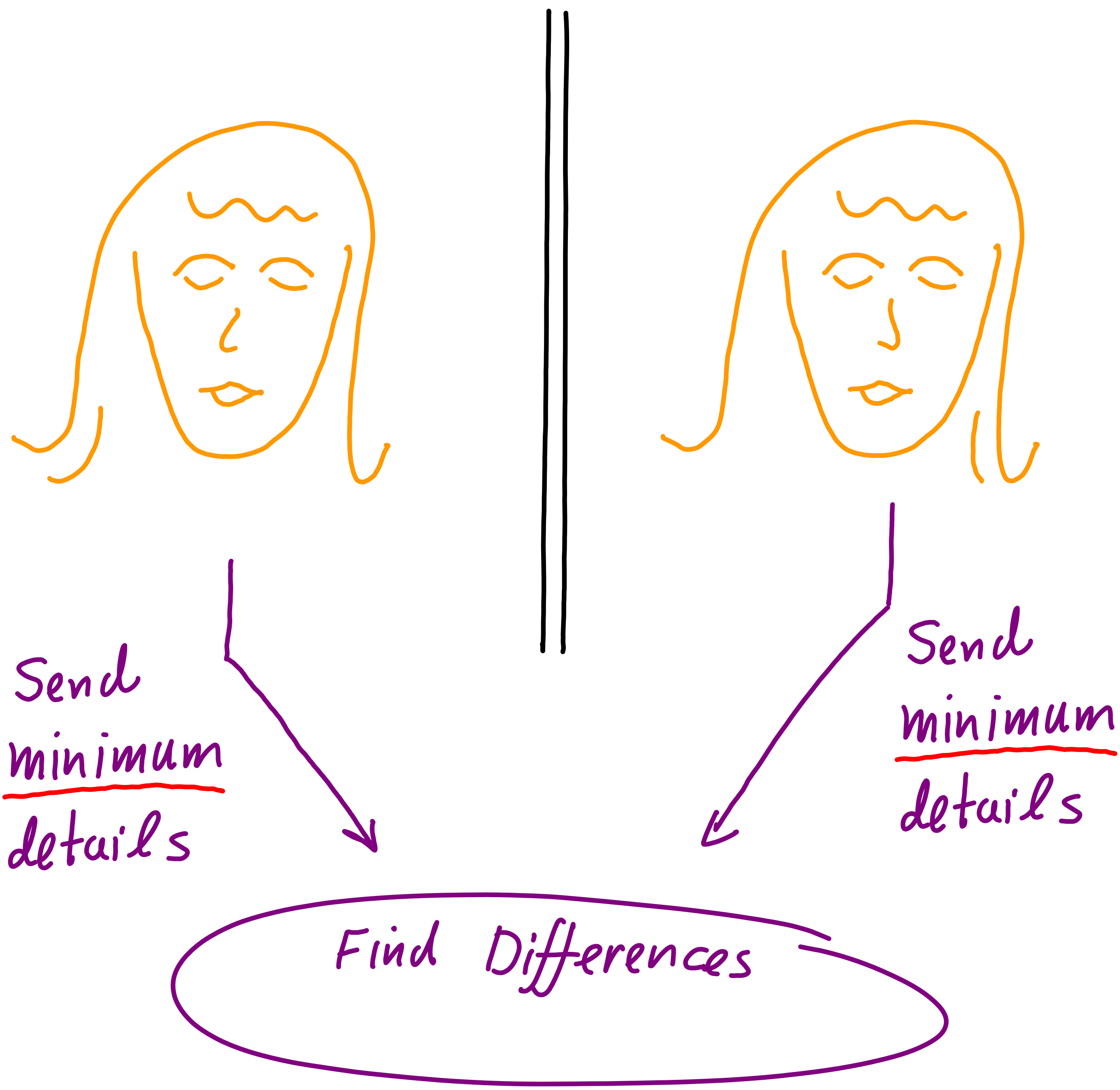
Find the Differences

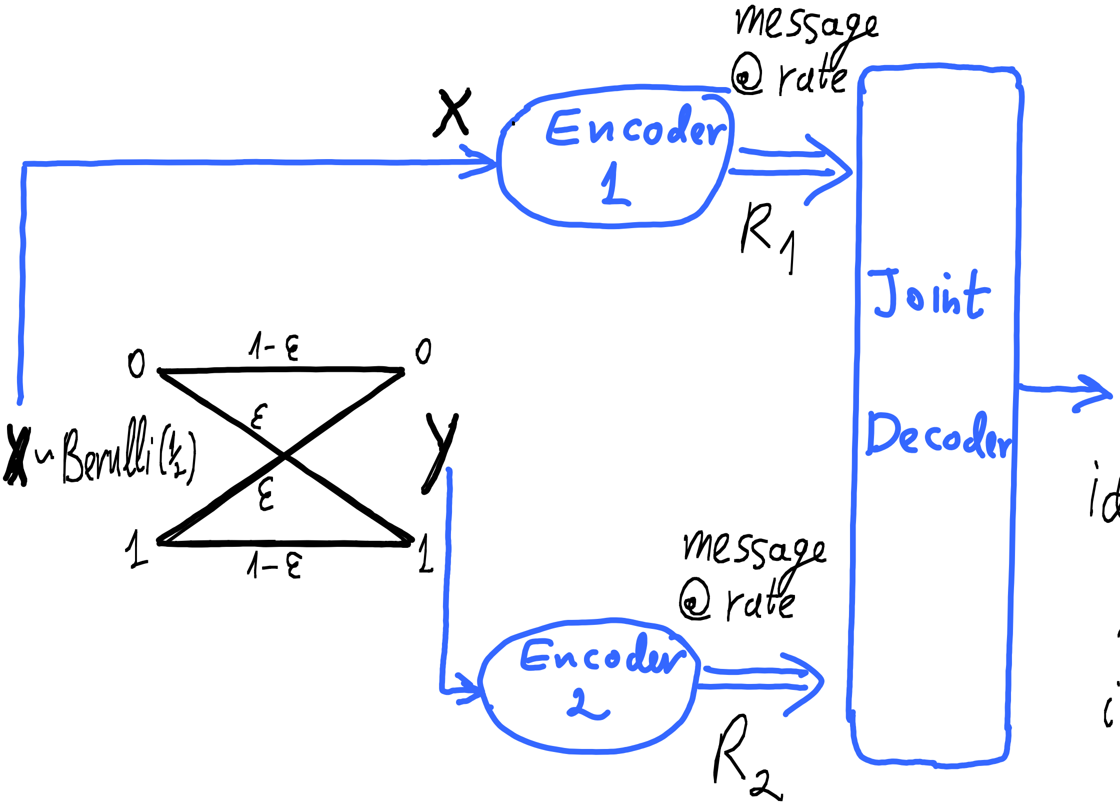


?

Communicate the Differences



The Korner-Marton Problem



$\widehat{X \oplus Y} =$

identify where

$X_i \neq Y_i$

$i = 1 \dots n$

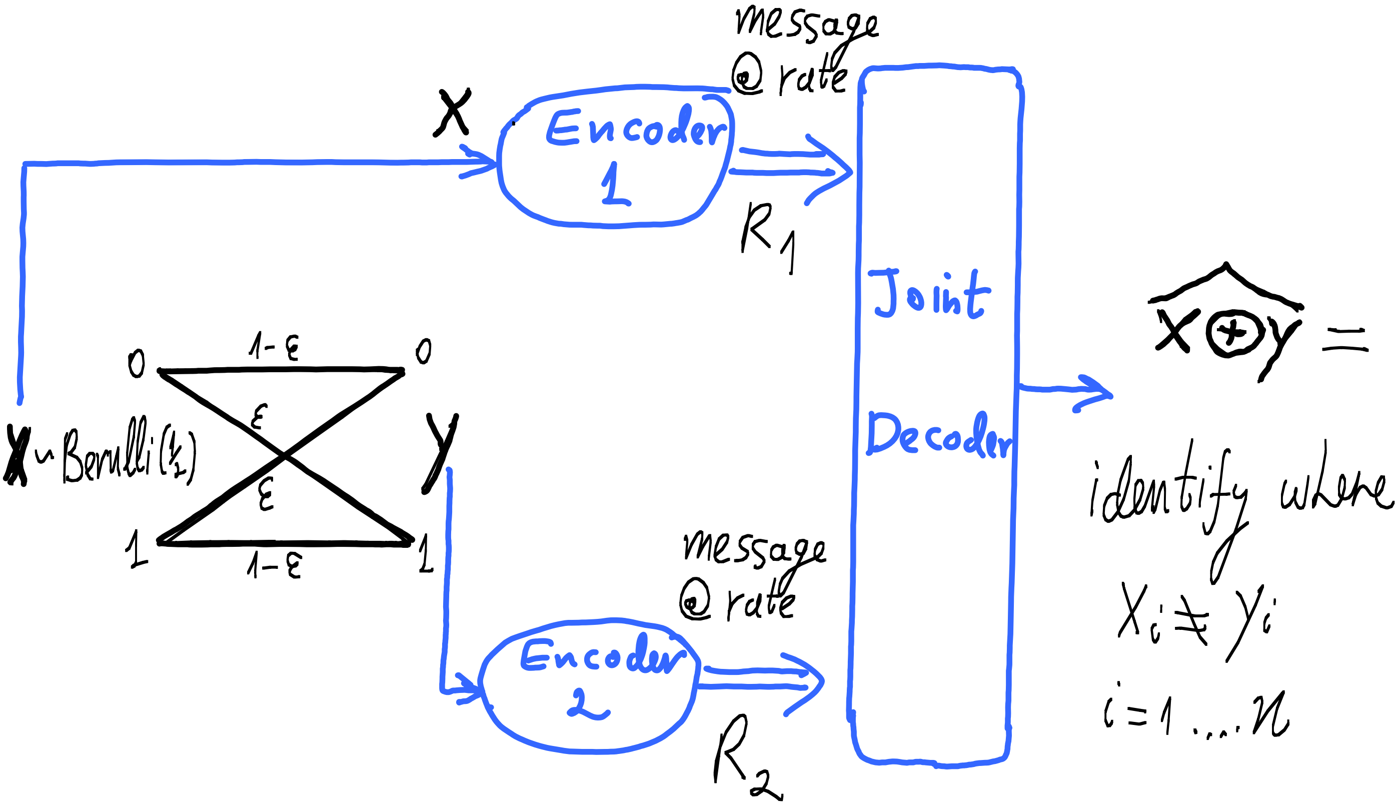
$Z = X \oplus Y$

Compress & estimate:

$H(X) + H(Y) = 1 + 1 = 2$ Bit

Rate $\stackrel{?}{=}$

The Korner-Marton Problem



$$Z = X \oplus Y$$

$$\text{Rate} \overset{?}{=}$$

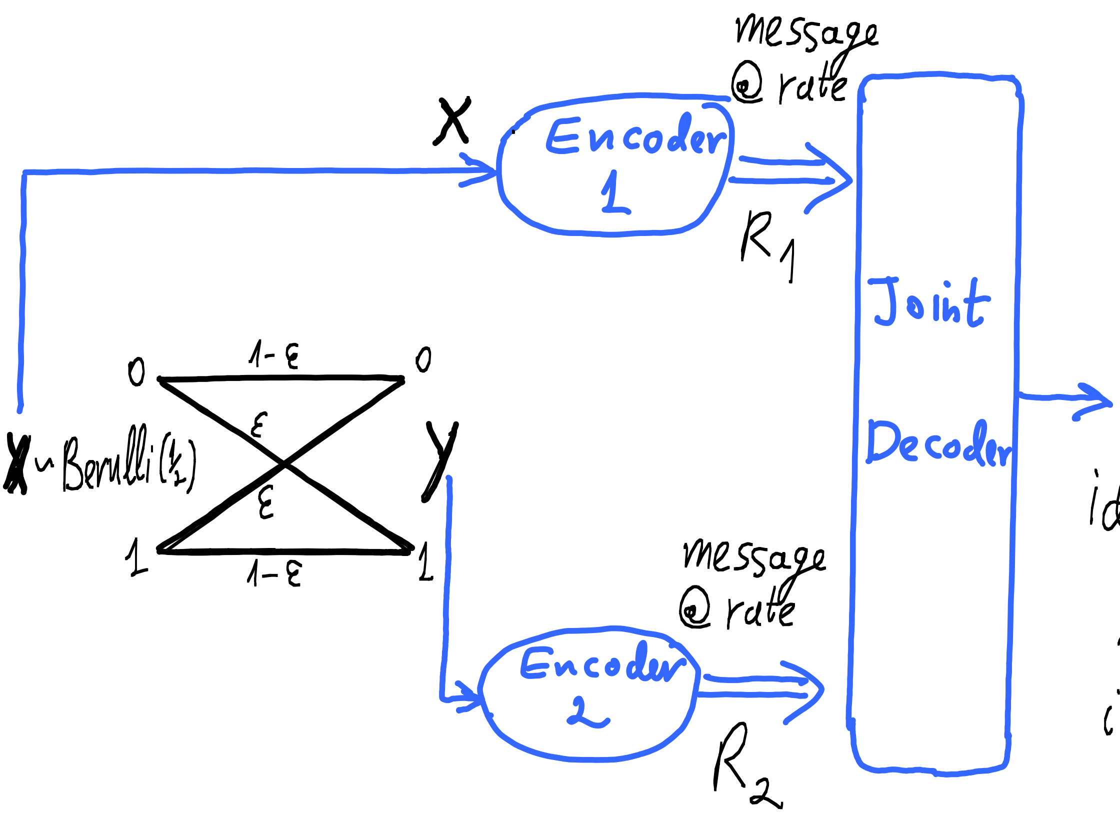
Compress & estimate:

$$H(X) + H(Y) = 1 + 1 = 2 \text{ Bit}$$

Compress **well** & estimate:

$$H(X, Y) = H(X) + H(Z) = 1 + H_B(\epsilon) = 1.1 \text{ Bit}$$

The Korner-Marton Problem



$\widehat{X \oplus Y} =$

identify where

$X_i \neq Y_i$

$i = 1 \ldots n$

$Z = X \oplus Y$

Rate $\overset{?}{=}$

compress & estimate:

$H(X) + H(Y) = 1 + 1 = 2$ Bit

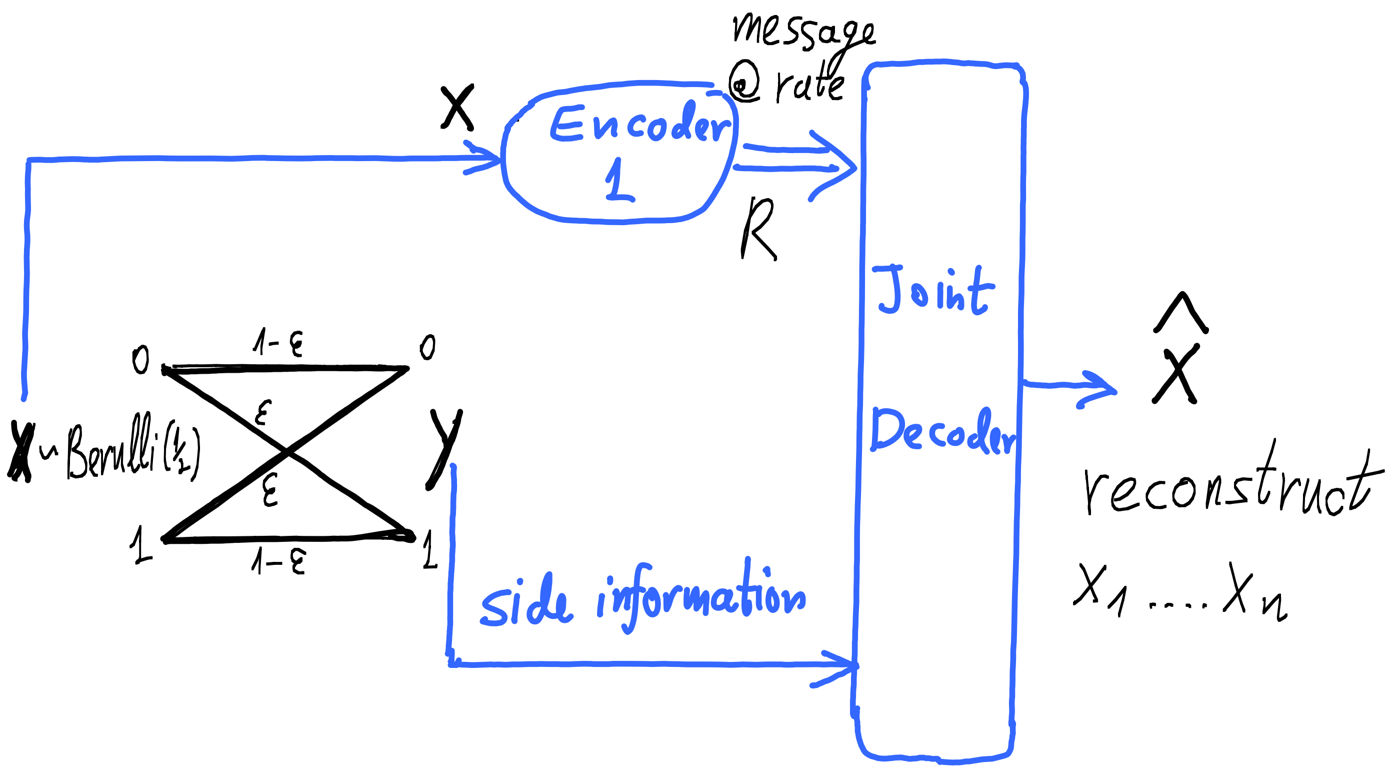
compress **well** & estimate:

$H(X, Y) = H(X) + H(Z) = 1 + H_B(\varepsilon) = 1.1$ Bit

estimate & compress:

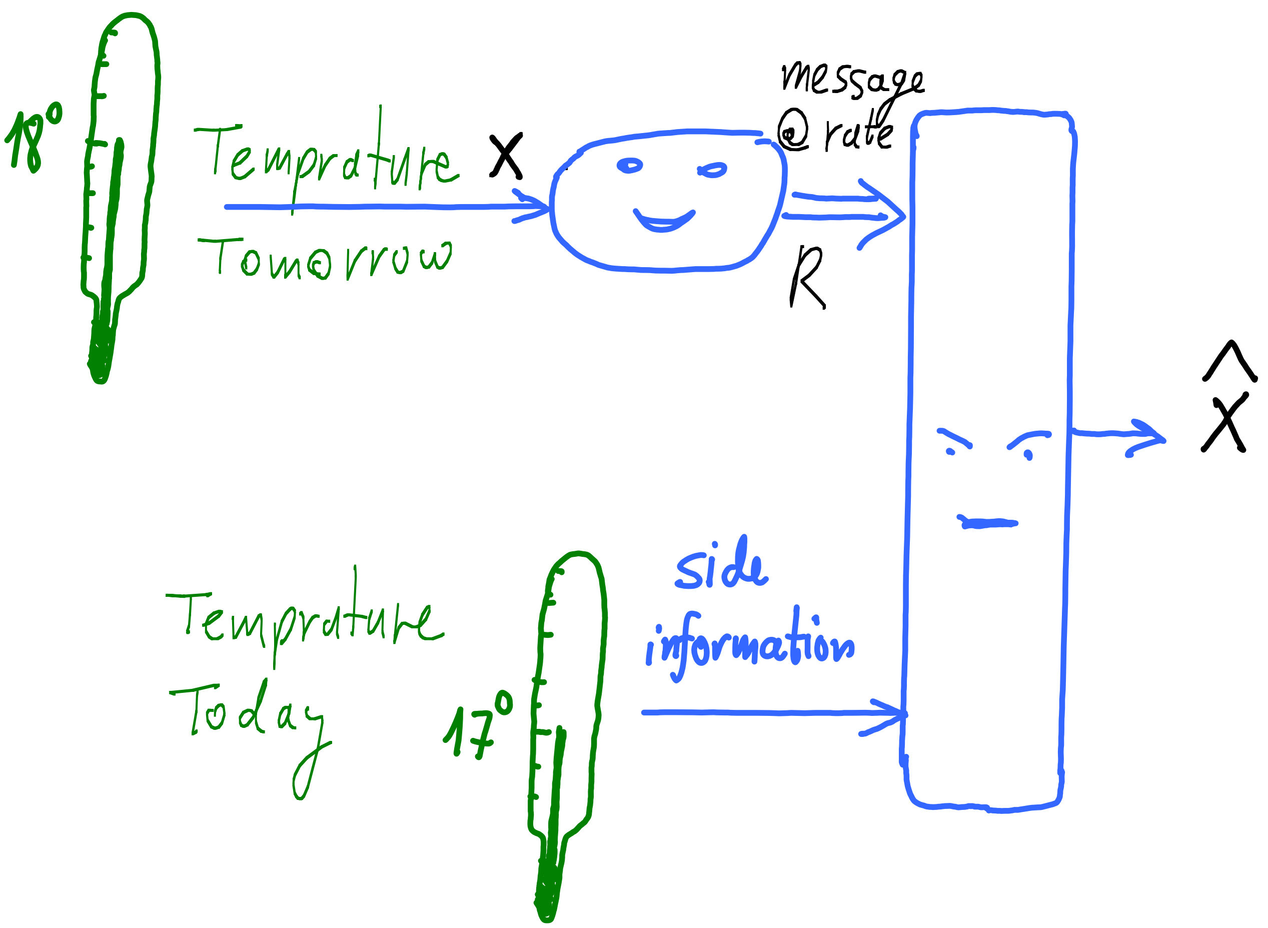
$H(Z) = H_B(\varepsilon) = 0.1$ Bit

The Slepian-Wolf Problem



$$R = H(X|Y) = H(Z) = H_B(\varepsilon) = 0.1 \text{ Bit}$$

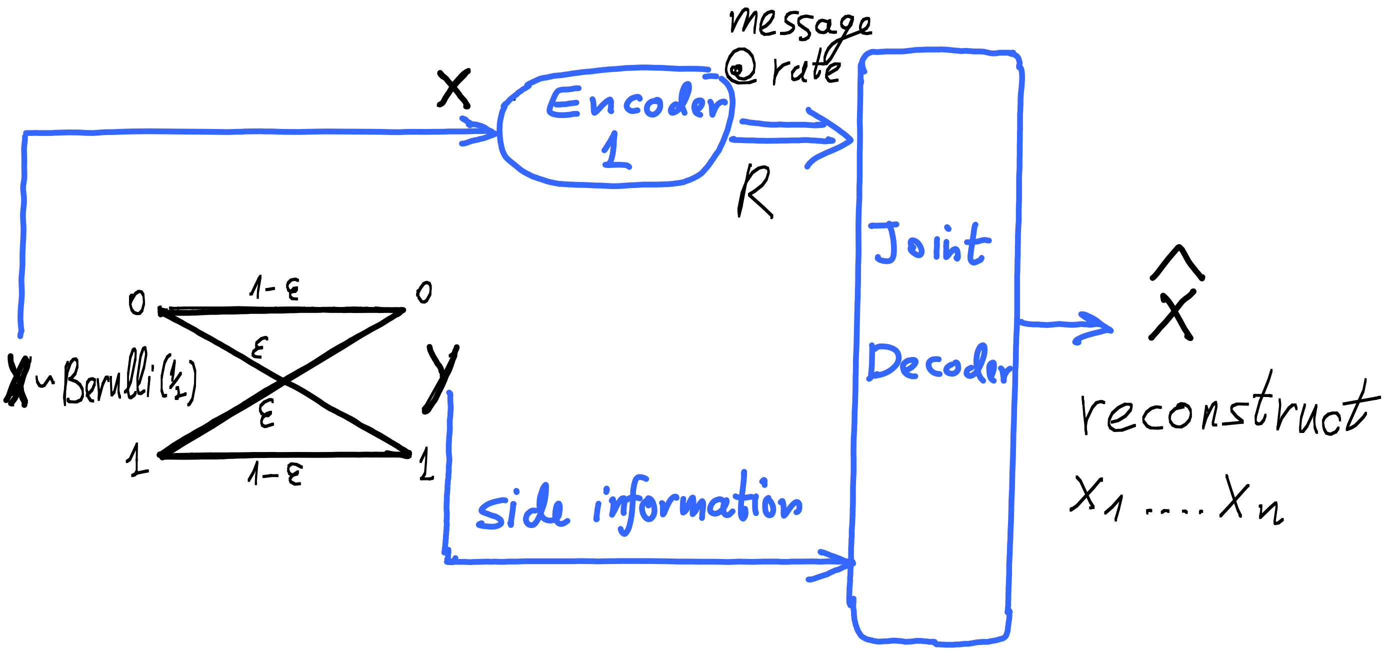
The Slepian-Wolf Problem

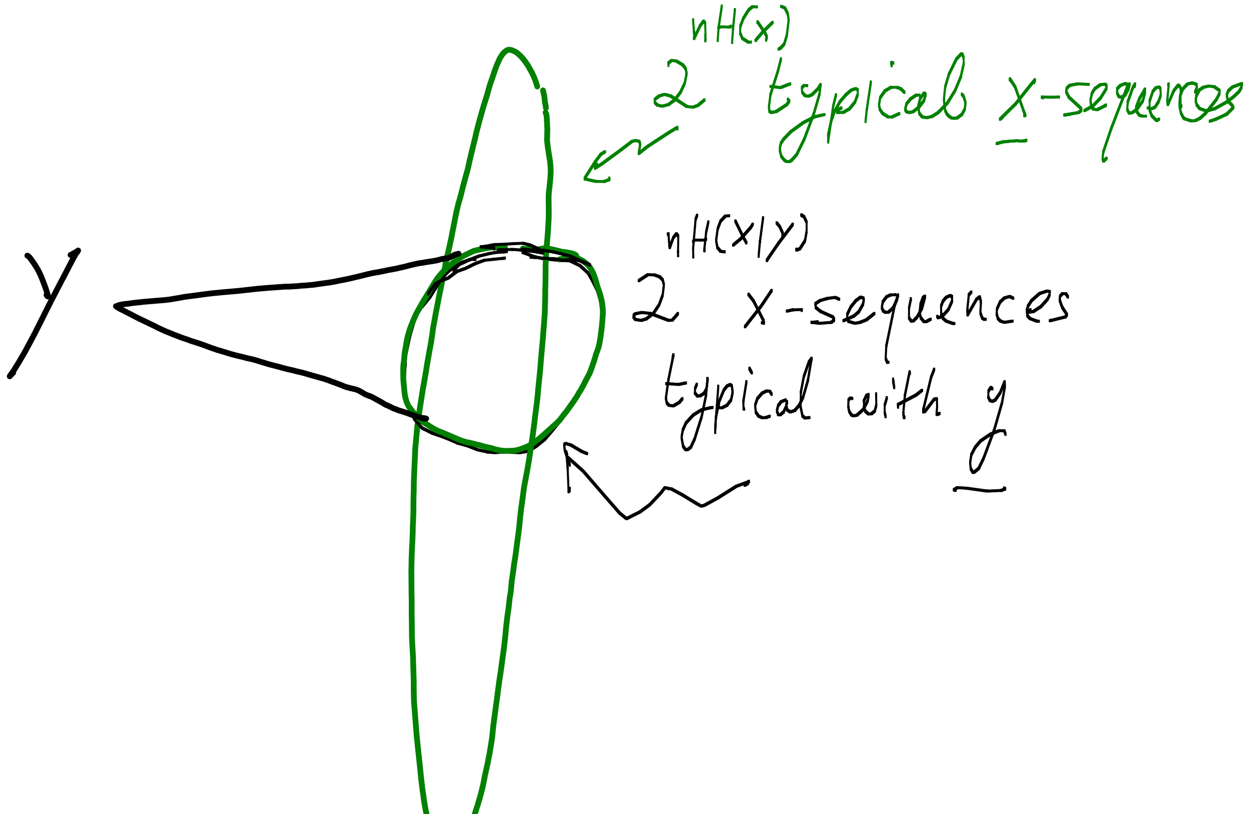


$$T_{tomorrow} = T_{today} \pm 1^\circ c$$

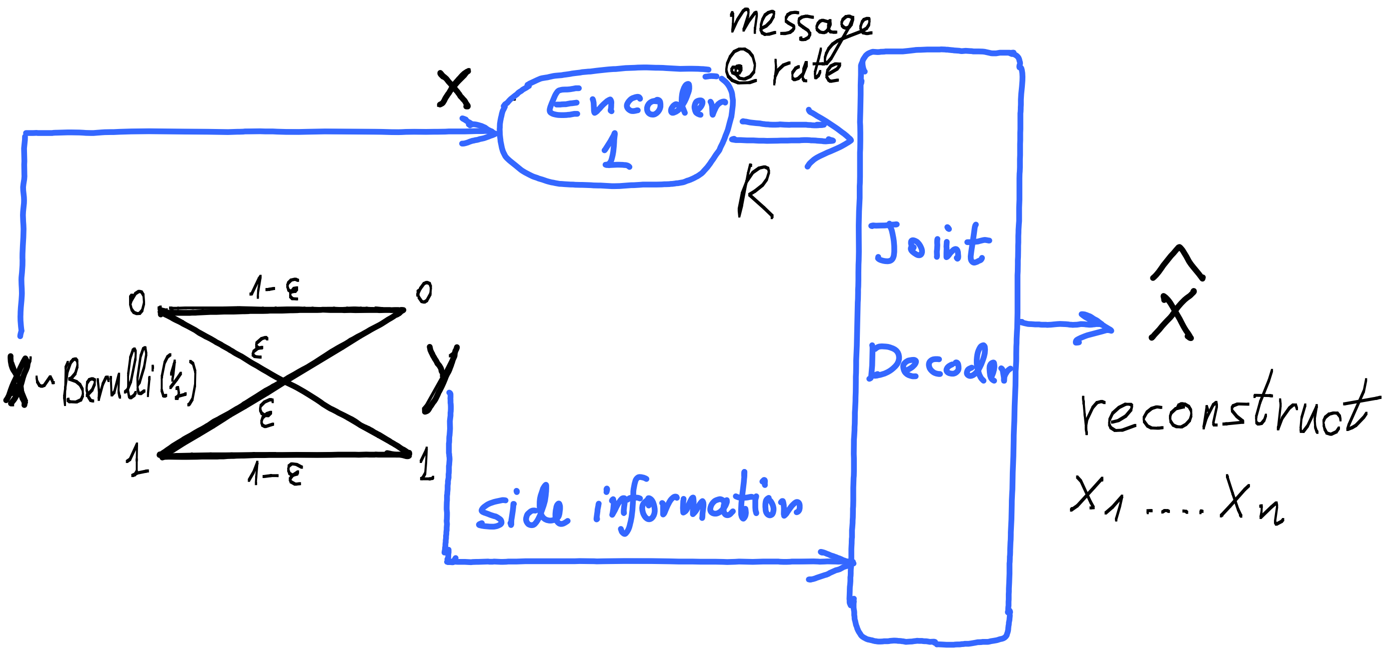
Can we send $T_{tomorrow}$ using only one bit ?

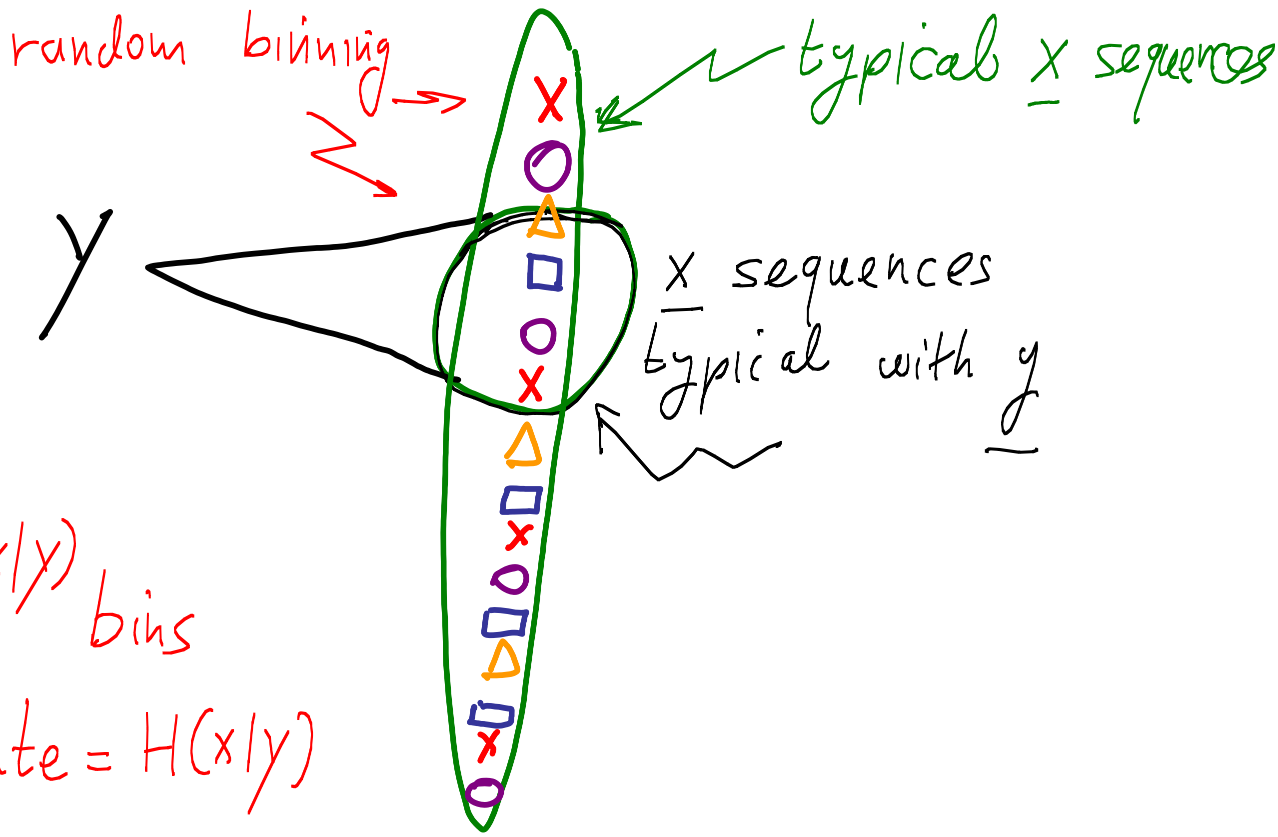
The Slepian-Wolf Problem





The Slepian-Wolf Problem





$2^{nH(x|y)}$ bins

$\Rightarrow$ Rate $= H(x|y)$

Back to Korner-Marton: Solution

$\mathbb{C}$ = (n,k) linear code for $B.S.C.(\varepsilon)$

general properties:

$k/n \simeq 1 - H_B(\varepsilon)$

generator matrix — parity-check

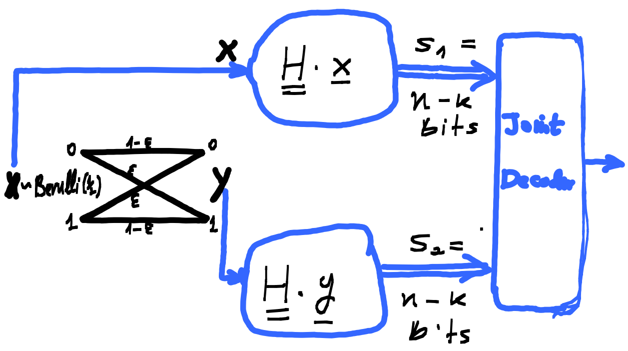
$$\underbrace{\underline{x}}_{n\times 1} = \underbrace{\underline{\underline{G}}}_{n\times k} \cdot \underbrace{i}_{k\times 1} \qquad \underbrace{\underline{\underline{H}}}_{(n-k)\times n} \cdot \underbrace{\underline{x}}_{n\times 1} = \underline{0} \text{ for } \underline{x} \in \mathbb{C}$$

If $\underline{y} = \underline{x} \oplus \underline{z}$, where $\underline{z} \sim \text{Bernulli}(\varepsilon)$, then

$$\hat{\underline{z}} \triangleq f(\underbrace{H \cdot \underline{y}}_{s = \text{syndrome}}) = \underline{z} \quad \text{with high prob.}$$

$P_e = P_r\{\hat{\underline{z}} \neq \underline{z}\}$

= the same $\forall\, \underline{x} \in \mathbb{C}$



Back to Korner-Marton: Solution

$\mathbb{C}$ = (n,k) linear code for B.S.C.(ε)

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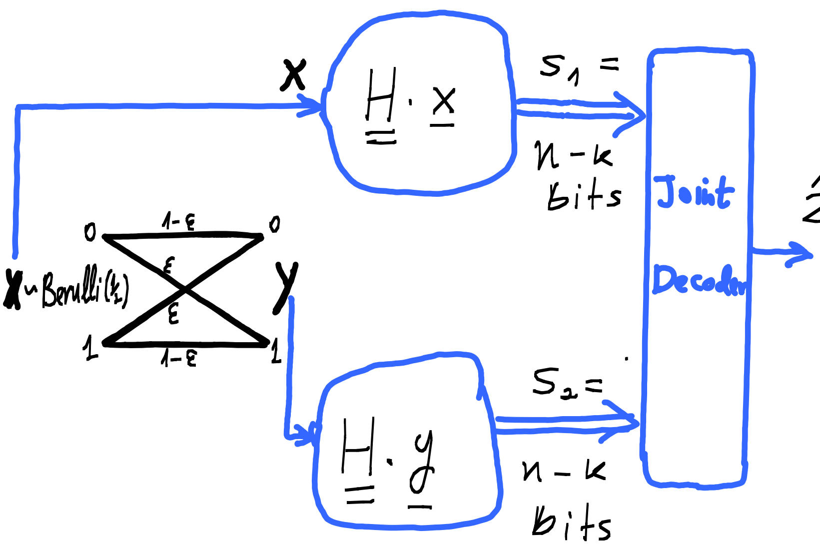


$$\hat{\underline{z}} = f(s_1 \oplus s_2)$$
$$= f(\underline{\underline{H}} \cdot (\underline{x} \oplus \underline{y}))$$
$$= f(\underline{\underline{H}} \cdot \underline{z})$$
$$= \underline{z} \quad \text{w. high prob.}$$

Total Rate $= 2 \times \frac{n-k}{n} = 2 \times H_B(\varepsilon) \simeq 0.2$ bits

$P_e = \Pr\{\hat{\underline{z}} \neq \underline{z}\}$ = independent of x

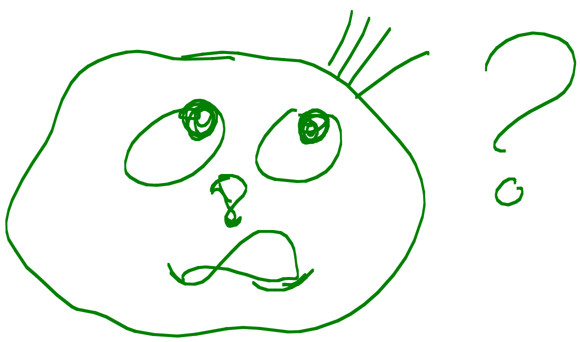
A comment by KM: best known "single letter" = SW = 1.1 bit

* Do we really need structured Codes?

* How do we extend to real signals?

* How do we measure code goodness?

{rate, error prob., distortion...}



Nature Knows his Way..

* picture editing
by Kessem Zamir

Lattice: Definition

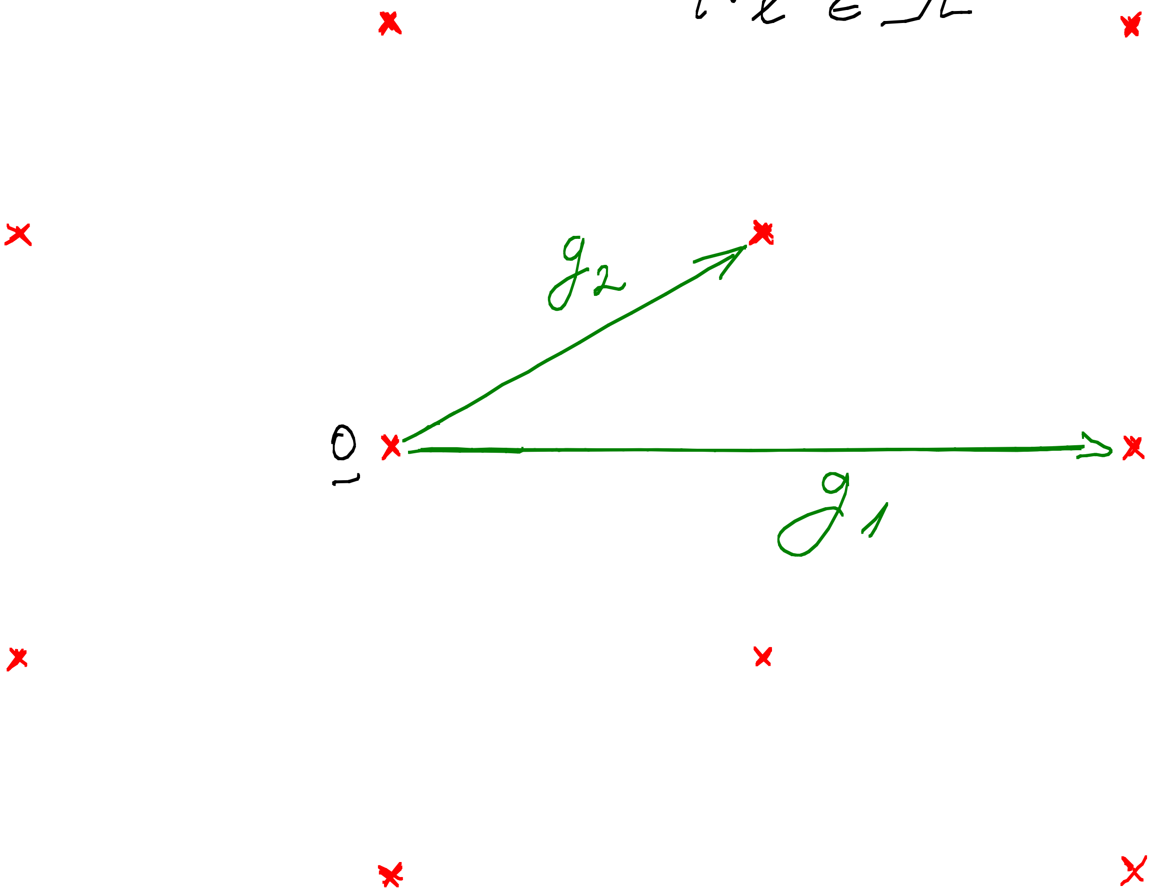
$$\Lambda = \{ \underline{\underline{G}} \cdot \underline{i} \; : \; \underline{i} = \text{vector of integers} \; (0, \pm 1, \pm 2, \ldots) \}$$

Lattice in $\mathbb{R}^k$

Generator Matrix $k \times k$

linearity: $\underline{l}_1, \underline{l}_2 \in \Lambda \Rightarrow \underline{l}_1 + \underline{l}_2 \in \Lambda$

$i \cdot \underline{l} \in \Lambda$



Lattice : Definition

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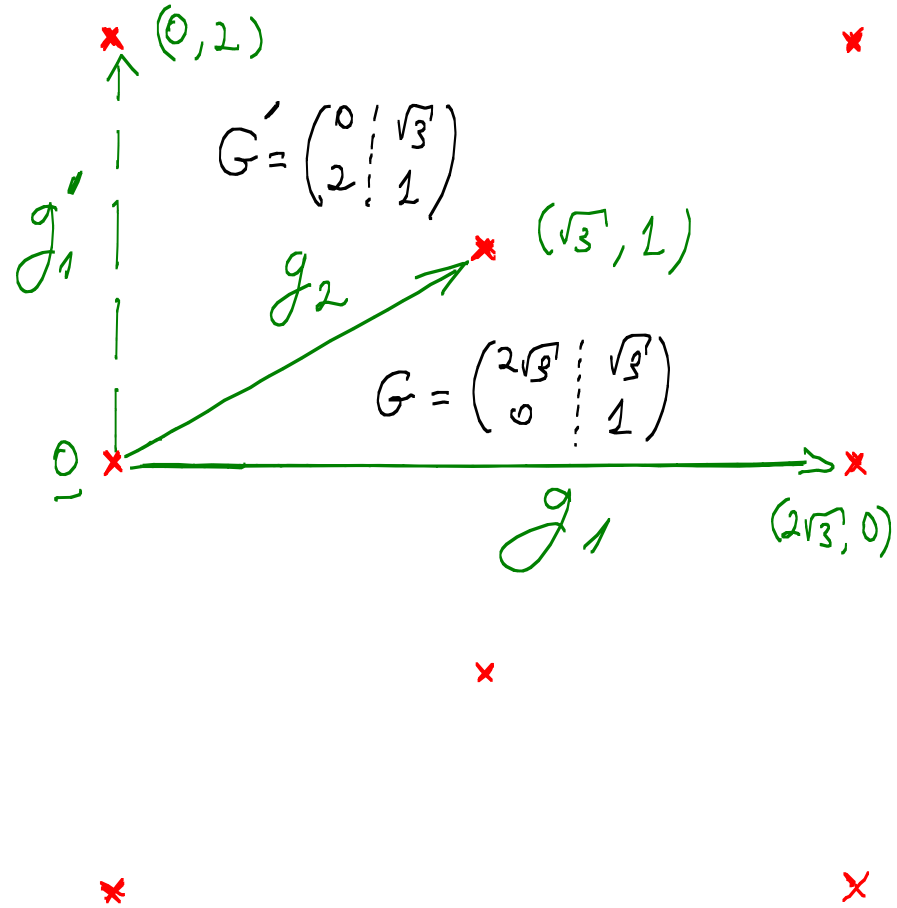
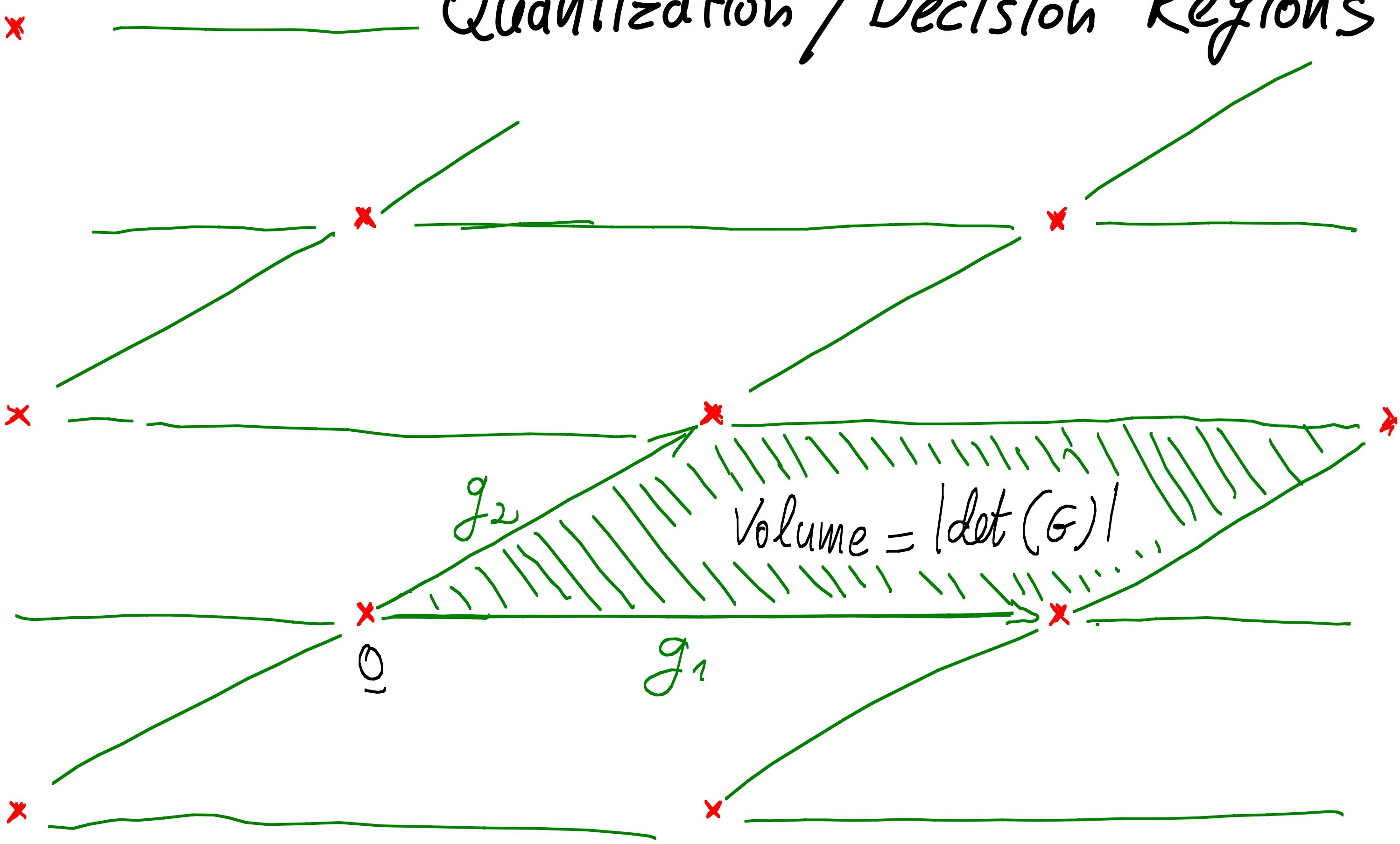


Lattice Partition:

Quantization/Decision Regions

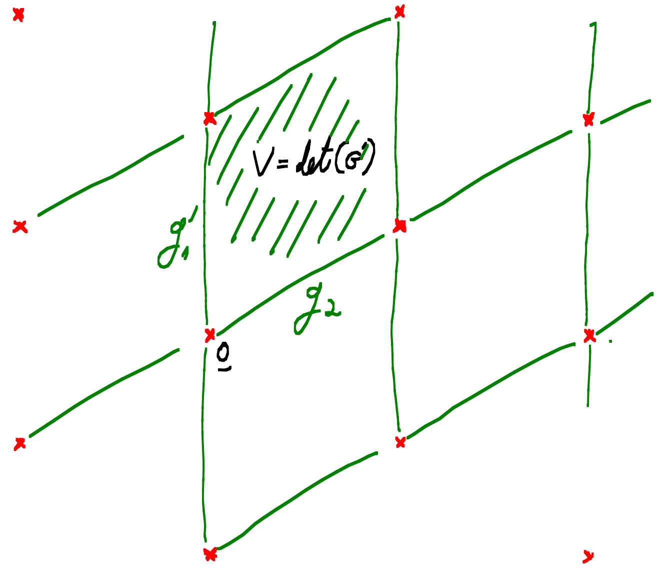


Parallelopipeds

$$P_0 = \{ \alpha_1 g_1 + \alpha_2 g_2 : \quad 0 \le \alpha_1, \alpha_2 \le 1 \}$$

$$\Lambda + P_0 = \mathbb{R}^k$$

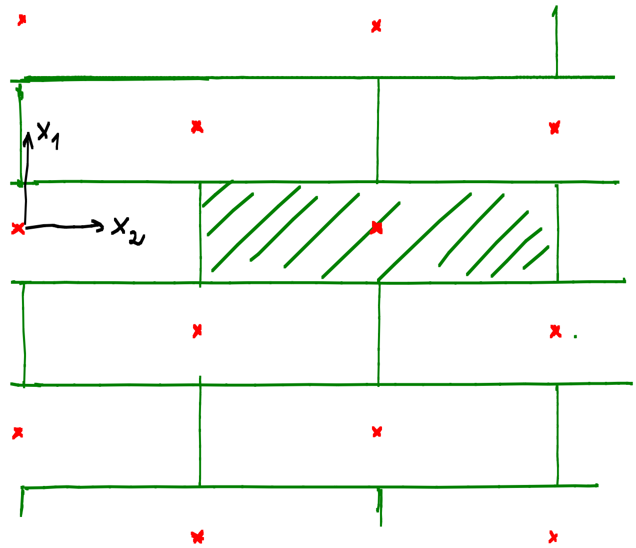
Lattice Partitions

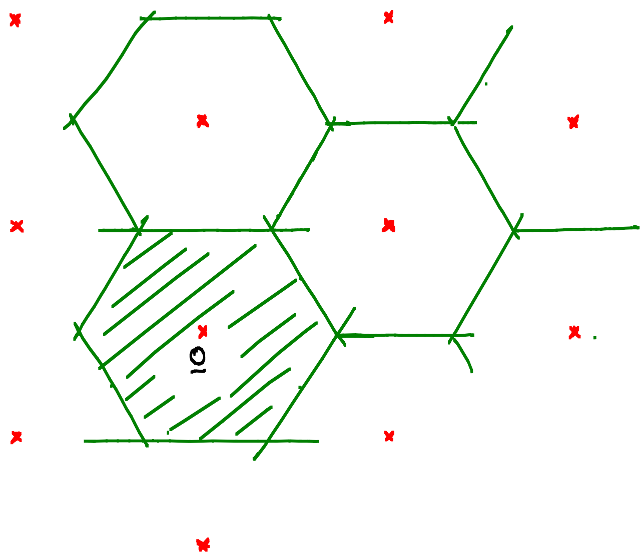


Other Basis ⇒ other Parallelepiped

⇒ Cell Volume V is invariant of partition

Sequential Quantization



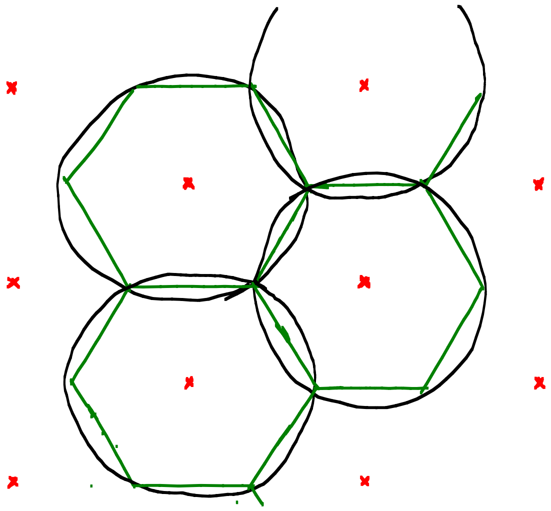


Voronoi Partition

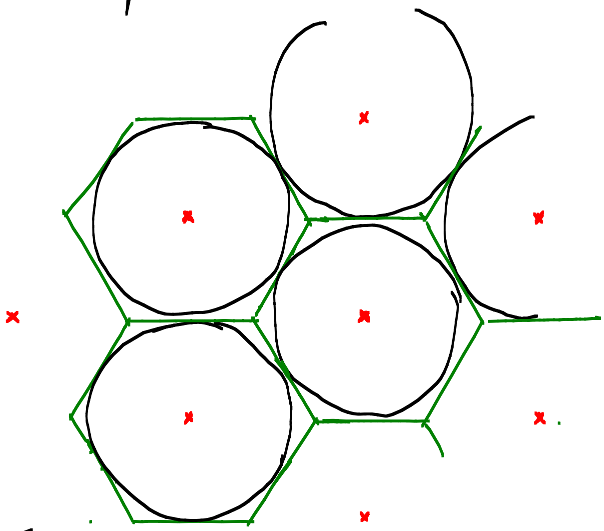
$$P_0 = \{ x : \|x\| \le \|x - l_i\| \quad \forall l_i \in \Lambda \}$$

Covering, Packing, Kissing Number & More ....

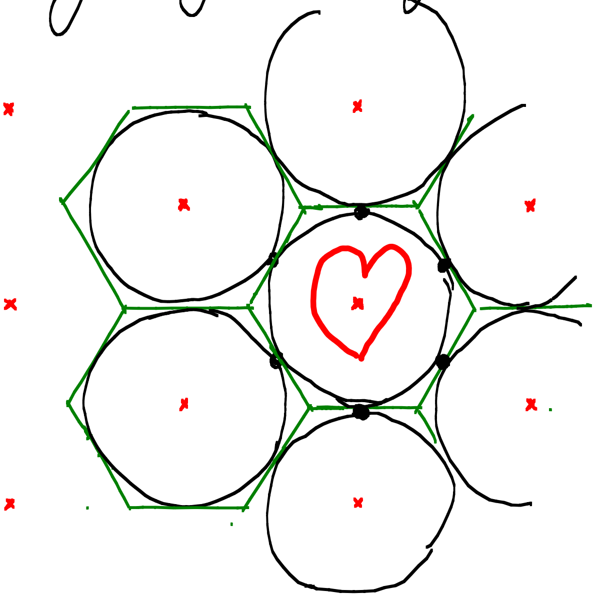
Covering $\mathbb{R}^k$ with (few) Spheres



Packing (many) spheres in $\mathbb{R}^k$



Kissing by (many) Spheres

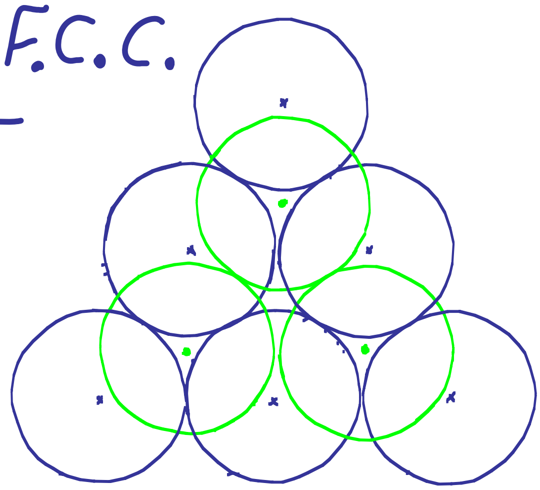


&

good arrangements for quantization and AWGN channel coding

Not an "All-Purpose" Lattice!

* Best 3-dim Packing: F.C.C.



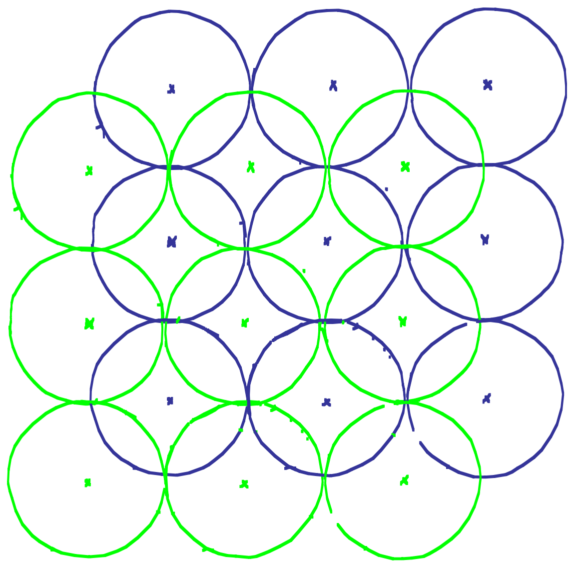
each layer = hexagonal Λ

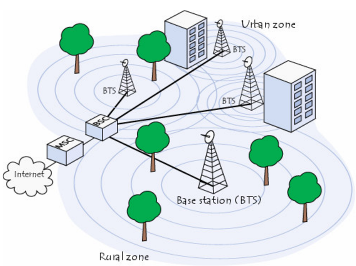
layers are staggered

* Best 3-dim Covering: B.C.C.

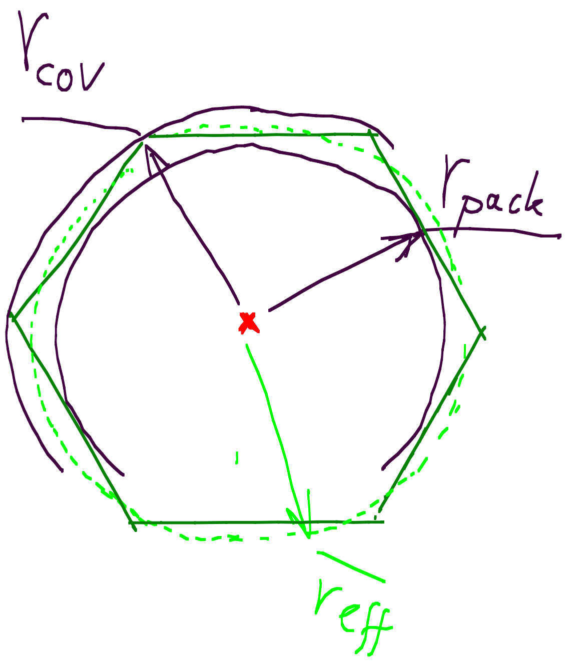
each layer = cubic Λ

layers are staggered





Figures of Merit



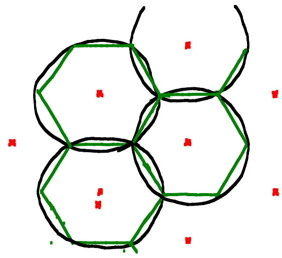
Radiuses:

r_{cov} = min sphere containing V_0

r_{pack} = max sphere contained in V_0

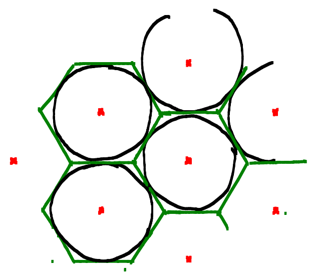
r_{eff} = Sphere with same volume

- Covering efficiency:



$$\rho_{cov}(\Lambda) = \frac{r_{cov}}{r_{eff}} > 1$$

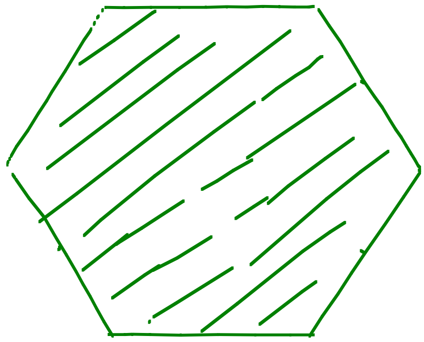
- Packing efficiency:



$$\rho_{pack}(\Lambda) = \frac{r_{pack}}{r_{eff}} < 1$$

Figures of Merit (Continued)

- **Quantization efficiency:**



$\underline{X} \sim \text{Uniform}(V_0)$

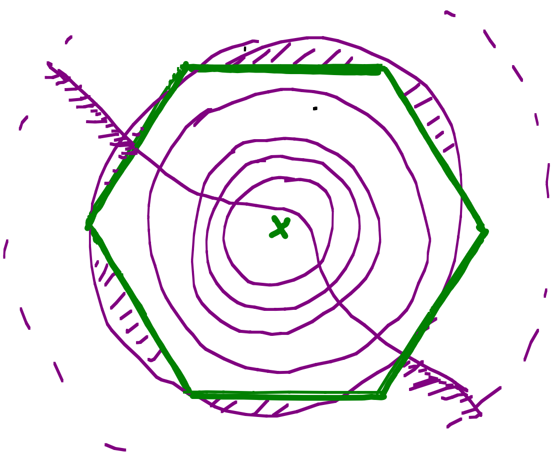
$$\sigma^2(\Lambda) \triangleq \frac{1}{k} E\|X\|^2$$

$$G(\Lambda) \triangleq \frac{\sigma^2(\Lambda)}{V^{2/k}}$$ = normalized second moment

- **AWGN coding efficiency:** $\underline{Z} \sim \text{AWGN } N(0, \sigma^2)$

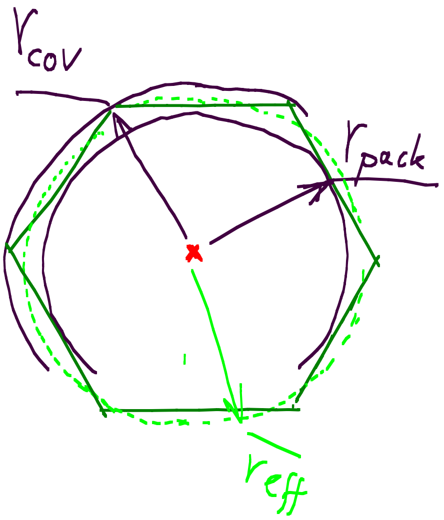
$$P_e \triangleq \Pr\{\underline{Z} \notin V_0\}$$

= "Poltyrev's error prob."



$$\mu(\Lambda, P_e) \triangleq \frac{V^{2/k}}{\sigma^2}\Big|_{@ P_e}$$ = Volume-to-Noise Ratio

Iso-perimetric Inequalities
(Sphere bounds)



Ball minimizes

* * *

over all bodies

of a fixed volume V!

$$\sigma^2(\Lambda) \geq \sigma^2(\text{ball with radius } r_{eff})$$

$$Pe(\Lambda) \geq Pe(\;\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\;)$$

$\Longrightarrow$

$$G(\Lambda) \geq \text{N.S.M. of } k\text{-dim ball}$$

$$\mu(\Lambda, pe) \geq \text{V.N.R.}\;\text{"}\quad\text{"}\quad\text{"}$$

G_k as a function of k ...

[Conway & Sloane Book 1988]

n.

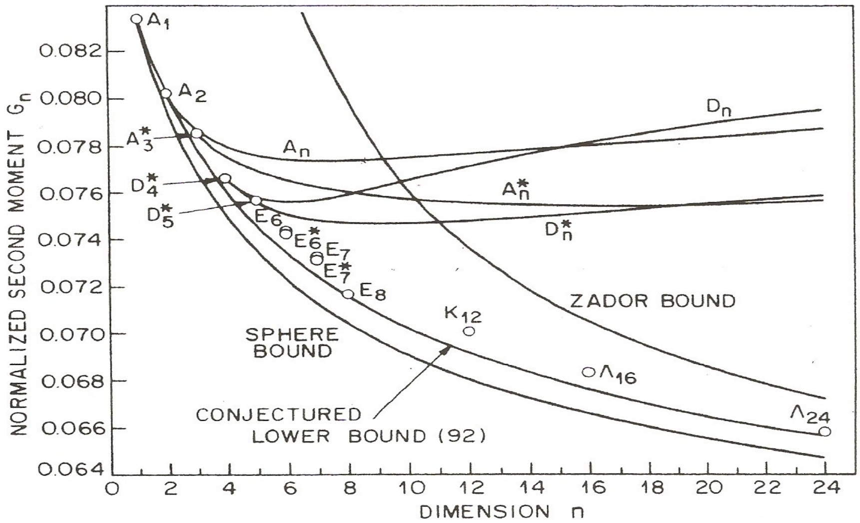
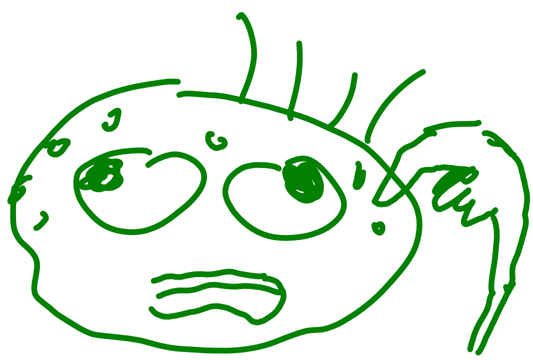


Figure 2.9. The best quantizers known in dimensions $n \leqslant 24$.

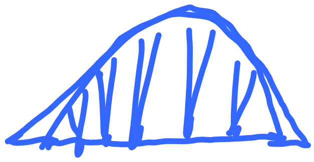
$\Lambda_k^{opt} \longrightarrow$

$G_k \longrightarrow$?

$\mu_k \longrightarrow$



Why Lattices in Communication?

1. a bridge from $n=1$  to $n=\infty$

   = non-asymptotic analysis per dimension

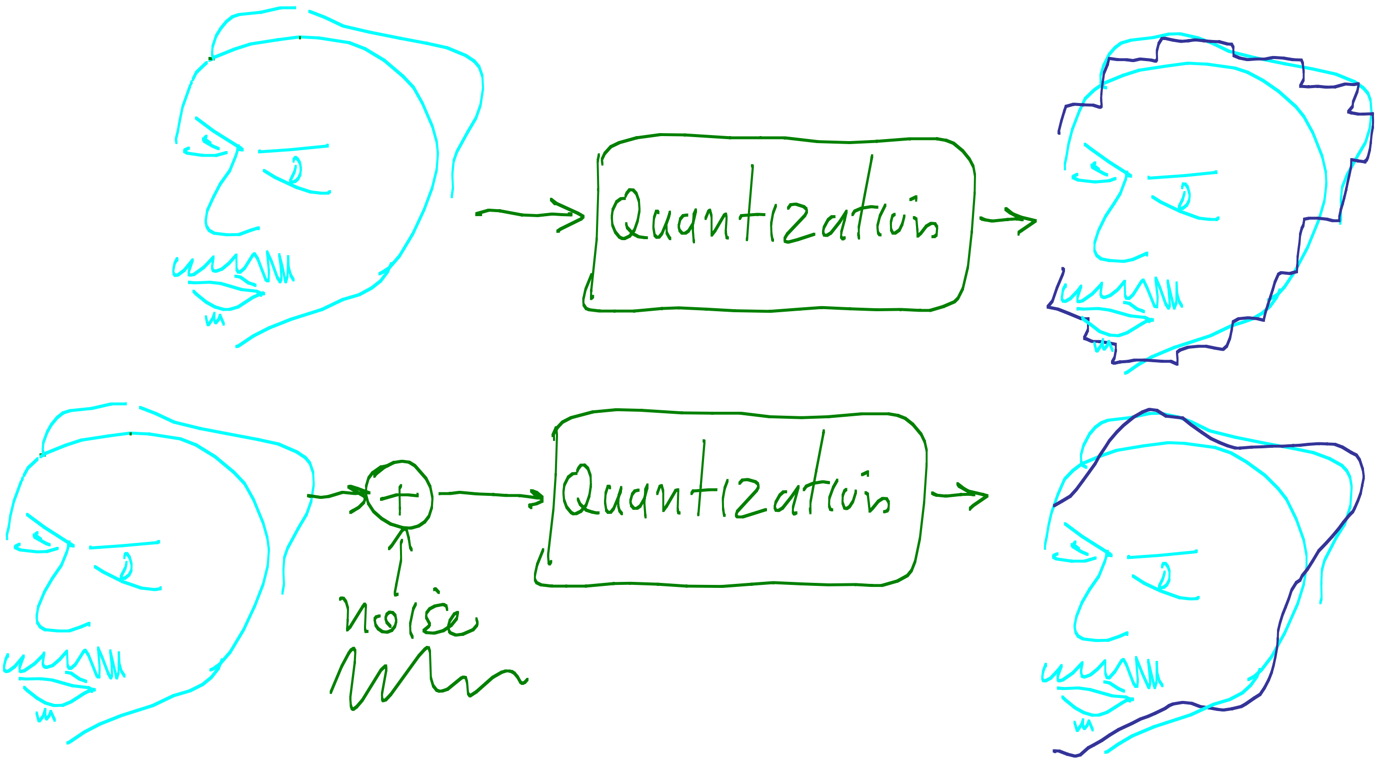
2.

3.

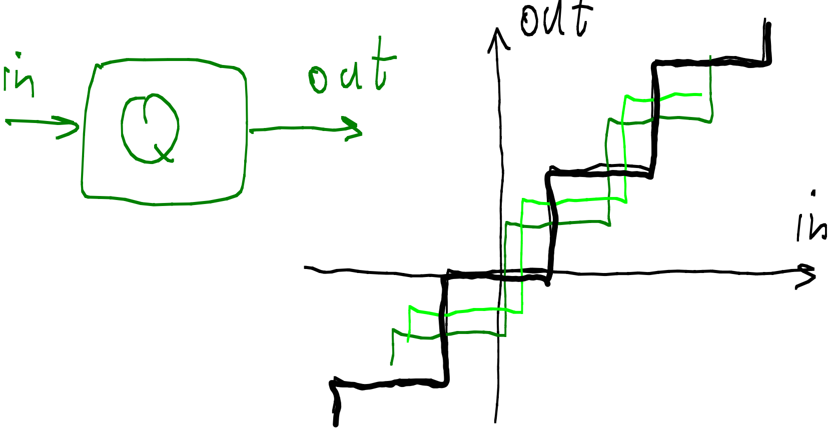
4.

Dithered Quantization

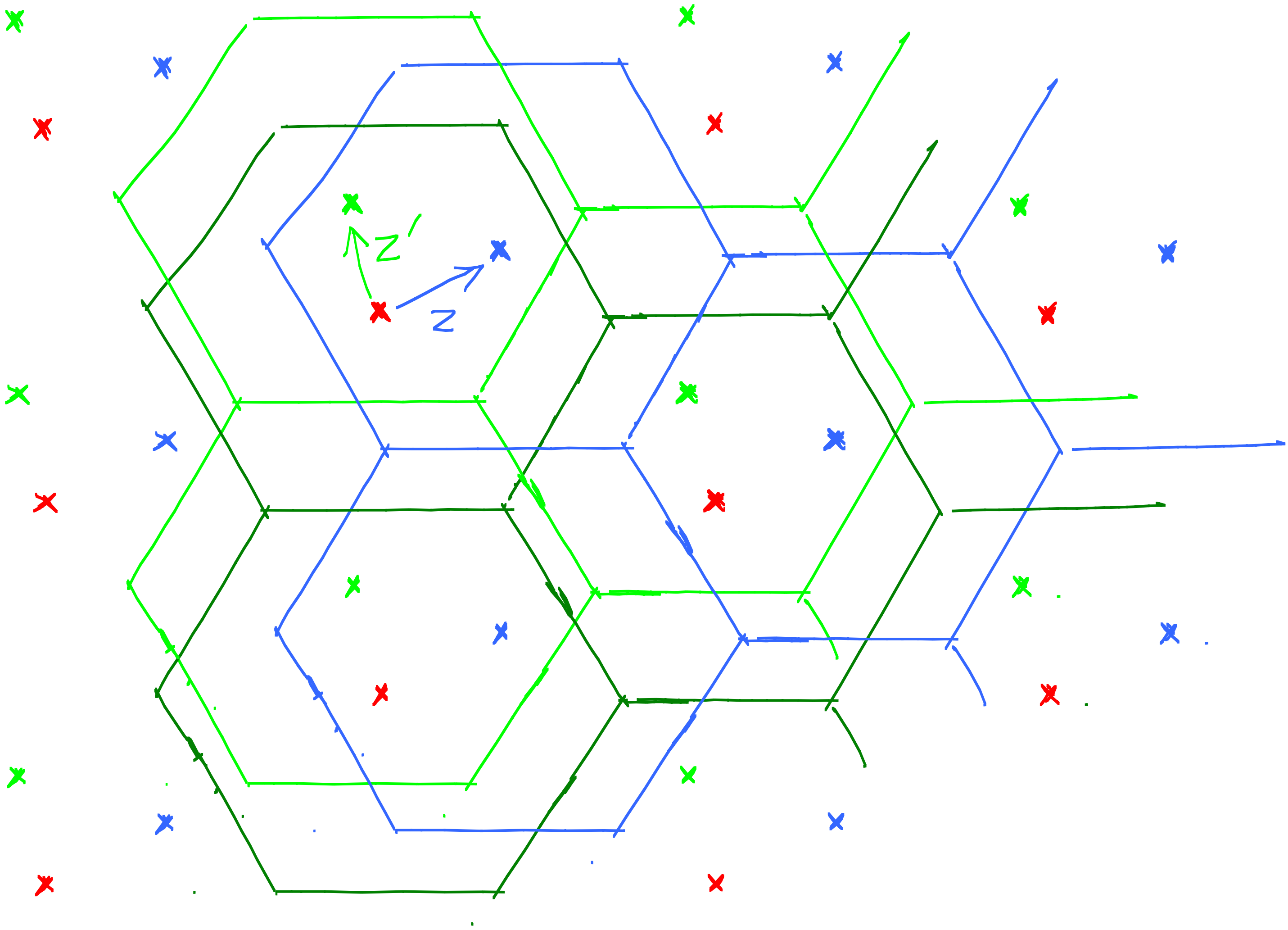
- dither for perceptual reasons:



- dither for analytical reasons:

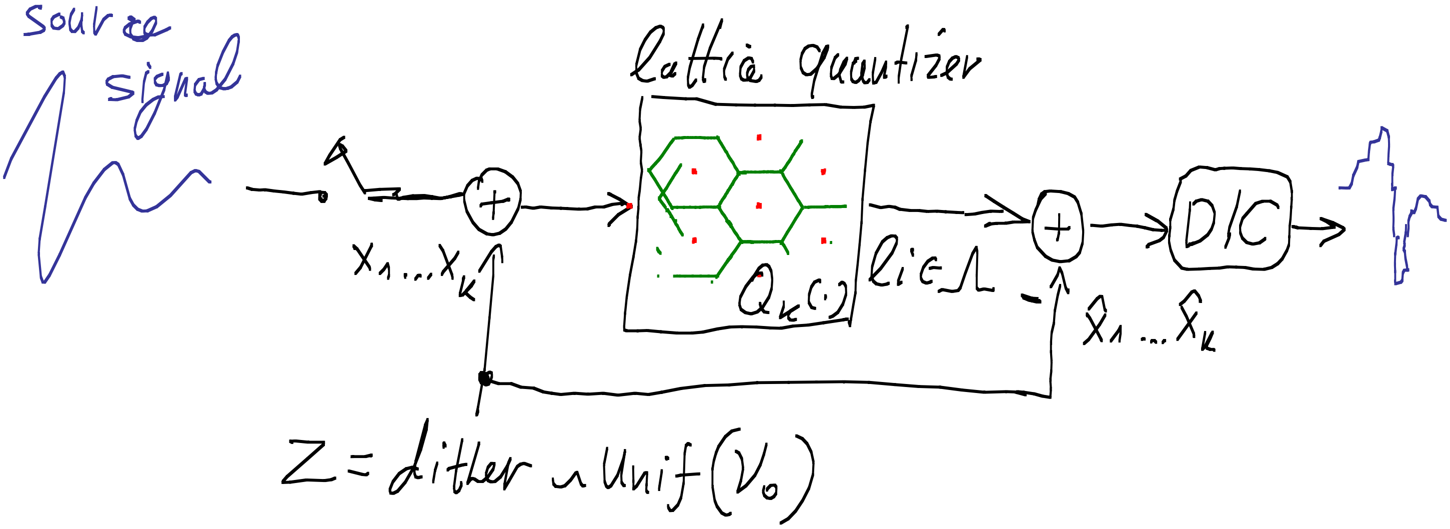


$$Q_\kappa(x+Z)-Z$$



$\Rightarrow$ Random shift of the lattice quantizer

Dithered Quantization Error



- Subtractive Dither (Pseudo Random Noise) $\Rightarrow$

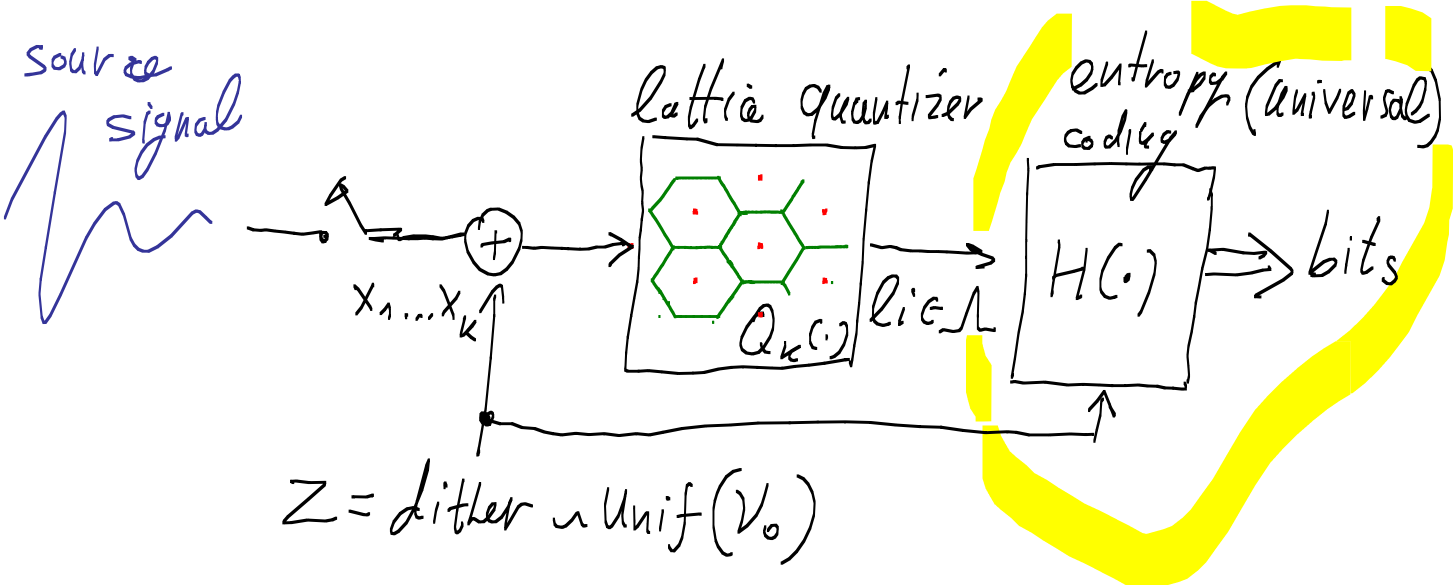
  - error: $Q_k(x+Z) - Z \sim \text{Unif}(-\mathcal{V}_0)$

  - distortion: $E\|Q_k(x+Z) - Z\|^2 = \sigma^2(\Lambda)$

    invariant of x

Entropy Coded Dithered Quantization

[ziv85]



- Rate Redundancy @ High Resolution ($D \to 0$):

$$H(Q_k(X+Z)|Z) - R(D) = \frac{1}{2}\log(2\pi e G(\Lambda))$$

Rate-Distortion Function

$$= \underbrace{D(Z \,\|\, N(0,\sigma^2(\Lambda)))}_{\text{Divergence of dither } Z \text{ from AWGN}}$$

Gersho's Conjecture

The best space-filling-polytope in $\mathbb{R}^k$ satisfies

$$G(\quad) \xrightarrow[k\to\infty]{} \frac{1}{2\pi e}$$

Note that...

(1) $G_k^* \triangleq G(\quad) \xrightarrow[k\to\infty]{} \frac{1}{2\pi e}$

(2) Iso-perimetric inequality:
<u>Ball</u> has the minimum <u>diameter</u> & <u>second moment</u> among all shapes of given <u>volume</u>!

Can "good" **lattice** cells approximate Balls (as $k\to\infty$)?...

Rogers & Minkowski Meet Shannon

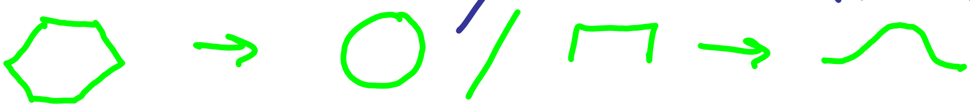
Rogers (1957):

for a sequence of "good" lattices Λ_k^*

$$\frac{V_{cov}}{V_{eff}}(\Lambda_k^*) \longrightarrow 1 \quad \text{as} \quad k \longrightarrow \infty$$

$$\Longrightarrow \quad G(\Lambda_k^*) \longrightarrow G(k\text{-ball}) \xrightarrow[k\to\infty]{} \frac{1}{2\pi e}$$

$\Longrightarrow$ Voronoi cell → Ball / Dither → AWGN !



Minkowski (1904):

for a sequence of "good" lattices Λ_k^*

$$\frac{V_{pack}}{V_{eff}}(\Lambda_k^*) \longrightarrow 1/2 \quad \text{as} \quad k \longrightarrow \infty$$

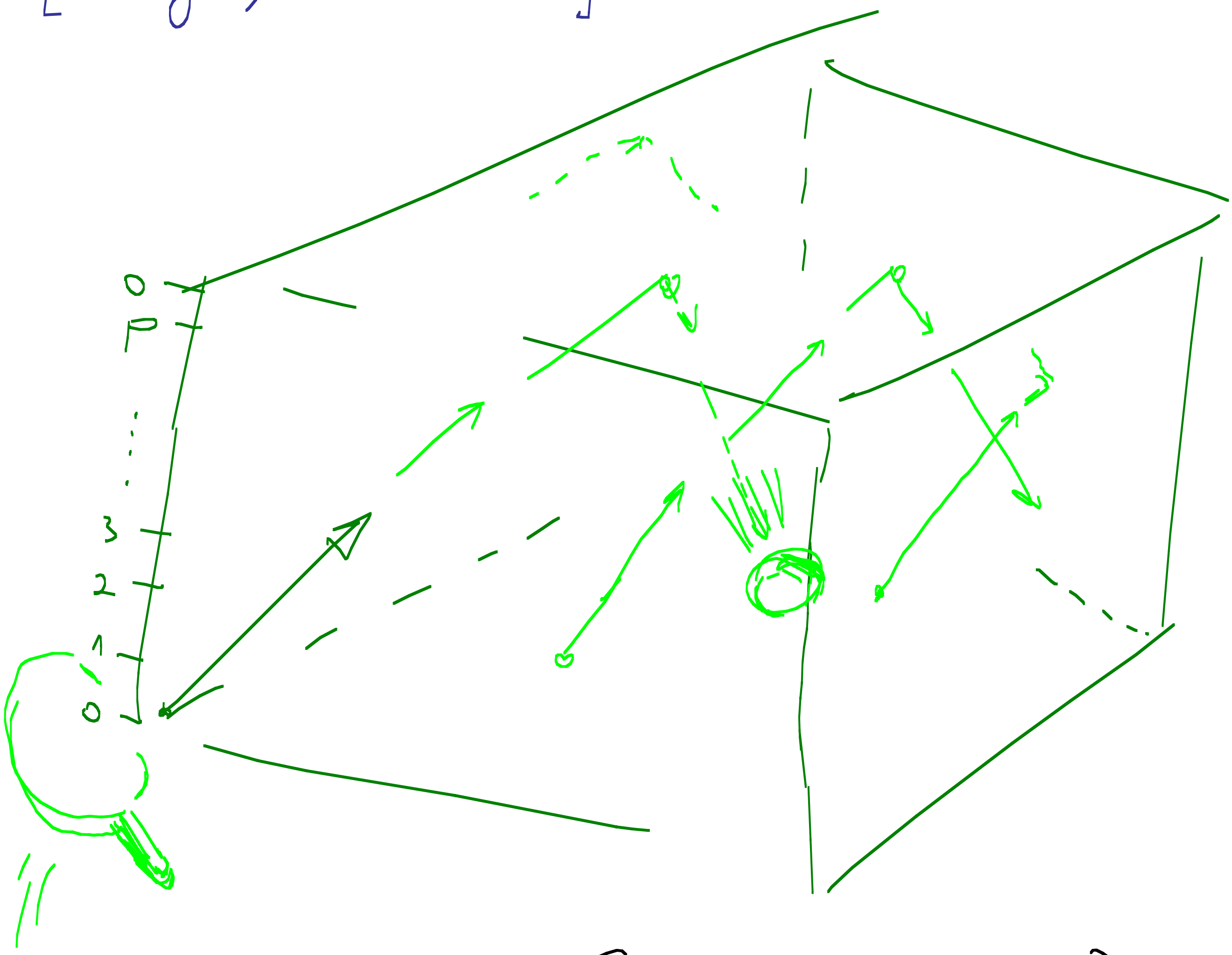
$\Longrightarrow$ $\mu(\Lambda_k^*, P_e)$ is asymptot. at most $4\pi e$

De-Buda $\Longrightarrow$ $\mu(\Lambda_k^*, P_e) \xrightarrow[k\to\infty]{} 2\pi e$

$\Longrightarrow$ Voronoi cell ~ packs AWGN !

How to get Structure (Λ_∞) from Random

Linear q-ary code → random matrix → construction-A
[Loeliger, Erez et al]



$$\Lambda \cap \text{cube} = \{ i \cdot \underline{X} \bmod p \}$$

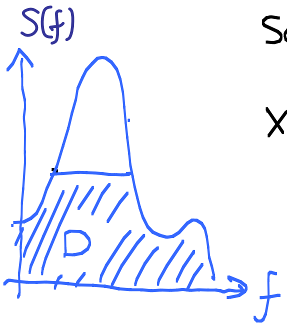
$$\Lambda = \text{construction } A$$

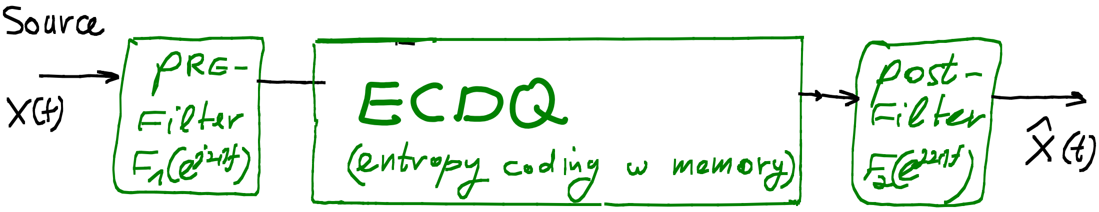
p = prime , $\underset{\sim}{X} \sim \text{unif}\{p \times p \times \cdots \times p\}$

ECDQ Applications

1. pre/post-filtered ECDQ:

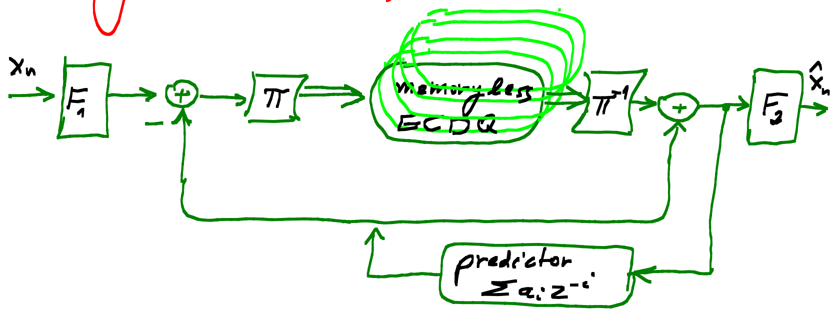
[Zamir - Feder]





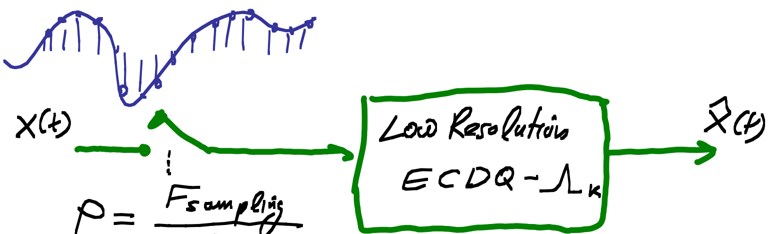
$$\bar{H}(Q_K|Z) = R^*(D) + \frac{1}{2}\log(2\pi e G_K)$$

2. Predictive Coding (DPCM):



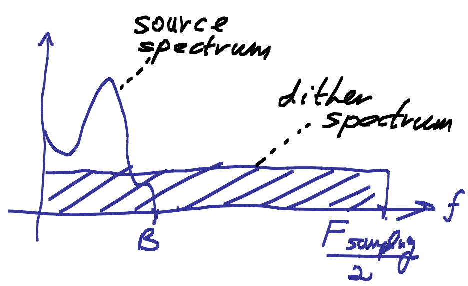
[Z-Kochman-Erez]

3. Oversampled ECDQ:



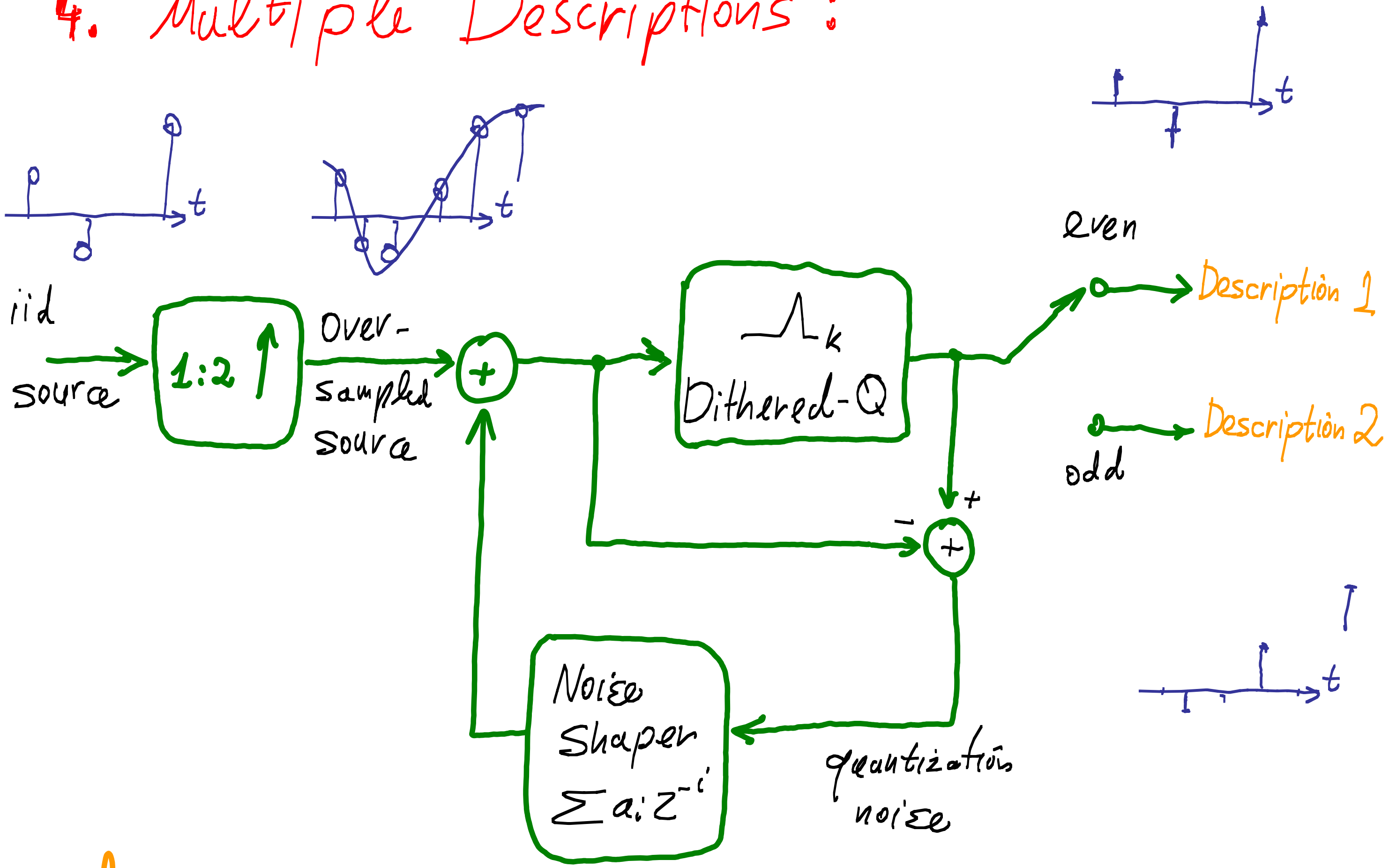
$$\rho = \frac{F_{sampling}}{2B}$$

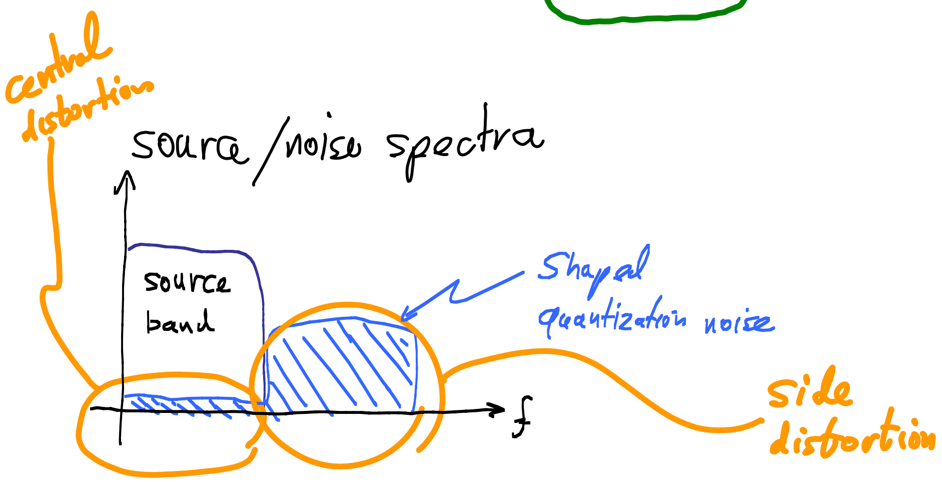
= oversampling ratio



ECDQ Applications (cont.)

4. Multiple Descriptions:



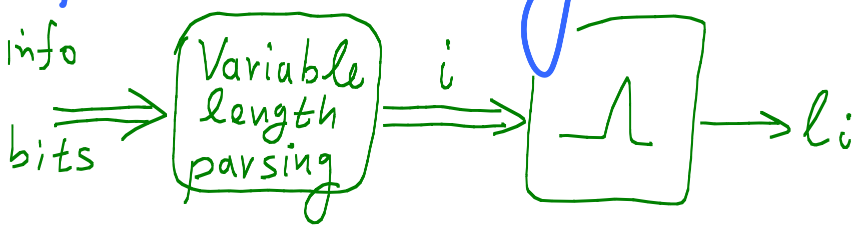


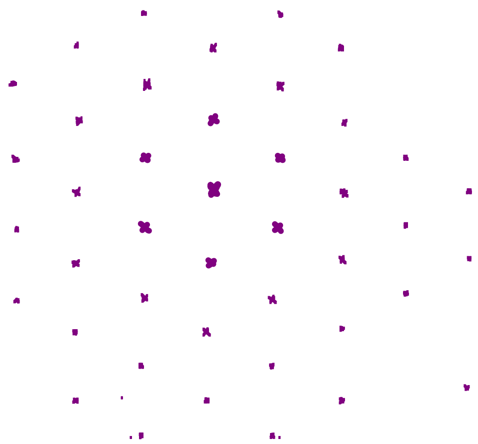
[Østergaard - Z]

The Channel-Dual of ECDQ

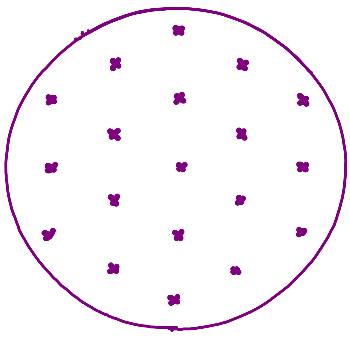
1. Unbounded lattice: capacity per unit volume [Poltyrev 93]

2. Probabilistic Shaping: [Gallager]

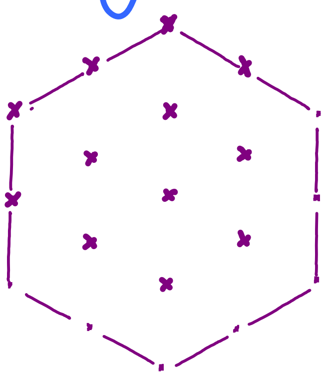




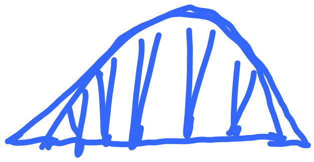
3. Deterministic Shaping - Spherical: [De Buda 89]



4. Deterministic Shaping - Voronoi Code: [Sloane, Forney 89]



Why Lattices in Communication?

1. a bridge from $n=1$  to $n=\infty$

   = non-asymptotic analysis per dimension

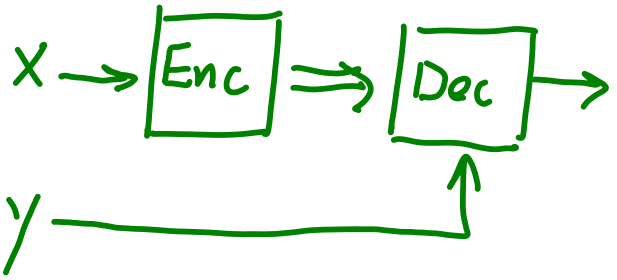
2. Algebraic (low complexity) Binning

   = structured coding schemes for networks

3.

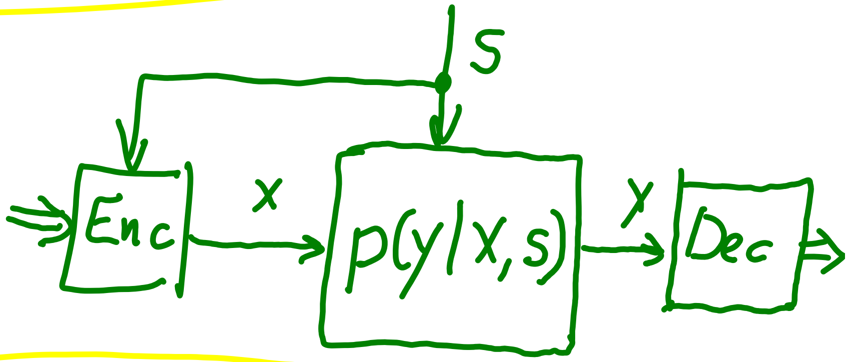
4.

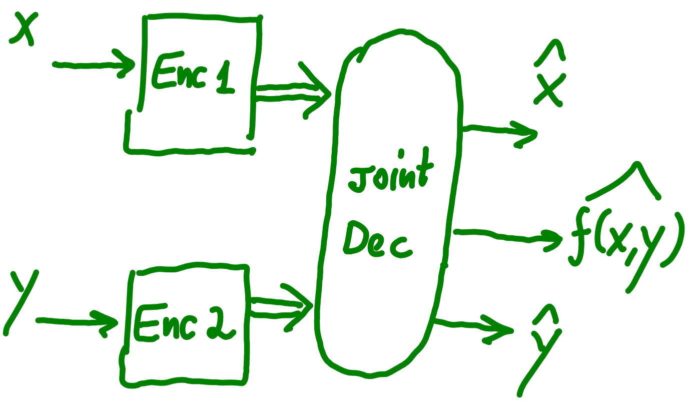
Lattices in Multi-Terminal Problems



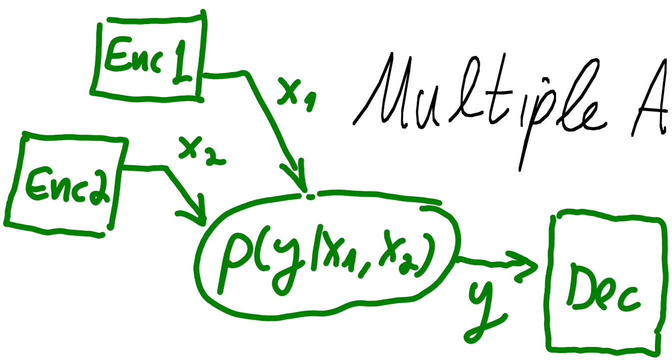
Source coding with Side Information

Channel Coding with Side Information

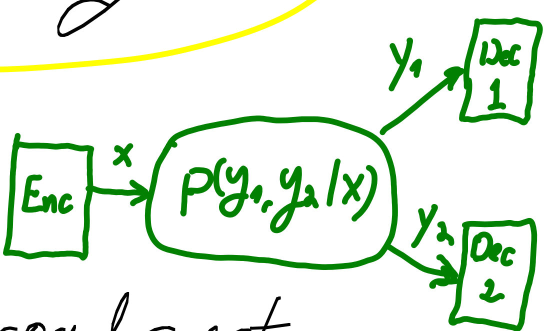




Multi-terminal Source coding



Multiple Access



& Broadcast Channels

Nested Lattices

$$\Lambda_2 \subset \Lambda_1 \Rightarrow \underline{\underline{G_2}} = \underline{\underline{G_1}} \cdot \underline{\underline{J}}$$

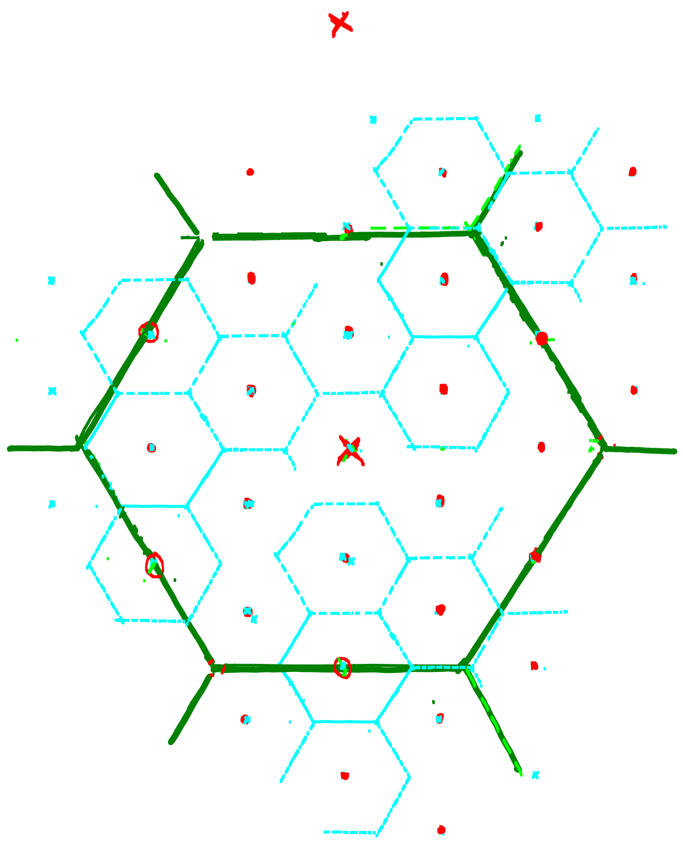
course lattice

fine lattice

integer matrix

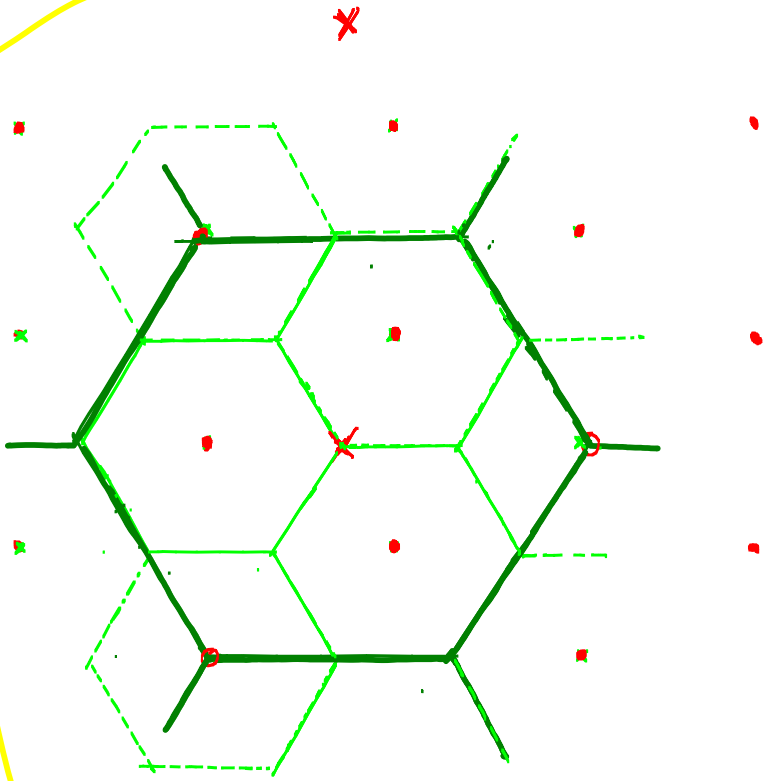
$$\text{Nesting Ratio} = \left(\frac{V_2}{V_1}\right)^{1/k} = \left|\det(J)\right|^{1/k}$$

4:1



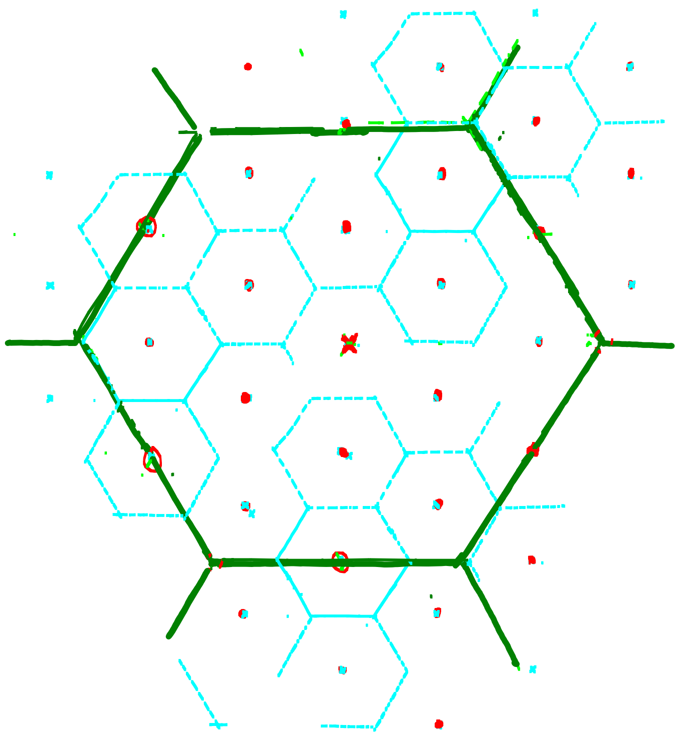
Not necessarily "Self Similar"!

$\Rightarrow V_{o2} \not\propto V_{o1}$



Relatively Periodic (non nested)

Nested & Self Similar



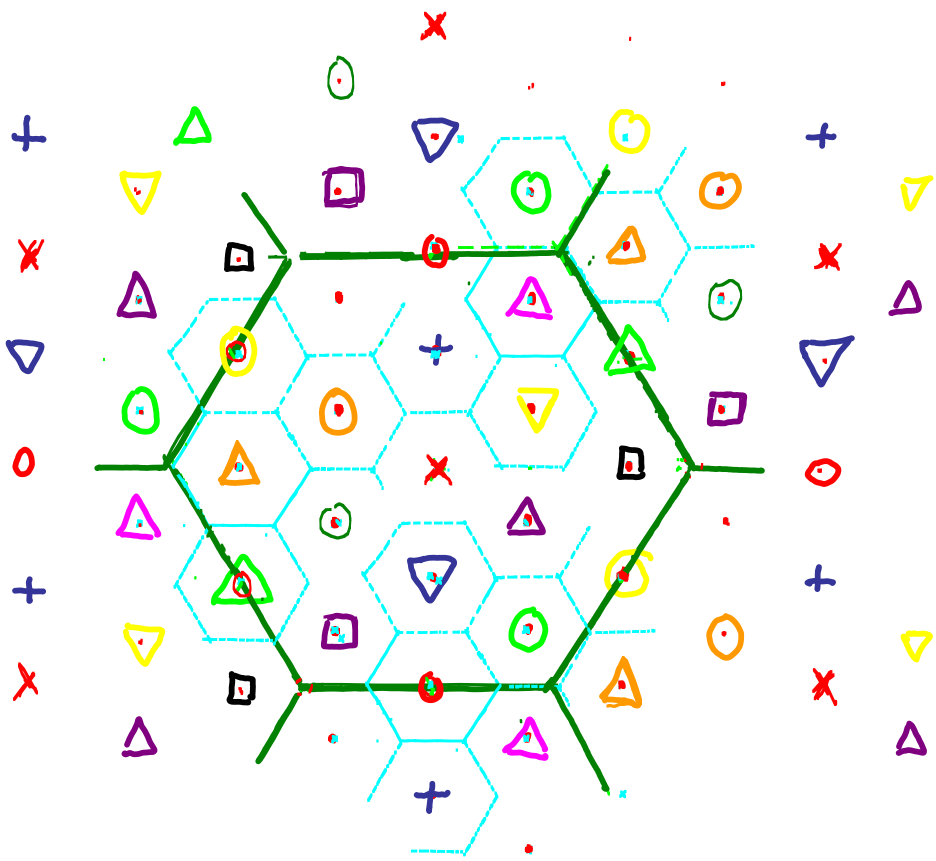
Nested Lattices

$$\Lambda_2 \subset \Lambda_1 \Rightarrow \underline{\underline{G}}_2 = \underline{\underline{G}}_1 \cdot \underline{\underline{J}}$$

$$\boxed{\text{Relative Cosets} = \Lambda_2/\Lambda_1}$$

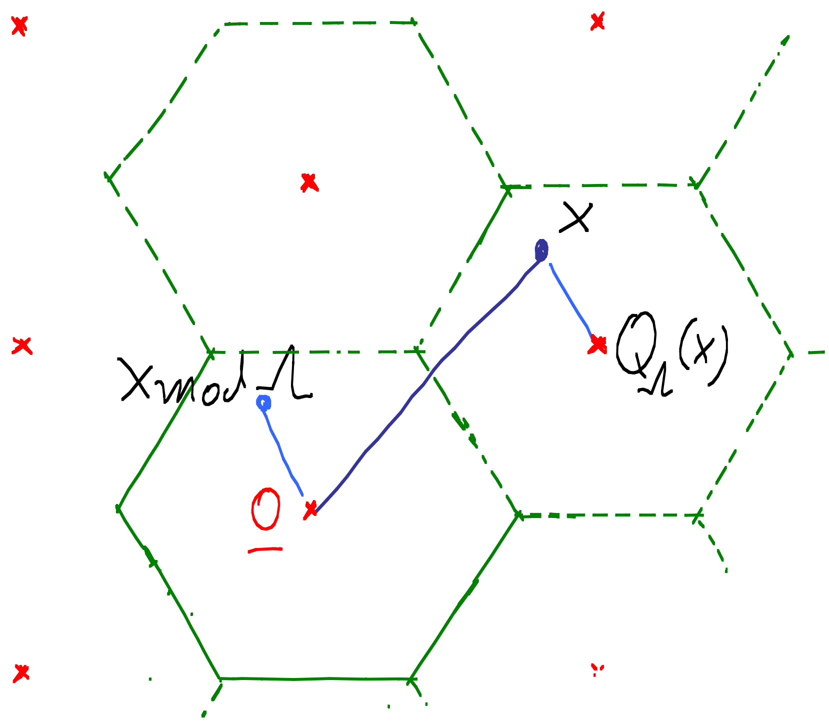
$$\text{Coset} \triangleq \ell_1 + \Lambda_2, \quad \text{for some } \ell_1 \in \Lambda_1$$

$$|\Lambda_2/\Lambda_1| = V_2/V_1 = |\det(J)|$$

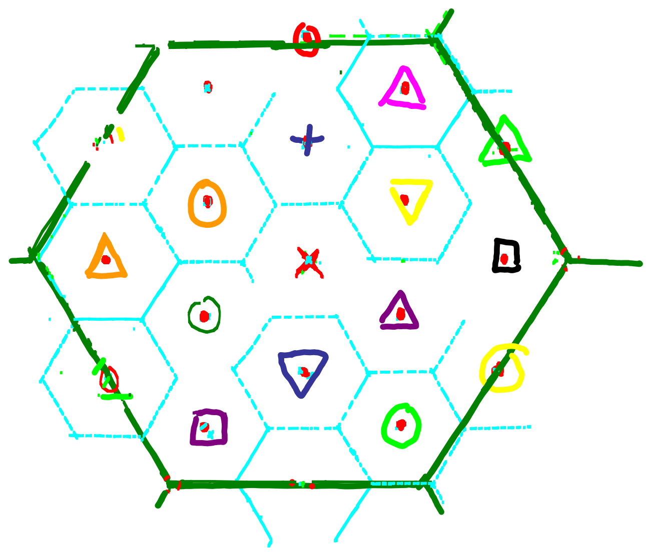


Modulo-Lattice Arithmetic

$$x \bmod \Lambda \triangleq x - Q_\Lambda(x)$$

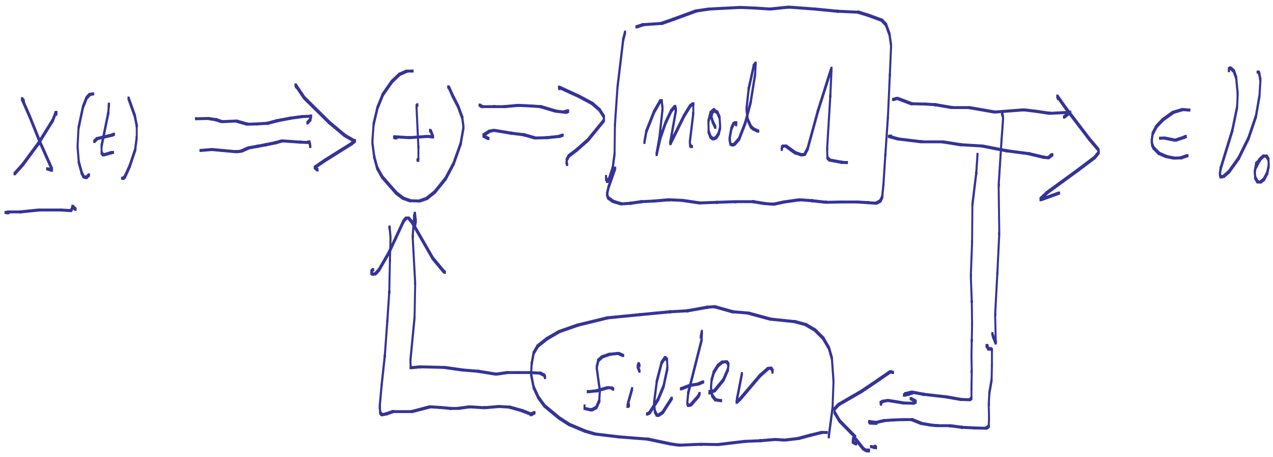


Coset Leader $= \lambda_1 \bmod \Lambda_2\,,\quad$ for $\lambda_1 \in$ coset

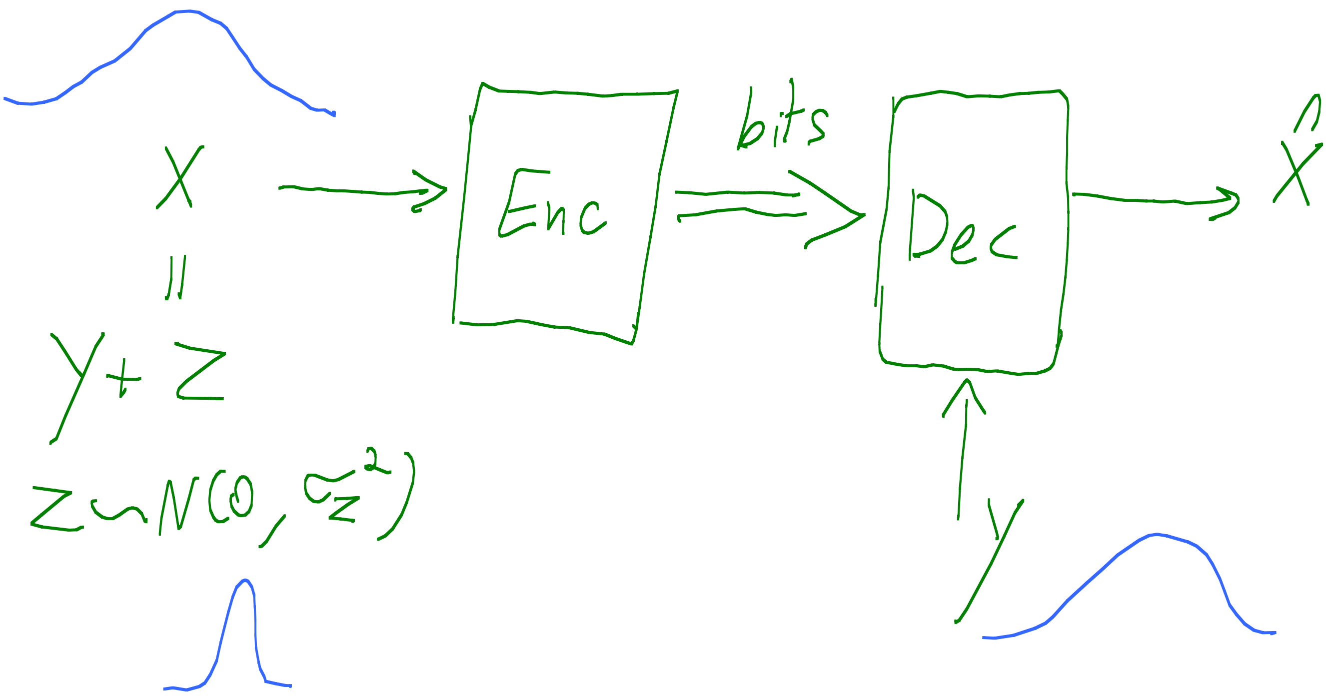


Modulo-Lattice Arithmetic

$$x \pm y \bmod \Lambda \in \mathcal{V}_0$$



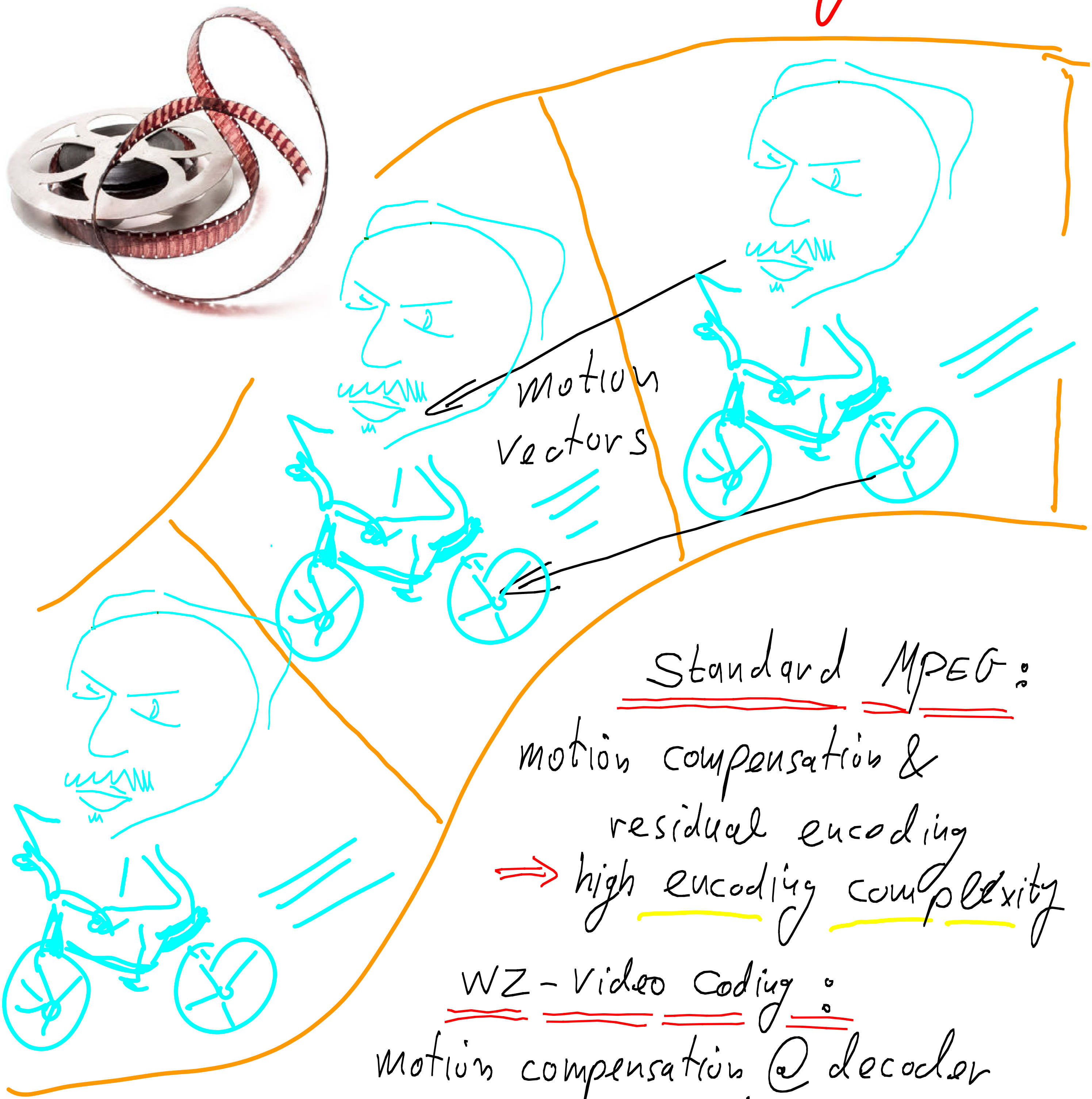
The Wyner-Ziv Problem
(Source coding with S.I. @ Decoder)



$$R^{WZ}_{X|Y}(D) = R_Z(D) = \frac{1}{2}\log\left(\frac{\sigma_Z^2}{D}\right) \ \frac{\text{bit}}{\text{source sample}}$$

Wyner-Ziv 1976
Wyner 1978

Wyner-Ziv Video Coding



Standard MPEG:

motion compensation &
residual encoding

⇒ high encoding complexity

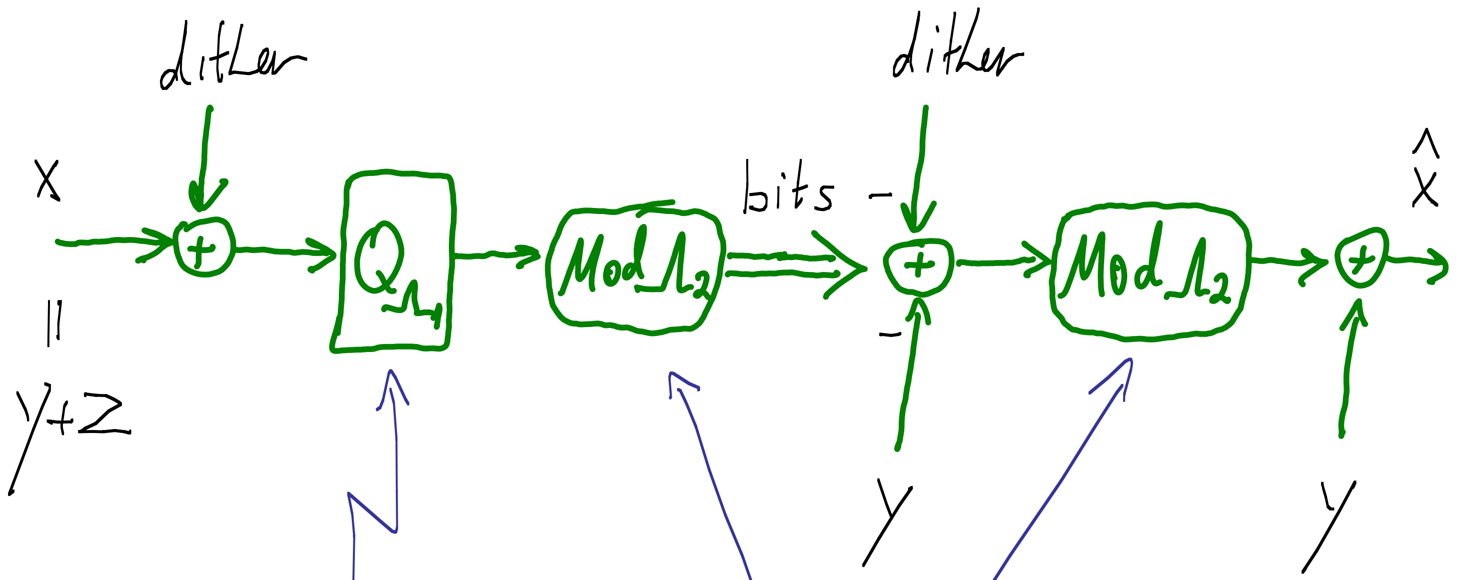
WZ-Video Coding:

motion compensation @ decoder

⇒ encoding = simple / deoding = complex

Lattice Wyner-Ziv Coding

[Z & Shamai Verdu]

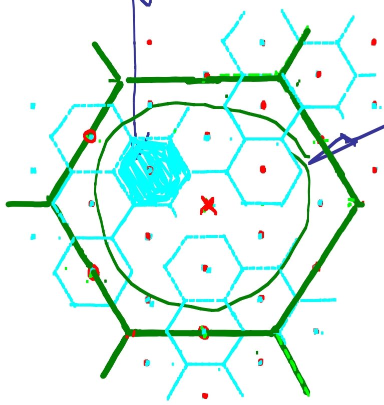


Good quantizer for desired distortion:

$\sigma^2(\Lambda_1) = D$

Good channel code for the noise Z:

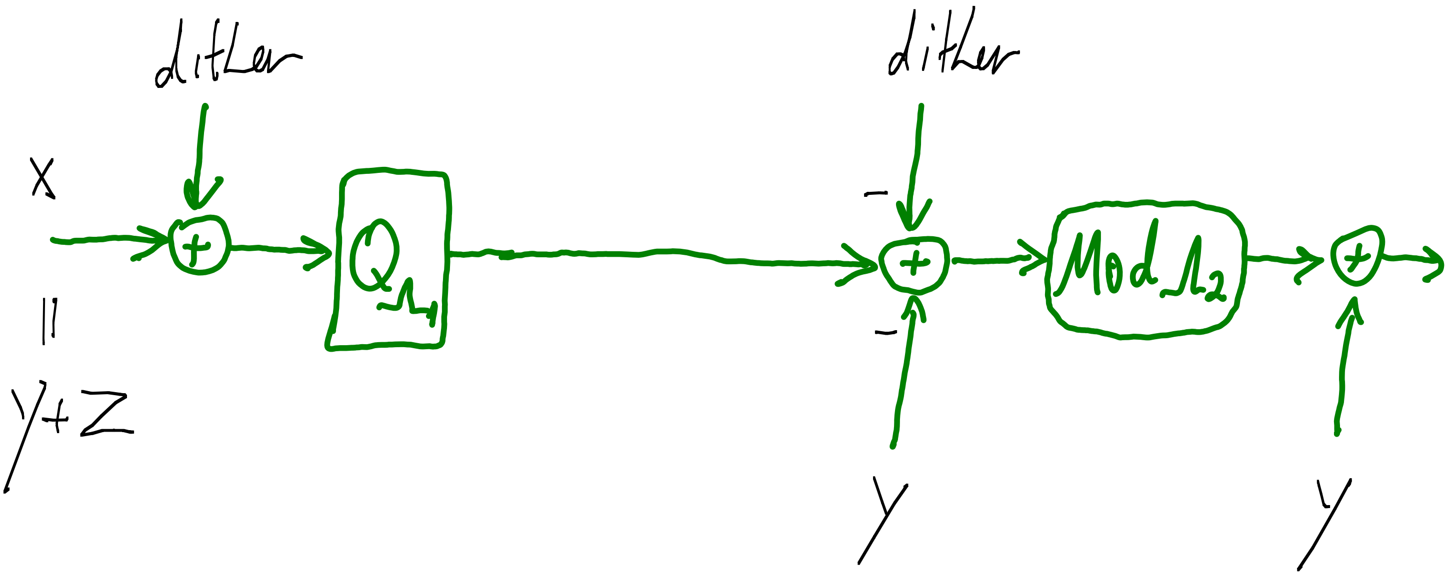
$P_e(\Lambda_2, \sigma_Z^2) < \varepsilon$



Lattice Wyner-Ziv Coding

$$(A \bmod \Lambda + B) \bmod \Lambda = (A+B) \bmod \Lambda$$

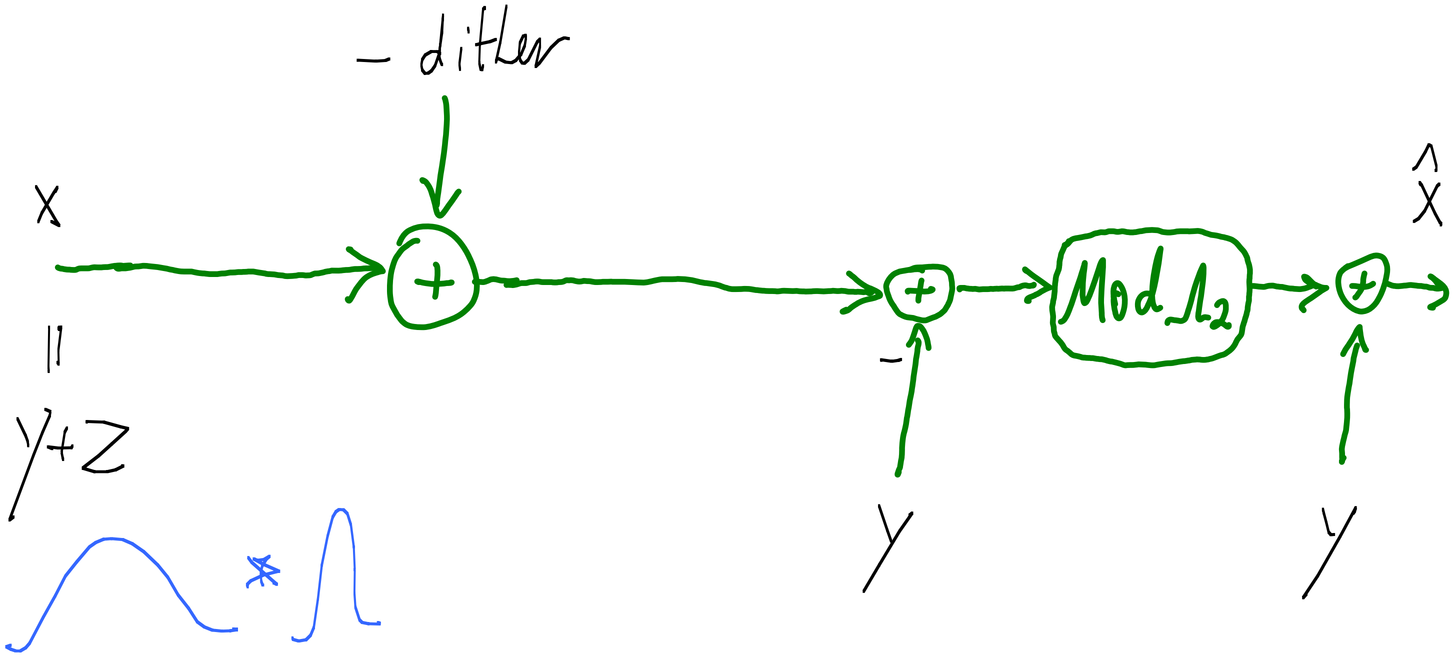
$\Longrightarrow$



Lattice Wyner-Ziv Coding

dithered quantization $\equiv$ additive noise

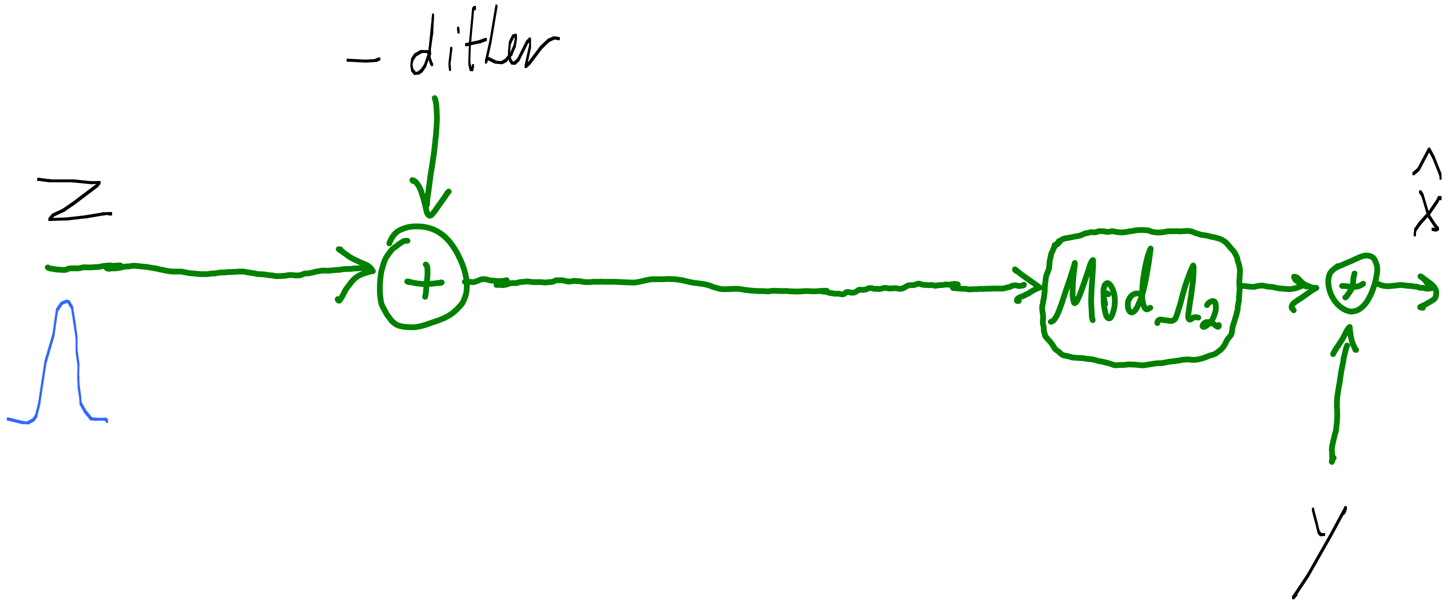
$\Rightarrow$



Lattice Wyner-Ziv Coding

dithered quantization $\equiv$ additive noise

$\Longrightarrow$

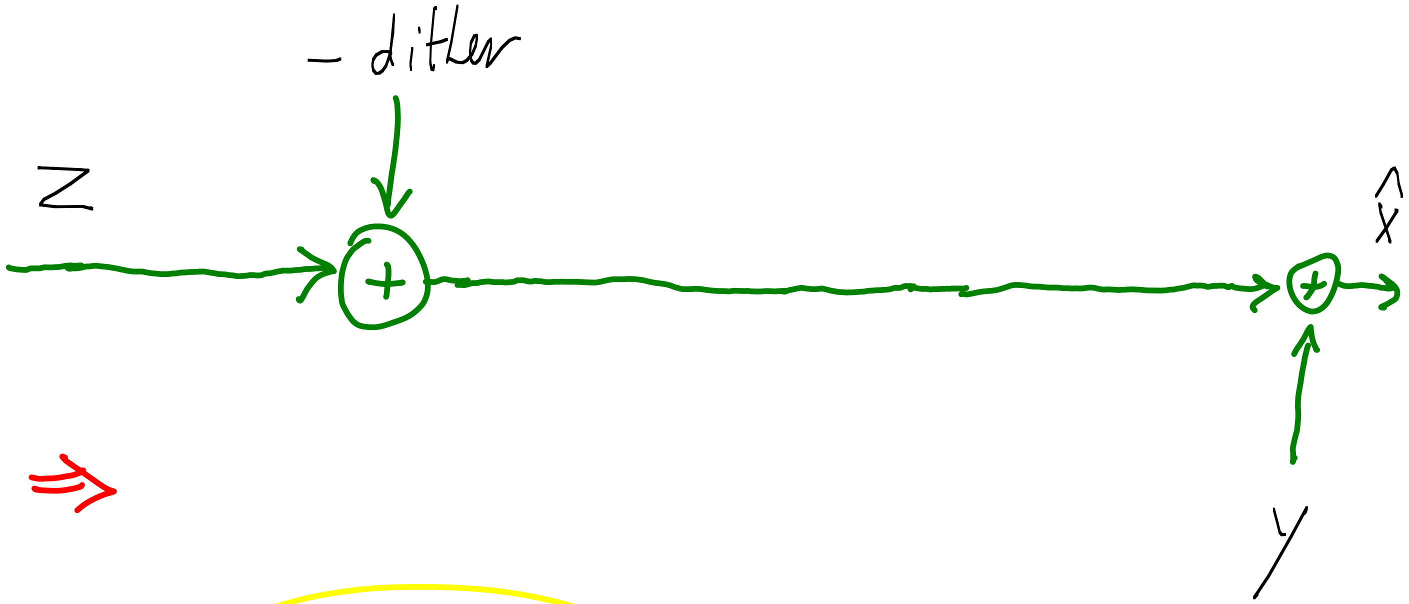


Lattice Wyner-Ziv Coding

Λ_2 = good channel code for $Z \sim \mathcal{N}(0, \sigma_Z^2)$.

$D \ll \sigma_Z^2$.

$\Rightarrow$ with prob. $> 1-\varepsilon$,



$\Rightarrow$

$\hat{X} = X - \text{dither}$, w.p. $> 1-\varepsilon$

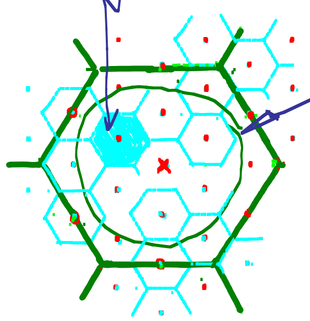
$\Rightarrow$ distortion $= \sigma^2(\Lambda_1) = D$

Lattice Wyner-Ziv Coding

$\sigma^2(\Lambda_1) = D$

$P_e(\Lambda_2, \sigma_Z^2) < \varepsilon$

Nesting Ratio:



$\Rightarrow$

Rate $= \frac{1}{k} \log\left(\frac{V_2}{V_1}\right)$ bit/sample

$$= \underbrace{\frac{1}{2}\log\left(\frac{\sigma_Z^2}{D}\right)}_{R_Z(D)} + \underbrace{\frac{1}{2}\log\left(G(\Lambda_1)\cdot \mu(\Lambda_2,\varepsilon)\right)}_{\text{Redundancy} \to 0}$$

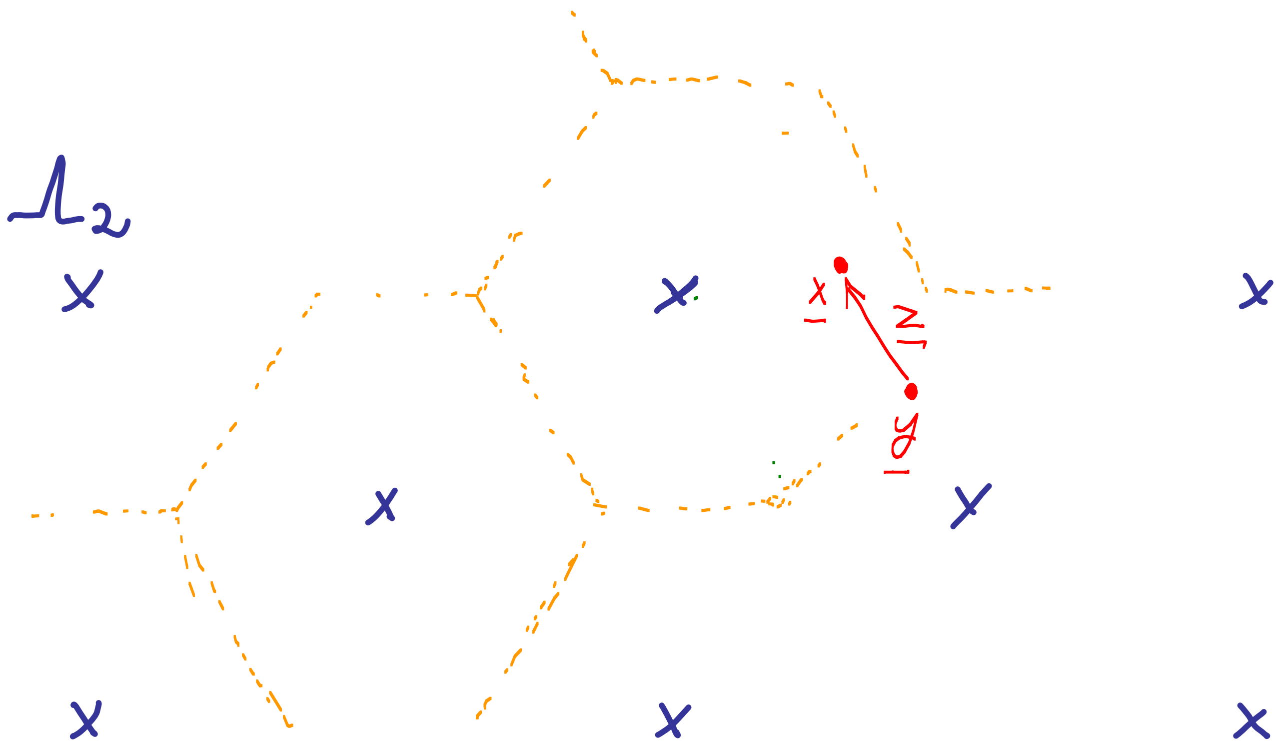
NSM (Λ_1)

VNR (Λ_2)

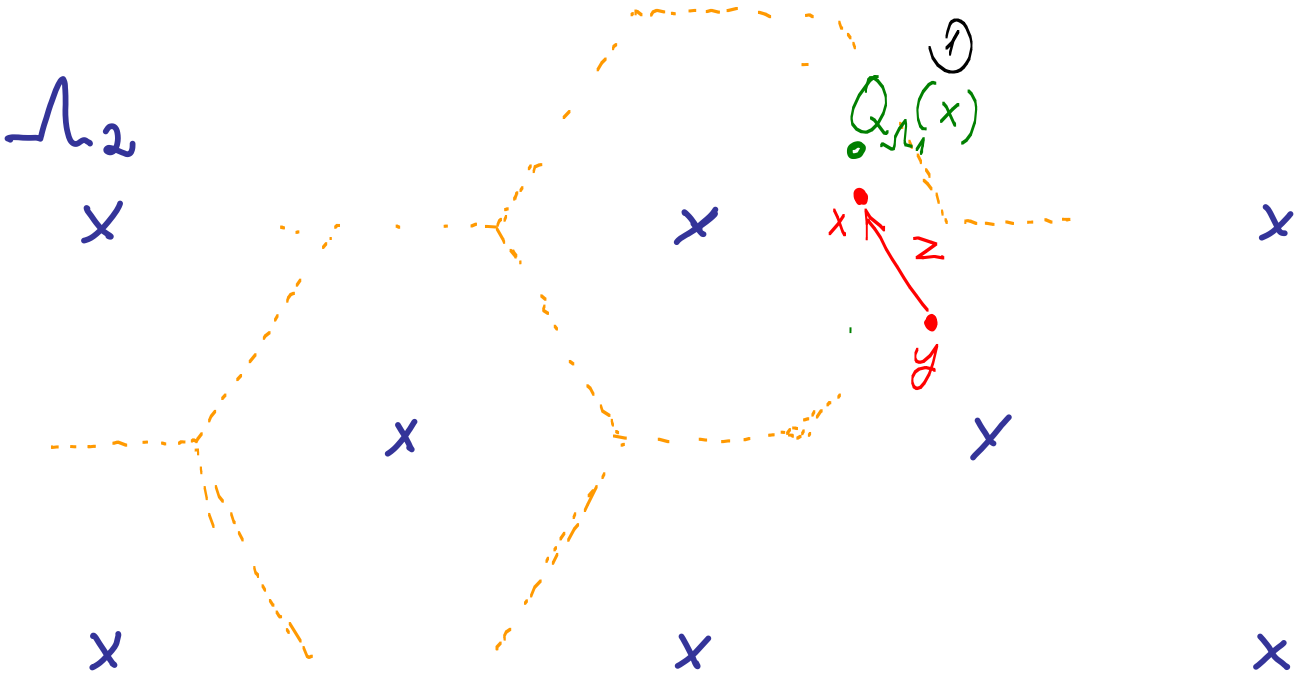
$k \to \infty$

for good lattices ....

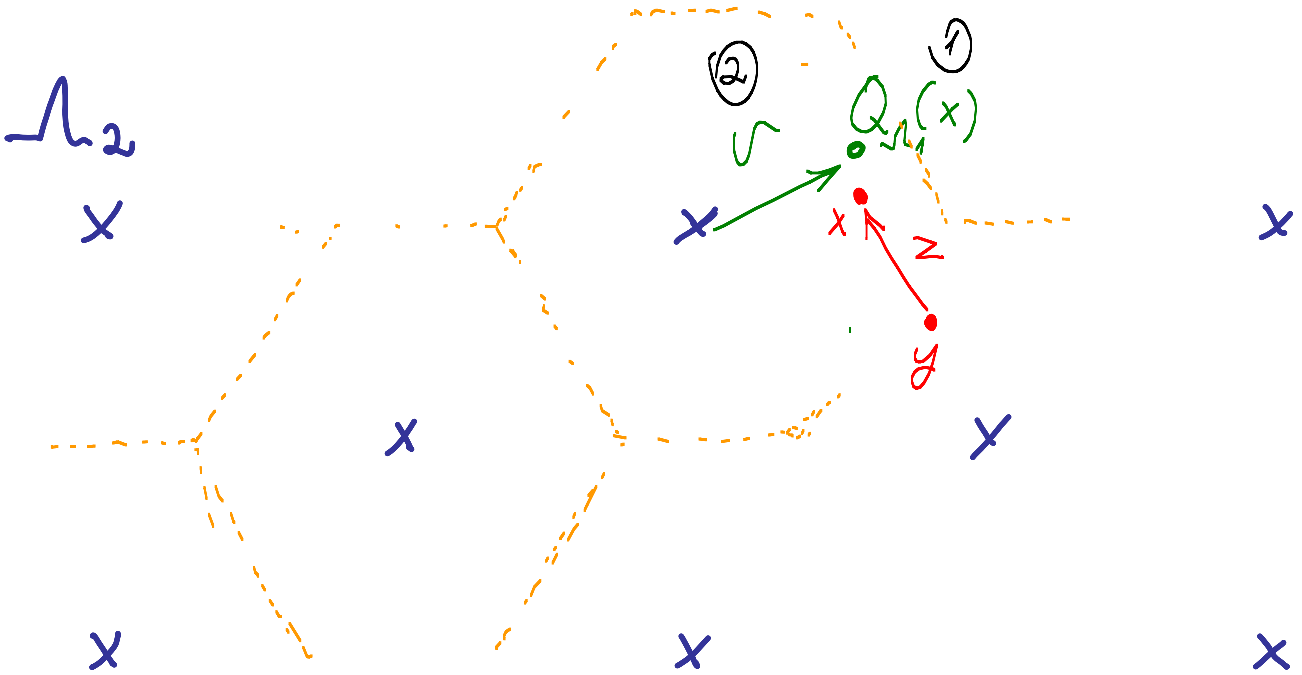
Geometric Picture in Signal Space



Geometric Picture in Signal Space

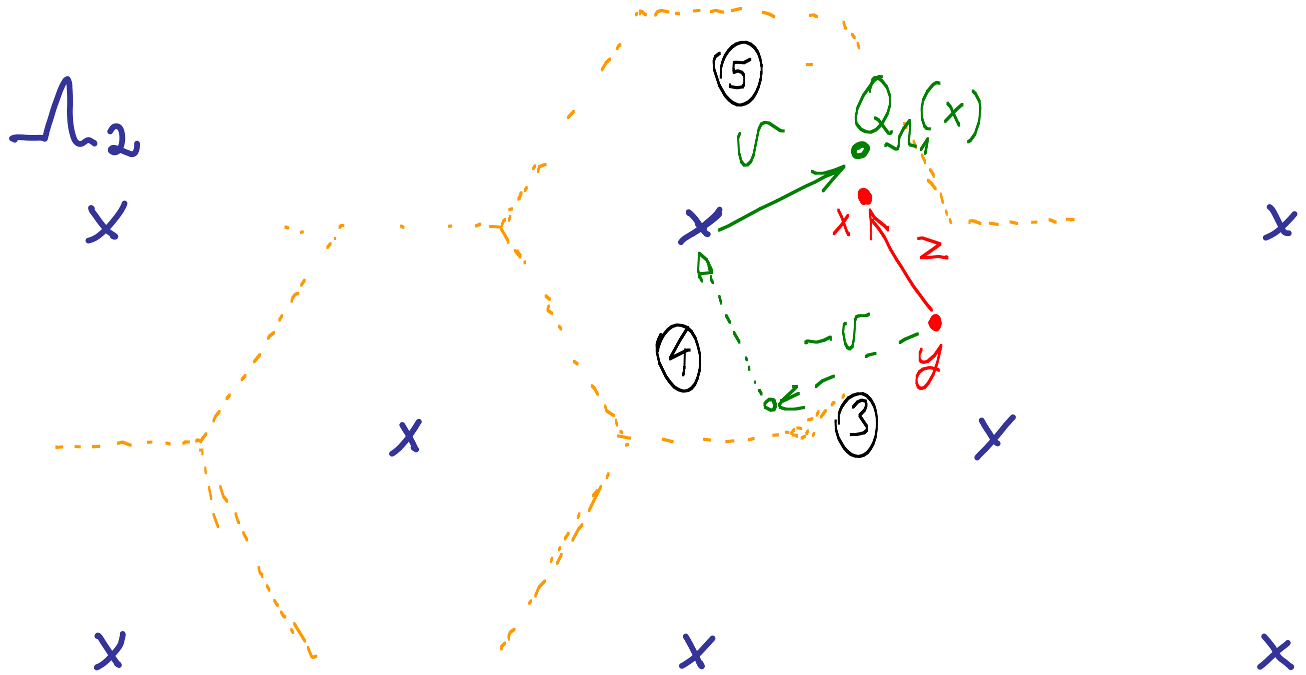


Geometric Picture in Signal Space

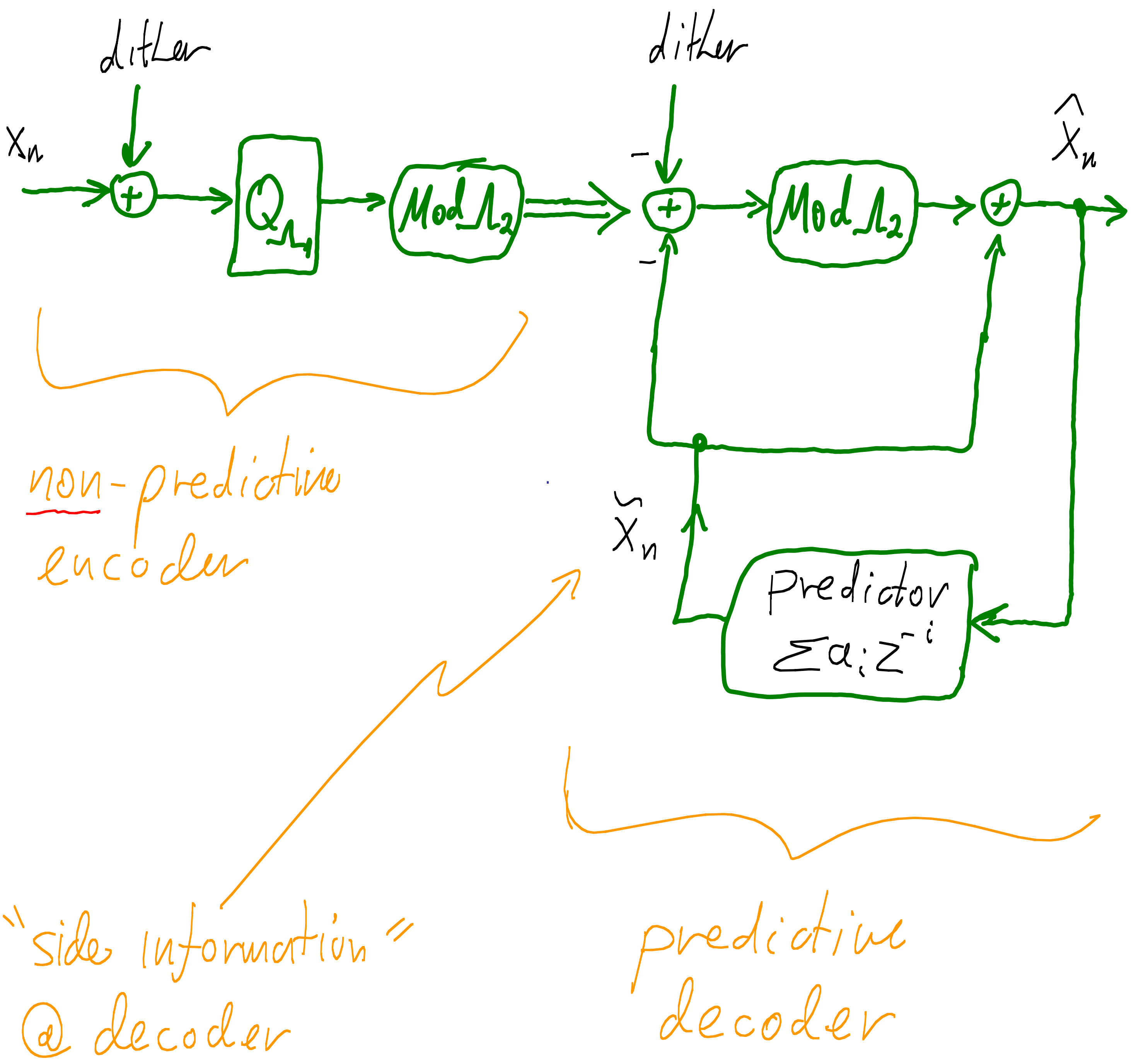


v = relative coset ("syndrome")

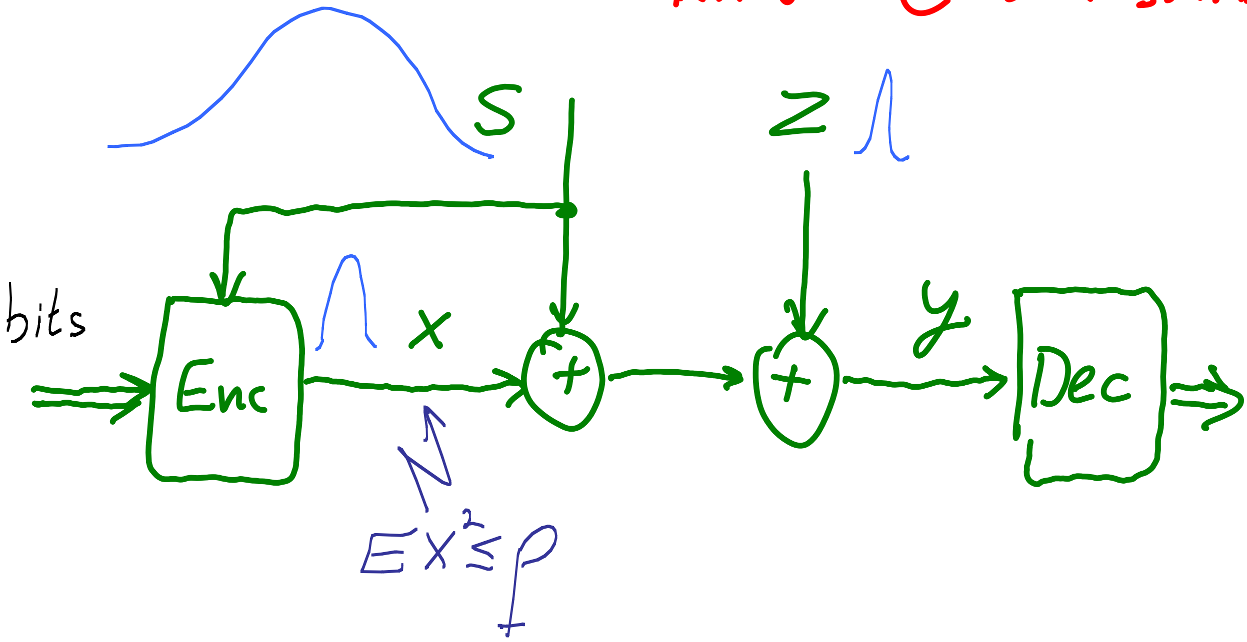
Geometric Picture in Signal Space



Wyner-Ziv - D.P.C.M.



"Writing on Dirty Paper"
(channel coding with Interference known @ transmitter)

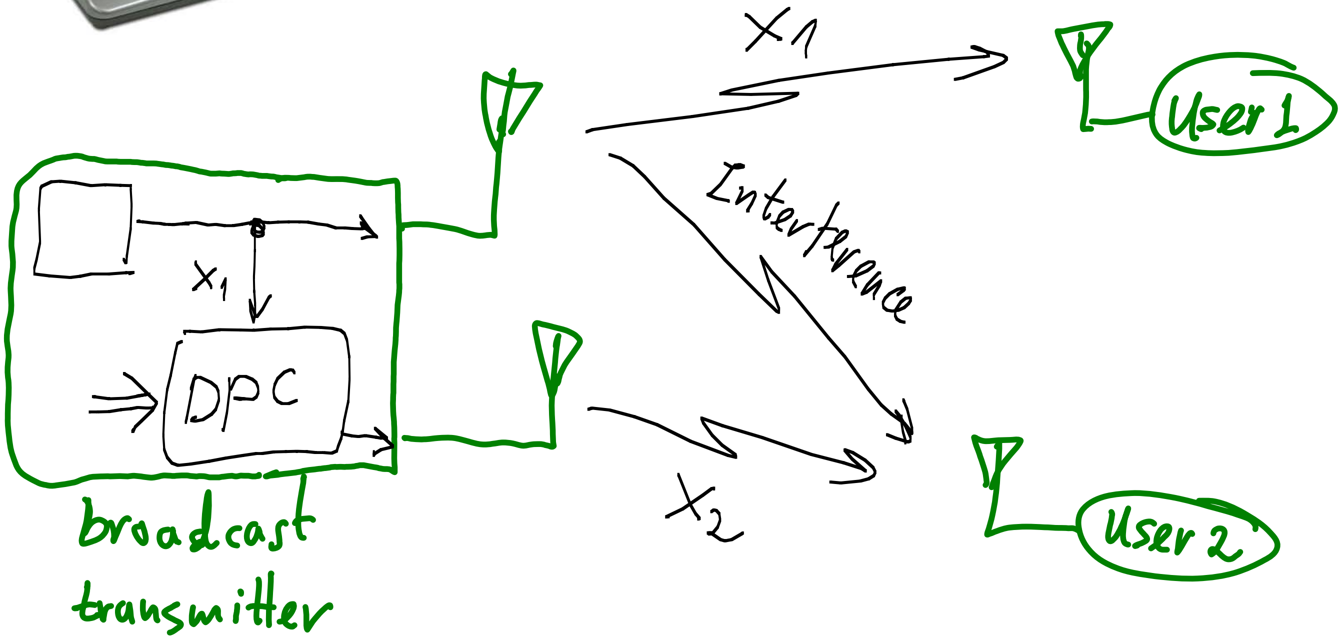


$$C_{SI@T_x} = \frac{1}{2}\log\left(1+\frac{P}{\sigma_z^2}\right) = C_{AWGN}$$

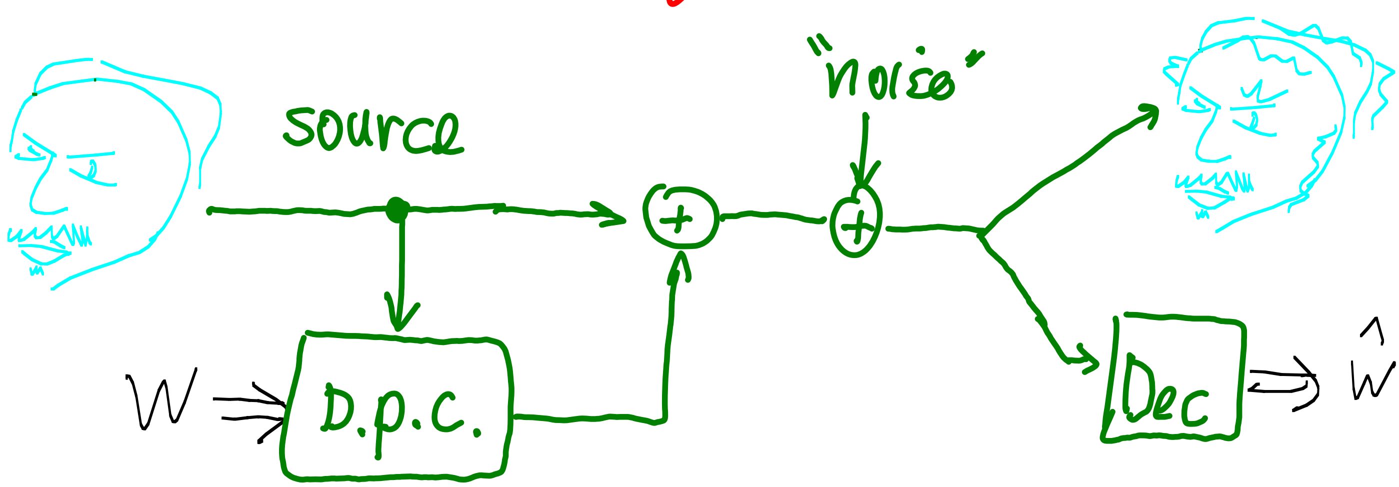
Gelfand-Pinsker 1980
Costa 1983

Surprising: interference cancellation with no power penalty ?

MIMO - Broadcast using D.P.C

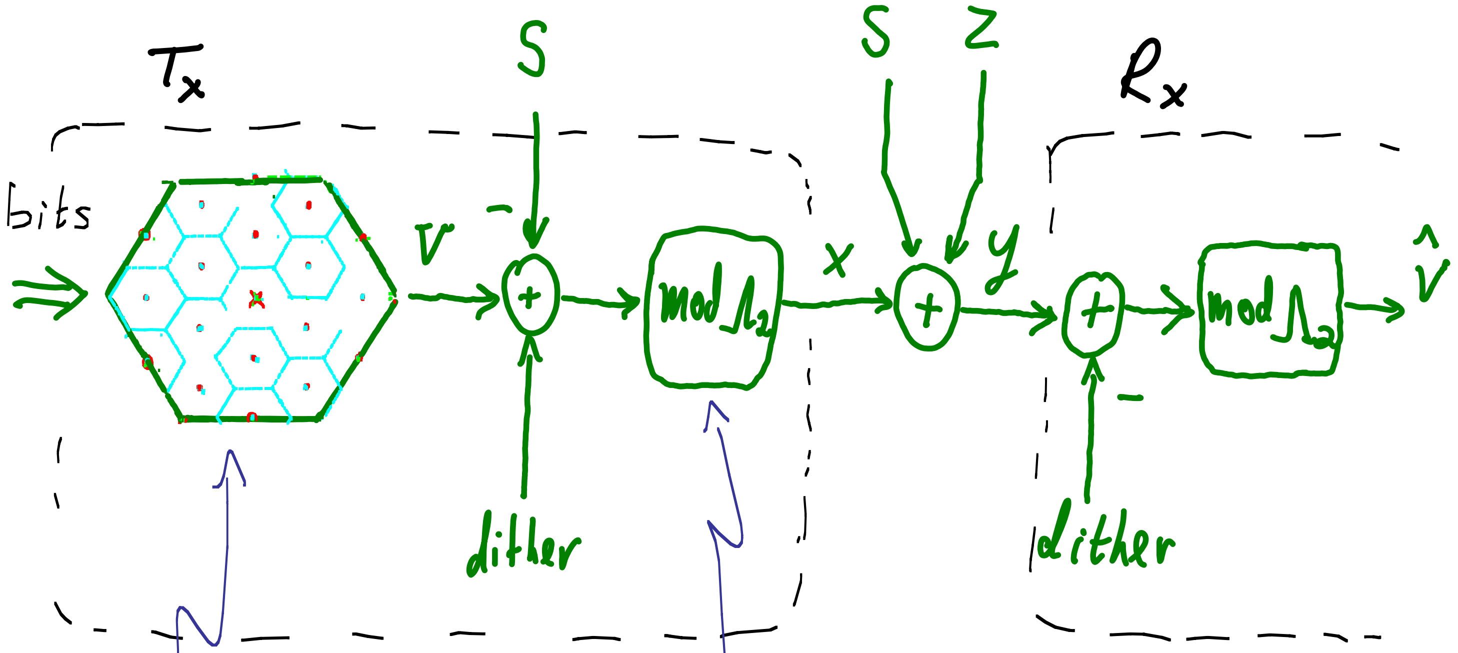


Information Embedding ("Watermarking")



Lattice Dirty Paper Coding

[Tomlinson - Harashima / Erez - Shamai - Zamir]

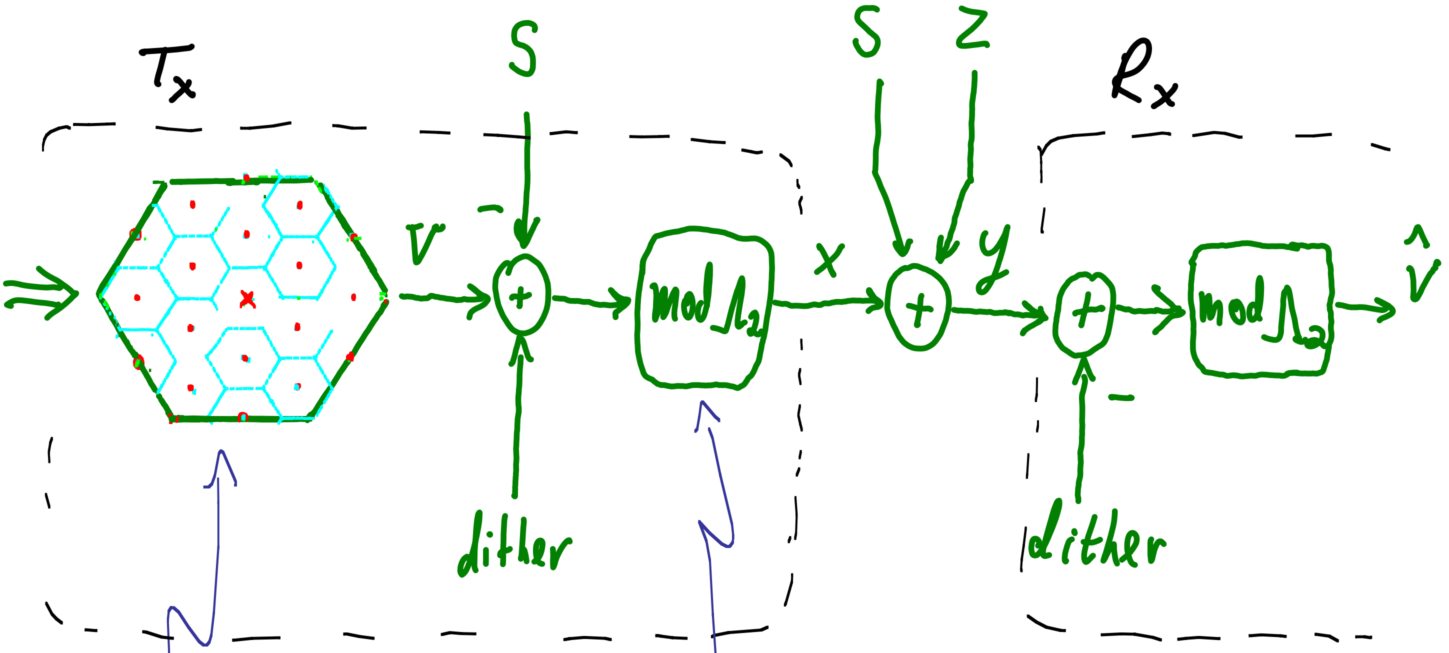


Λ_1 / Λ_2
Voronoi
Constellation

$\Lambda_2 =$ Good quantizer
$\sigma^2(\Lambda_2) = P$

$\Lambda_1 =$ good channel
code for $N(0, \sigma_Z^2)$

Lattice Dirty Paper Coding



Λ_1/Λ_2
Voronoi
Constellation

Λ_1 = good channel code for $N(0,\sigma_Z^2)$

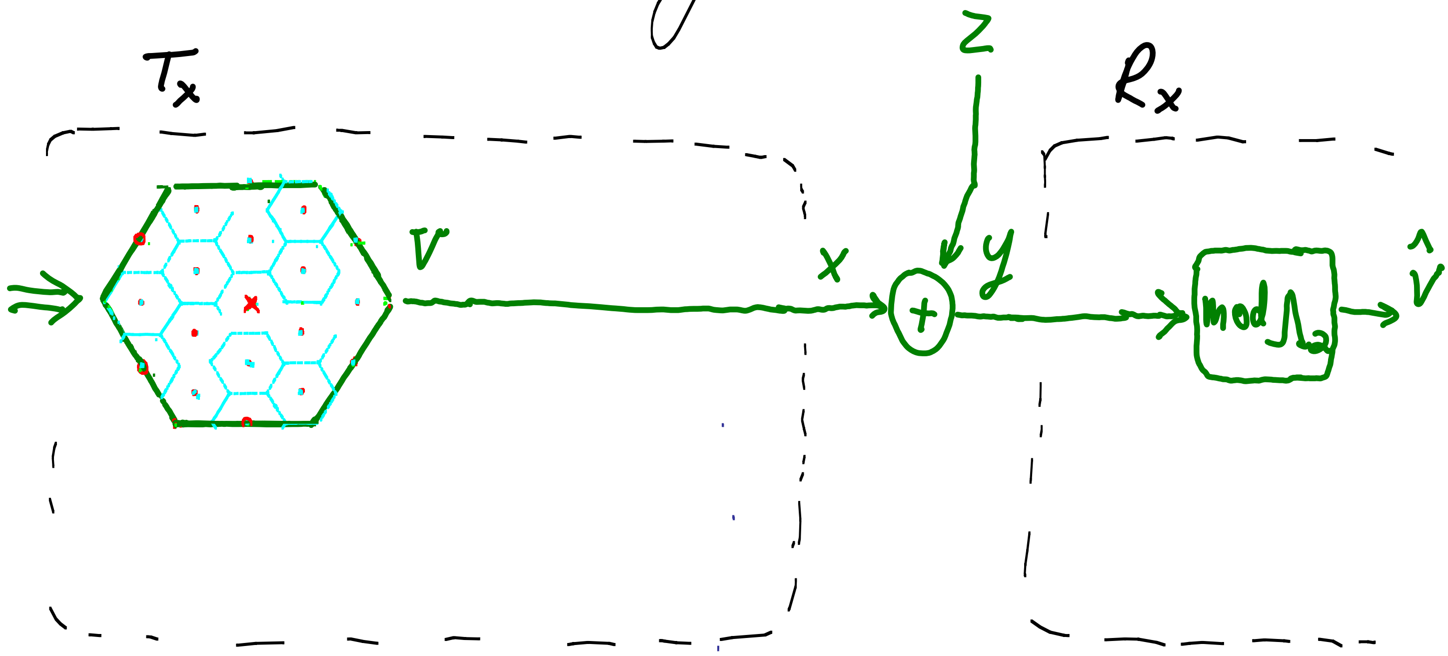
Good quantizer
$\sigma^2(\Lambda_2) = P$ + dither

$$E\,\frac{1}{k}\|X\|^2 = P$$

for any codeword!

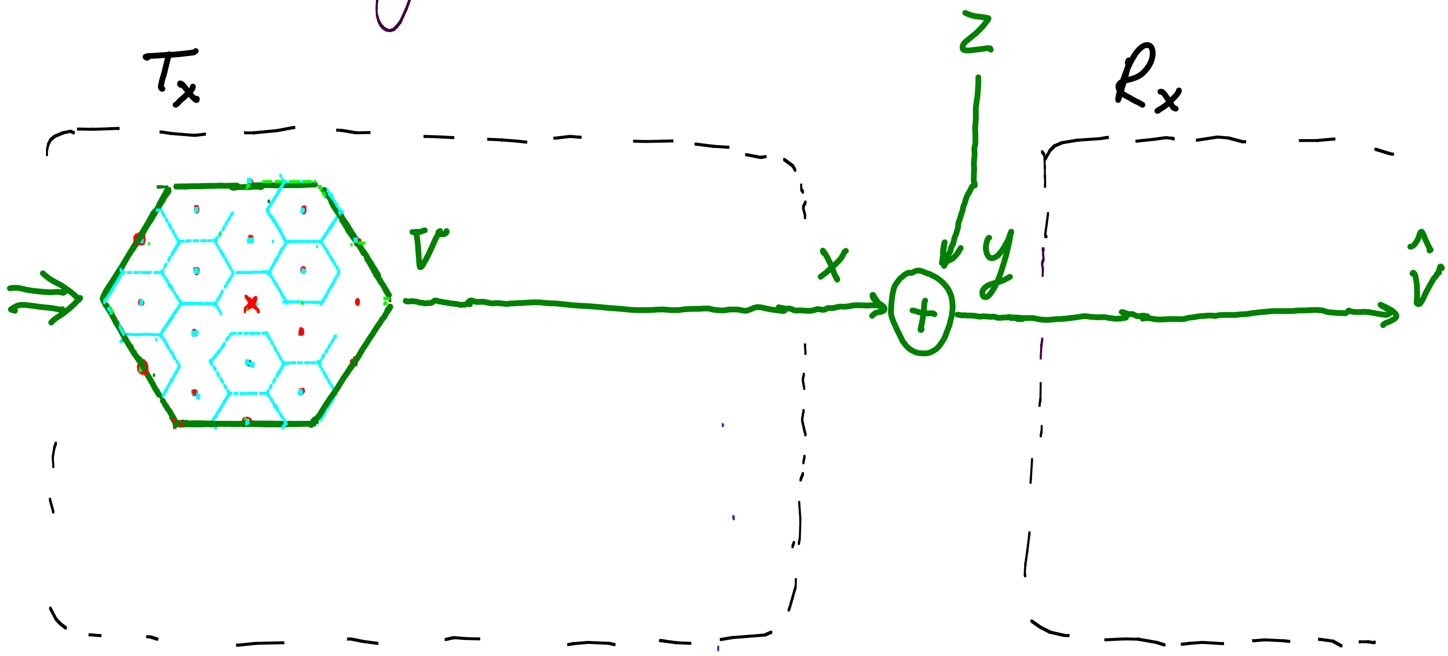
Lattice Dirty Paper Coding

Modulo property $\Rightarrow$



Lattice Dirty Paper Coding

Λ_1 = good for $N(0,\sigma_z^2)$ $\Rightarrow$ $P_e < \varepsilon \ \forall v$



$$\text{Rate} = \frac{1}{k}\log\left(\frac{V_2}{V_1}\right) \quad \text{bit / channel use}$$

NSM(Λ_2)

VNR(Λ_1)

$$= \underbrace{\frac{1}{2}\log\left(\frac{P}{\sigma_z^2}\right)}_{\text{AWGN capacity @ High SNR}} - \underbrace{\log\left(G(\Lambda_2)\cdot\mu(\Lambda_1,\varepsilon)\right)}_{\text{Redundancy} \to 0}$$

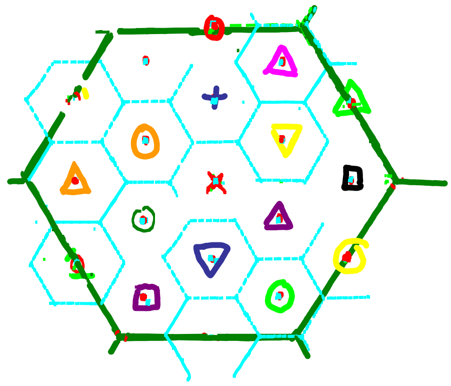
$k \to \infty$

for good lattices....

Costa (Random Binning) $\Rightarrow$ Lattice Coding

1. Code design

binning $\Rightarrow$ relative cosets



2. Transmission

message $\Rightarrow$ coset

typicality encoding $\Rightarrow$ quantization @ MSE $= P$

"u" $= Q_{\Lambda_2}(S + V - \text{dither}) - V + \text{dither}$

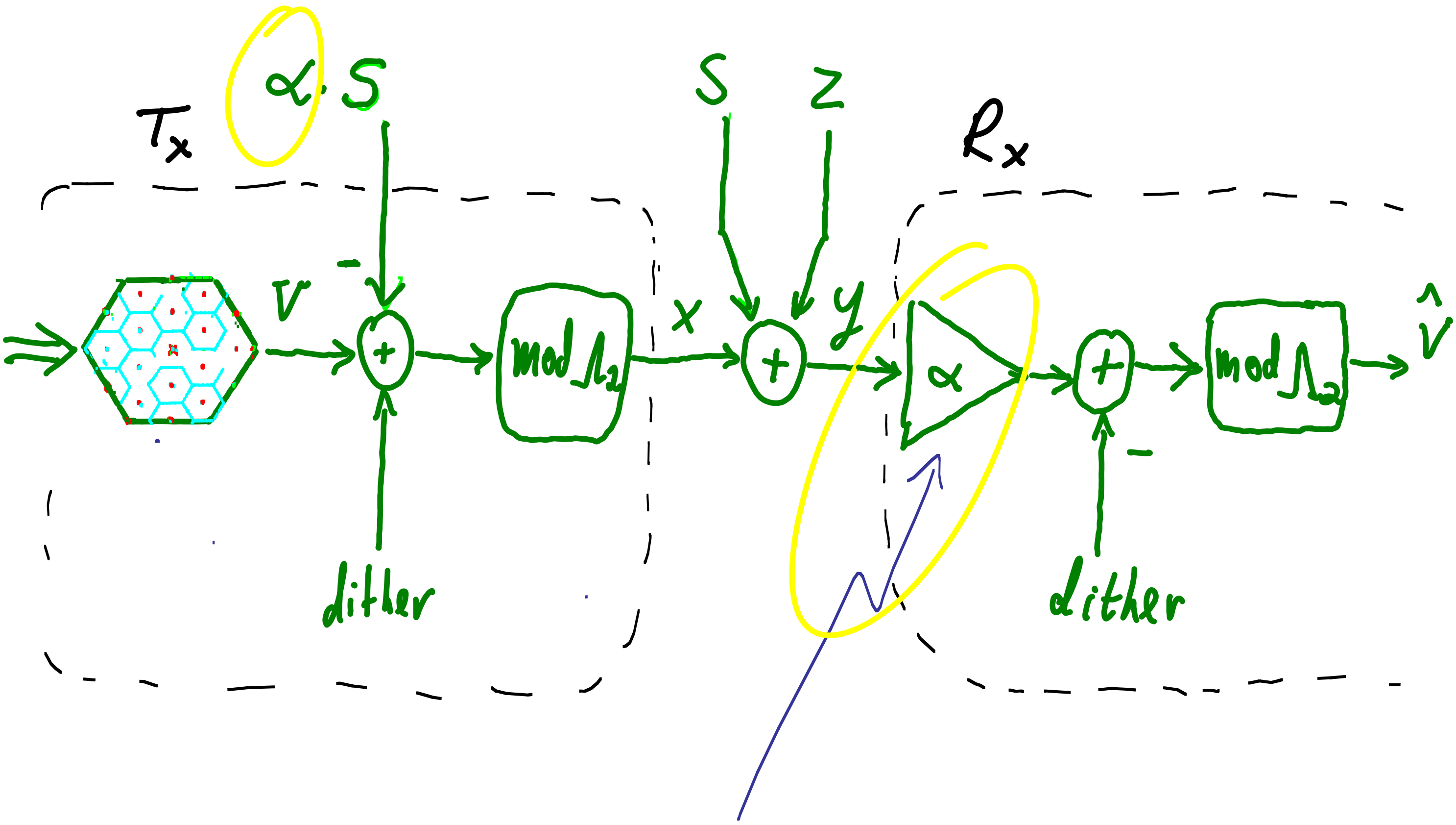
3. Reception

typicality decoding $\Rightarrow$ lattice decoding

"$\hat{u}$" $= Q_{\Lambda_1}(y - \text{dither} \bmod \Lambda_2)$

Achieving $\frac{1}{2}\log(1+SNR)$ @ general SNR

$(SNR = P/\sigma_z^2)$

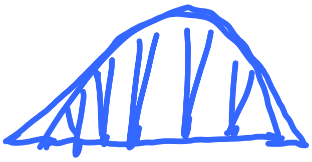


Where a good choice for α is:

$$\alpha = \text{MMSE (Wiener) Coefficient}$$

$$= \frac{P}{P+\sigma_z^2} \approx 1 \text{ @ HSNR}$$

Why Lattices in Communication?

1. a bridge from $n=1$  to $n=\infty$

   = non-asymptotic analysis per dimension

2. Algebraic (low complexity) Binning

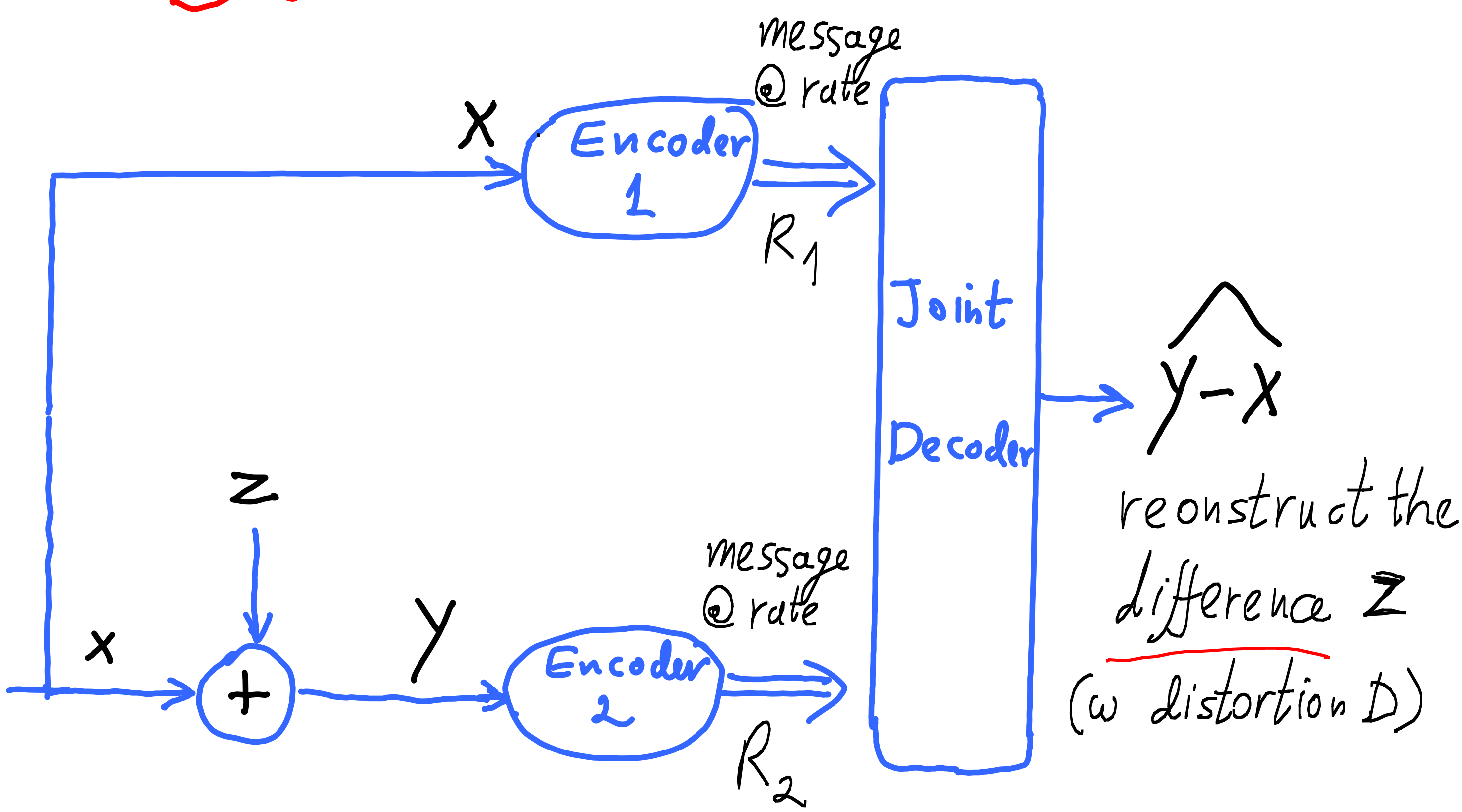
   = structured coding schemes for networks

3. Better than Random-Coding!

   in distributed side-information problems

4.

The Gaussian Korner-Marton Problem

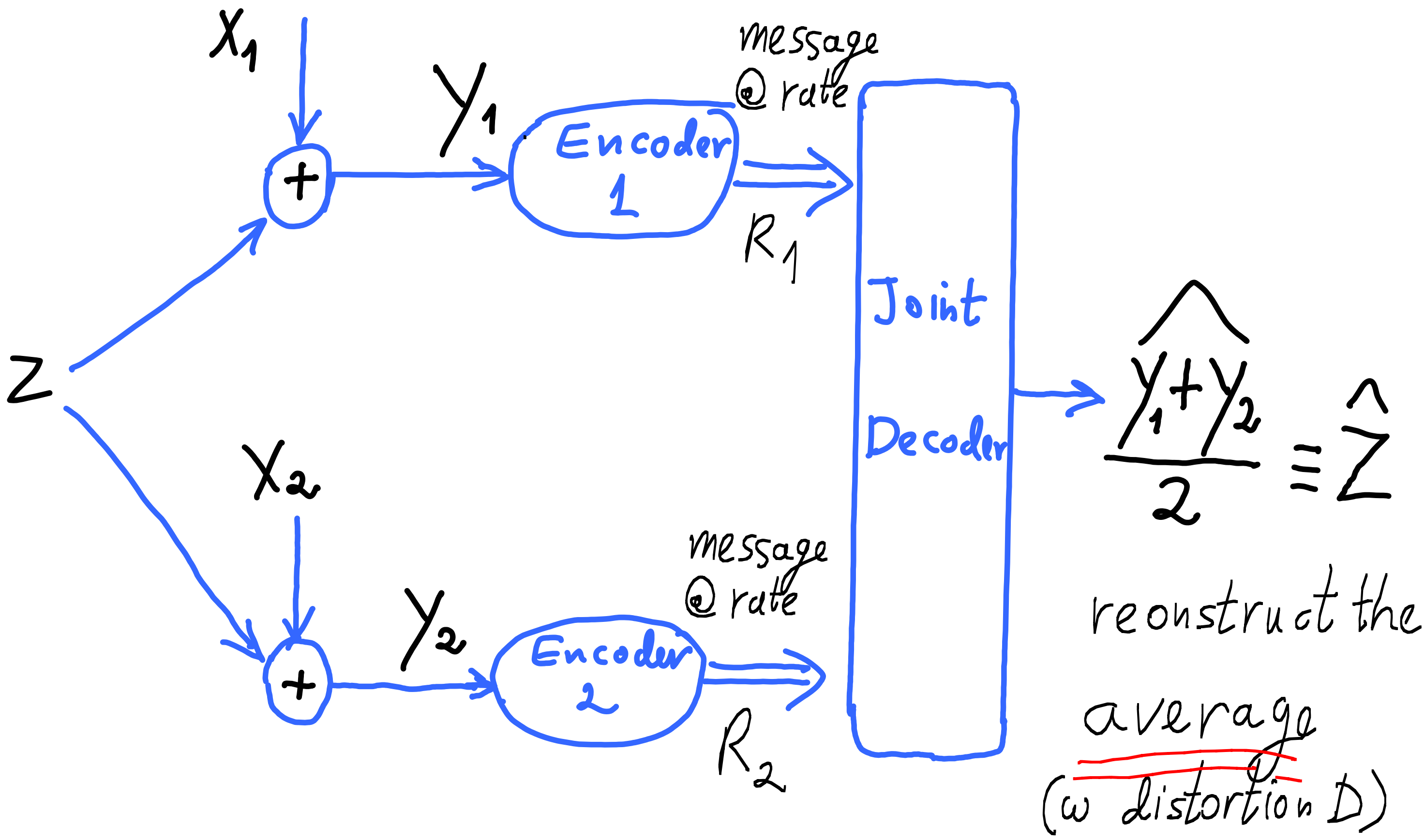


Rate $=$? $R_{X,Y}(D_1, D_2)$ where $D_1 + D_2 = D$ — Berger-Tang

$R_Z(D)$ — over optimistic

$2R_Z(D)$, $2R_Z(D/2)$ ... ? — outer / inner

Compare : The CEO Problem



Optimum rate:

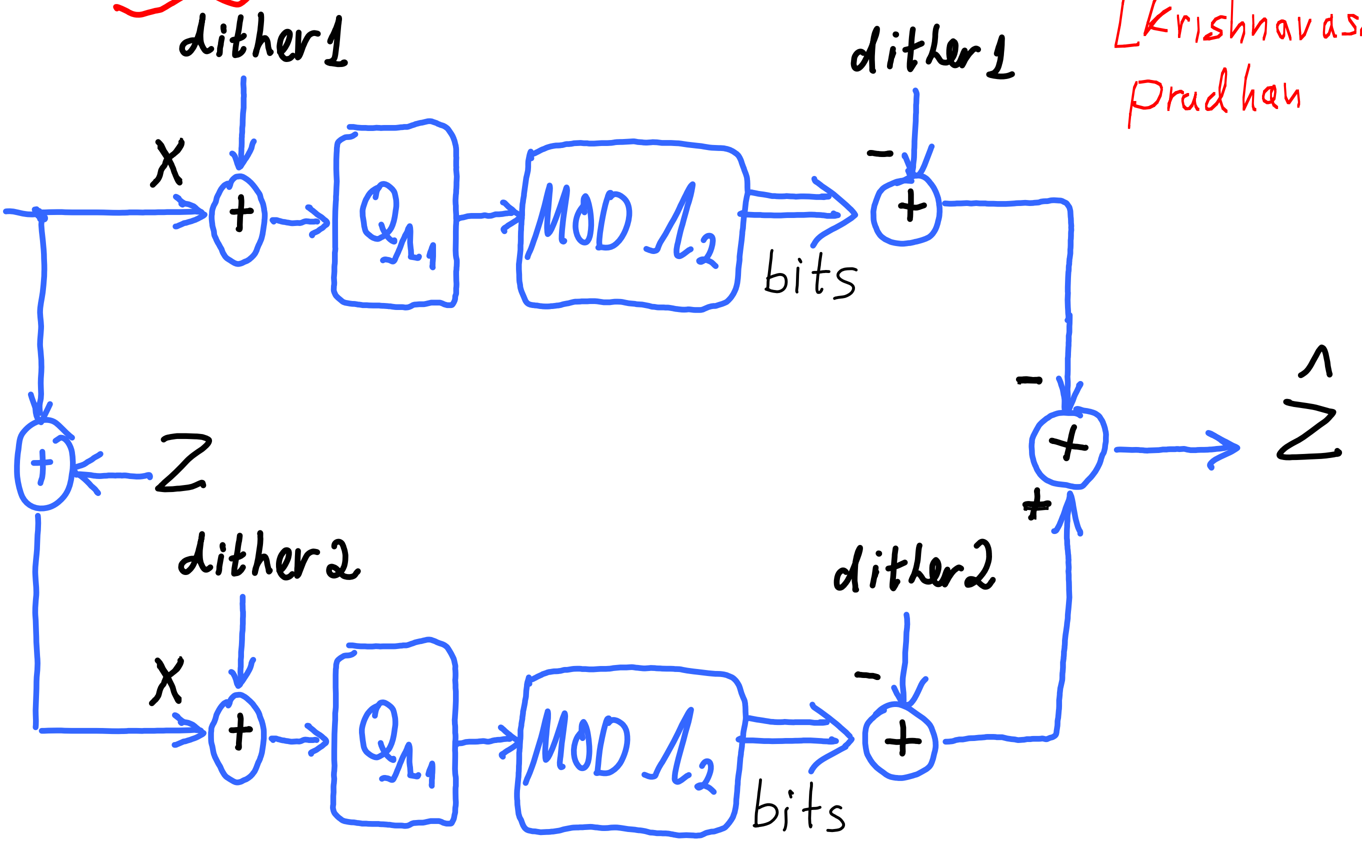
$$\text{Rate} = R_{Y_1 Y_2}(D_1, D_2) \quad \text{where } D_1 = D_2 = 2D$$

Berger-Tung

achieved by random codes!

The Gaussian Korner-Marton Problem

[Krishnavasan Pradhan]



* modulo distributive law ⇒

$$\hat{Z} = Z + \widetilde{\text{dither }1} + \widetilde{\text{dither }2} \quad \text{w.h.p}$$

$$\Rightarrow R_1 = R_2 = R_Z(D/2) + \tfrac{1}{2}\log(NSM_1 * VNR_2)$$

gap of ½ bit from outer bound

redundancy → 0 @ dim → ∞

Why Random Loses ?

Distributed Coding $\Rightarrow$ Need Commutativity:

$$\boxed{\text{Binning}(Y) - \text{Binning}(X) = \text{Binning}(Y-X)}$$

$\Rightarrow$ Binning should be aligned

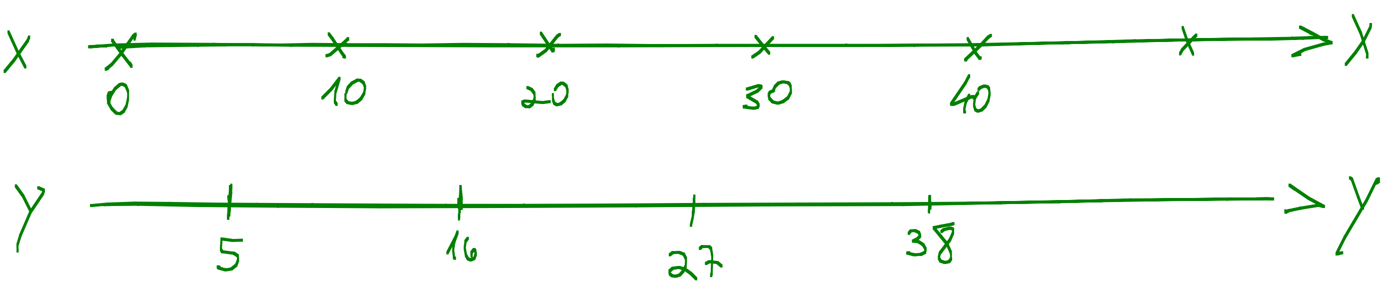
Example of mis-aligned ("pseudo random") binning:

$Y = X + Z$, $Z \sim \text{unif}[0,9]$, $X \sim \text{unif}[0,999]$

code 1 = $X \bmod 10$

code 2 = $Y \bmod 11$ (= higher rate but mis aligned)

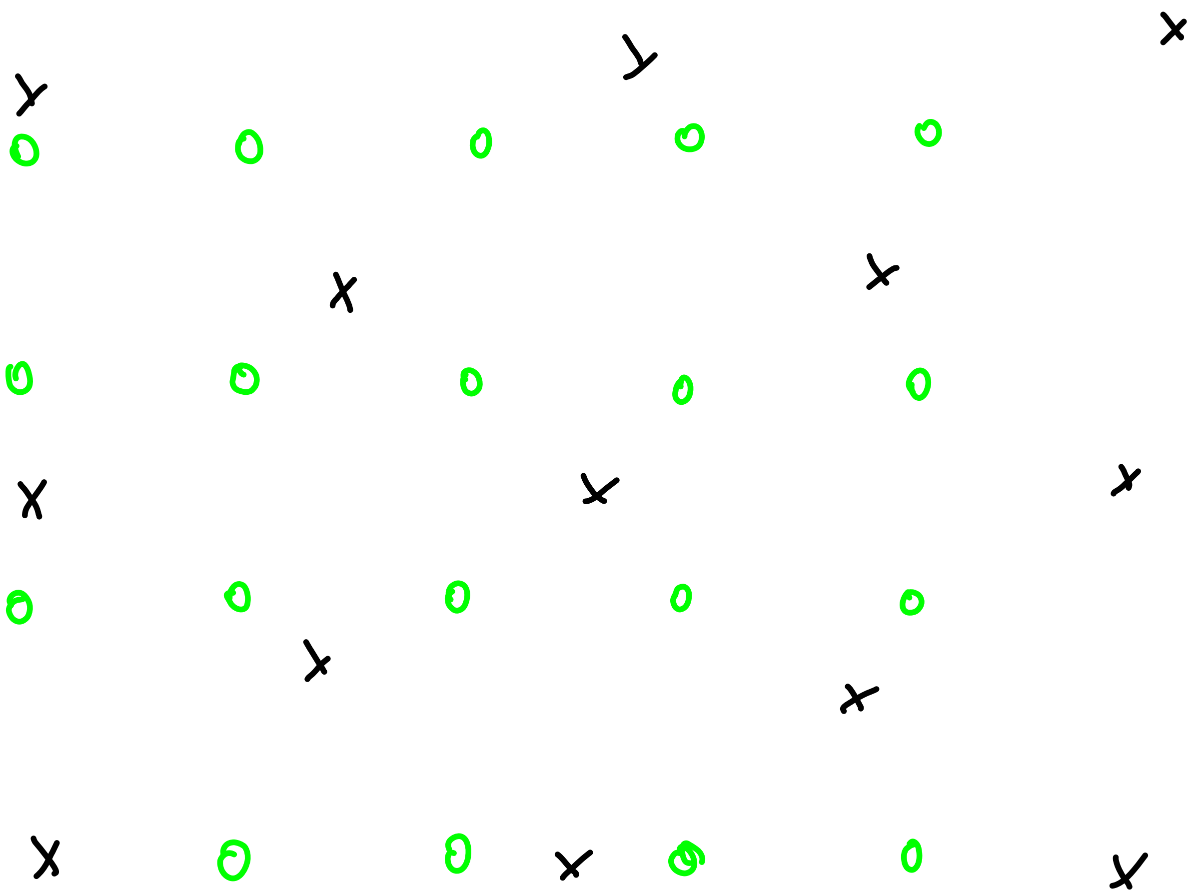
If code 1 = 0 code 2 = 5



$\Rightarrow Z = 5$ or 6 or 7 or ..... ?

Why Random Loses ?

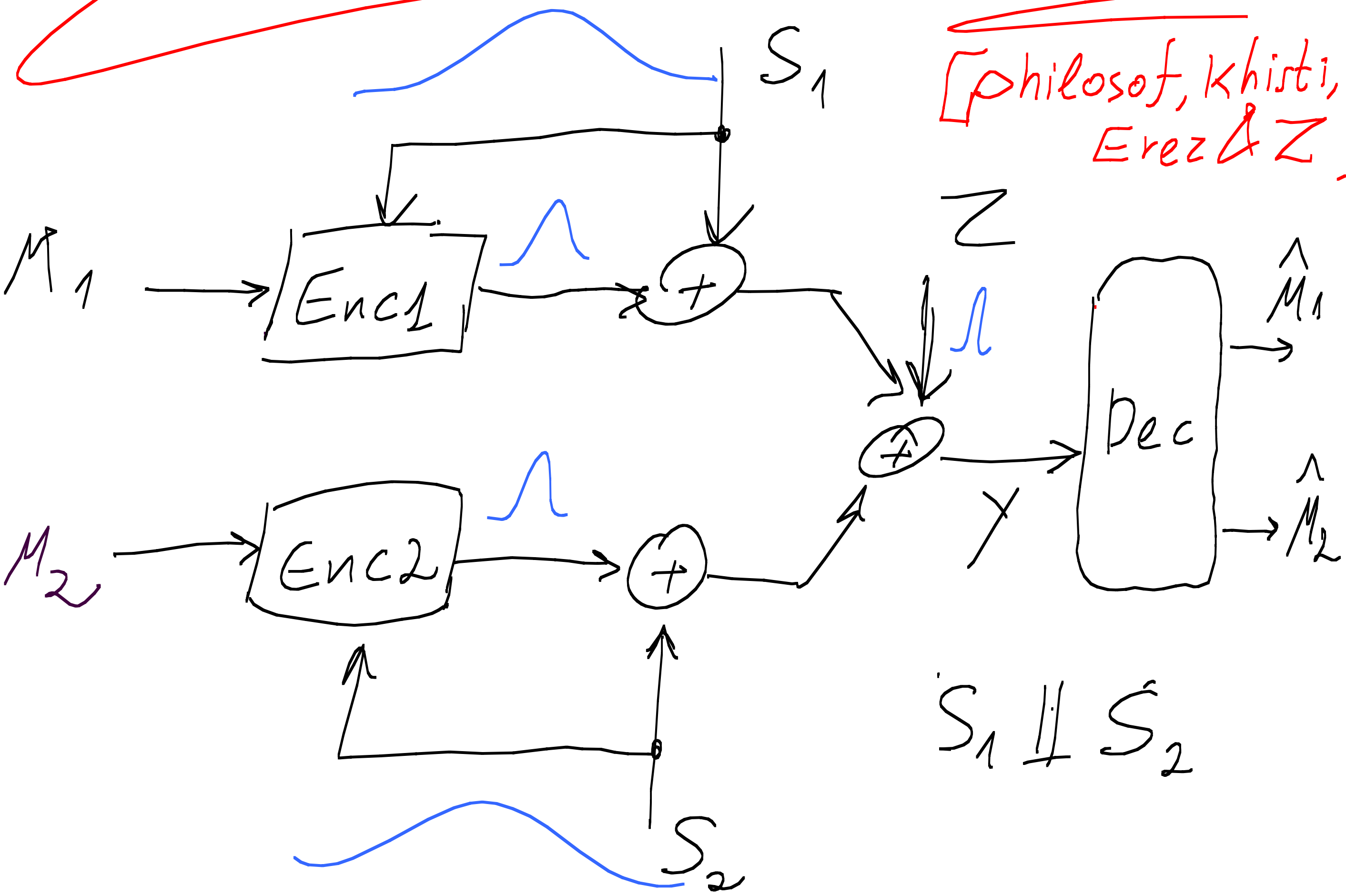
2-dim example of mis-aligned binning:



$$\boxed{\det(\Lambda_1) = \det(\Lambda_2)}$$

The Doubly-Dirty Multiple Access Channel

[Philosof, Khisti, Erez & Z]



$S_1 \perp\!\!\!\perp S_2$

Knowledge of the interference (S_1, S_2) is split between two independent encoders

Results

Costa (Random Binning) $\Rightarrow$ $C \simeq 0$

(for strong interference)

Lattice Strategies $\Rightarrow$ $C \simeq C_{\text{clean-MAC}}$

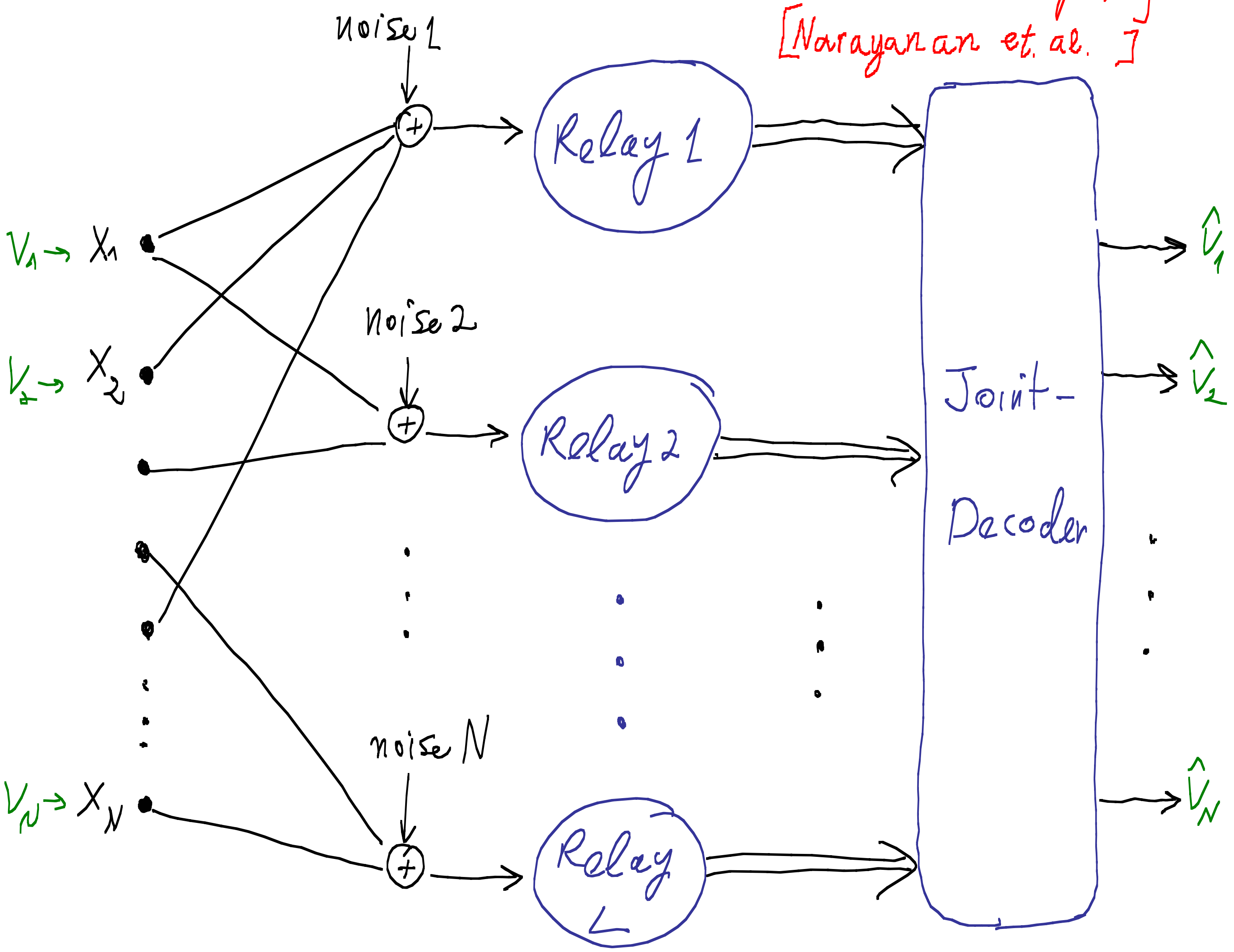
!

Lattice Alignment

	Align	Can be Random
KM	reference signals => coarse lattice	fine (quantize) code
DMAC	i concentration points => coarse lattice	fine (channel) code
CO&F	desired codewords => fine lattice	coarse (shaping) code
IC	interefer codewords => fine lattice	coarse (shaping) code

Lattice Coding for Noisy-Linear-Networks

[Nazer-Gastpar]
[Narayanan et. al.]

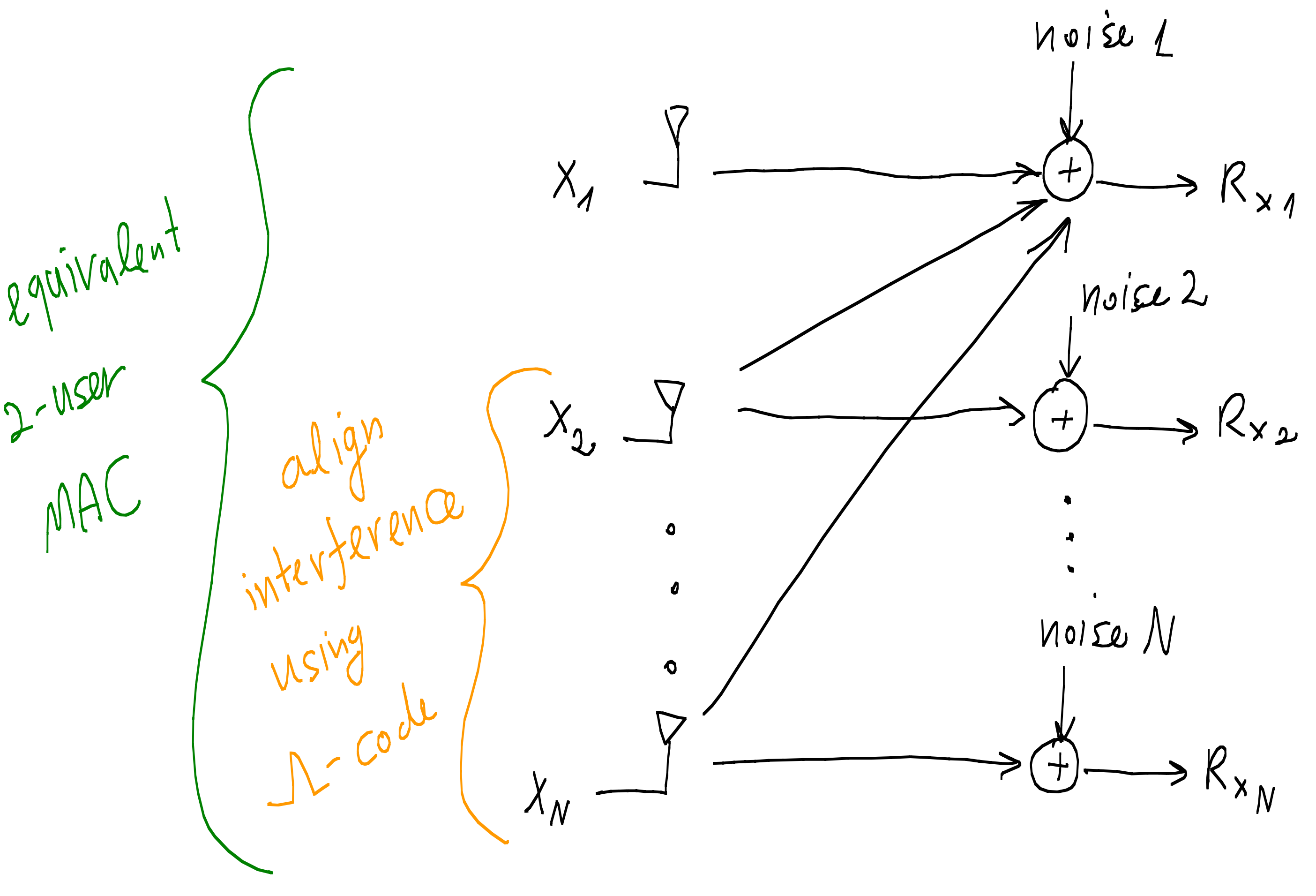


"digital" network coding : noiseless, bit pipes, random\linear code

"analog" network coding : noisy, linear channels (interference), ~~random~~\lattice code

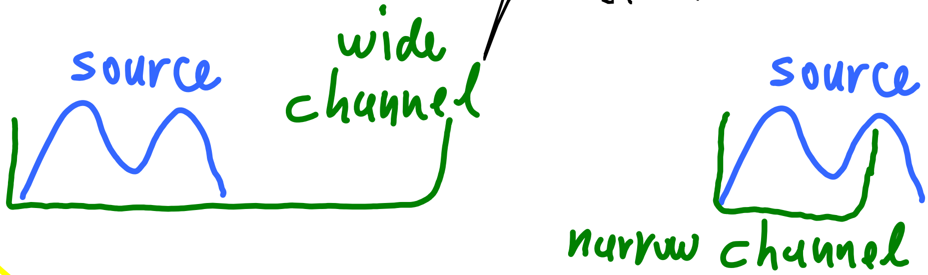
Interference Alignment
(in amplitude domain)

[Bresler - Parekh - Tse]

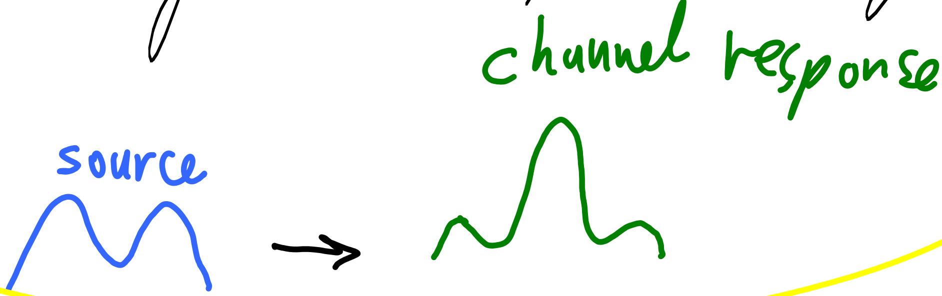


Joint Source-channel Coding

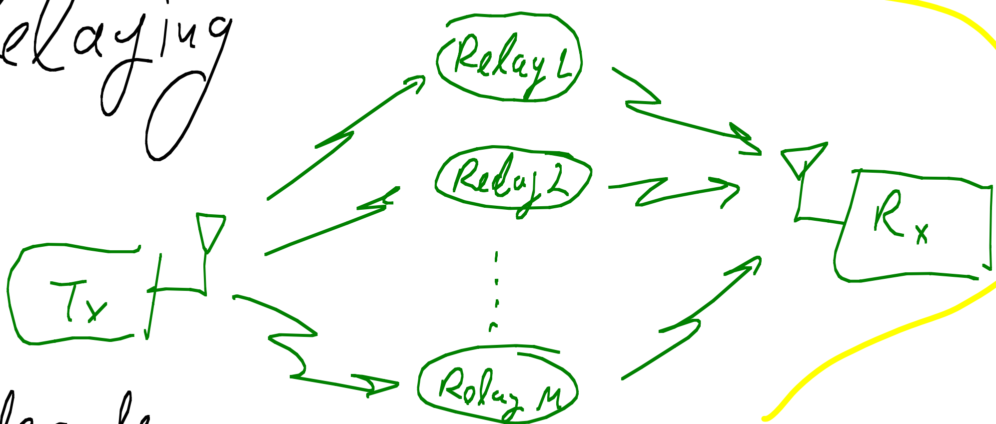
Bandwidth Expansion & Compression



Analog (colored) Matching



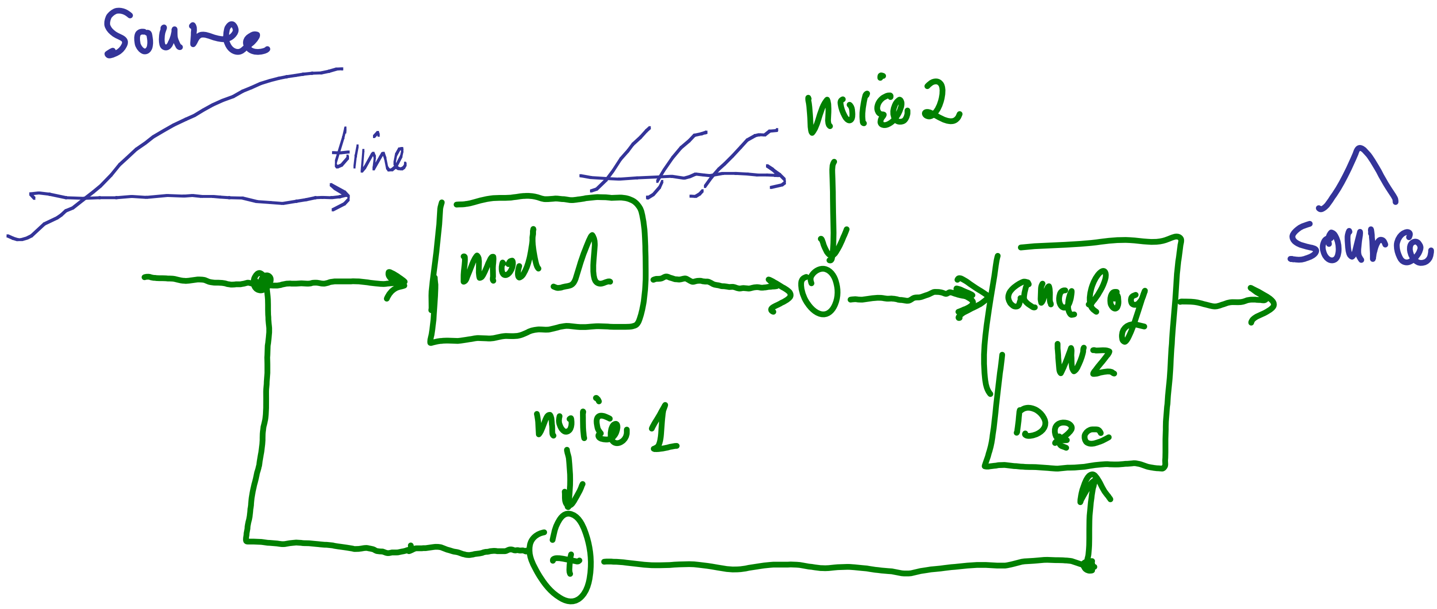
Analog Relaying



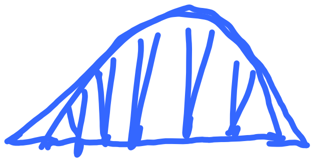
relays cannot decode ...

Modulo-Lattice Modulation for Bandwidth Expansion

[Reznic Feder Z, Kochman Z]



Why Lattices in Communication?



1. a bridge from $n=1$ to $n=\infty$

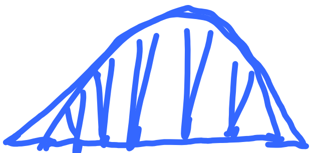
   = non-asymptotic analysis per dimension

2. Algebraic (low complexity) Binning

   = structured coding schemes for networks

3. Better than Random-Coding,

   in distributed side-information problems



4. a bridge from Analog-to-Digital

   = Robust joint source-channel coding

Concluding Remarks

- Classical FOMs : $G(\Lambda)$, $\mu(\Lambda, p_e)$ & properties $G(\Lambda_k^*) \to \frac{1}{2\pi e}$ $\mu(\Lambda_k^*, p_e) \to 2\pi e + p_e$

- New FOMs :
  $G(\Lambda_2)\cdot\mu(\Lambda_1, p_e)$ for DPC
  $G(\Lambda_1)\cdot\mu(\Lambda_2, p_e)$ for WZ
  $\Lambda_2 \subset \Lambda_1$
  $G(\Lambda)\cdot\mu(\Lambda, p_e)$ for Joint WZ-DPC

- Trellis code as Λ_∞

- Low complexity lattice encoding & decoding
  - Construction-A via q-ary L.D.P.C.
  - Low Density Lattice Codes

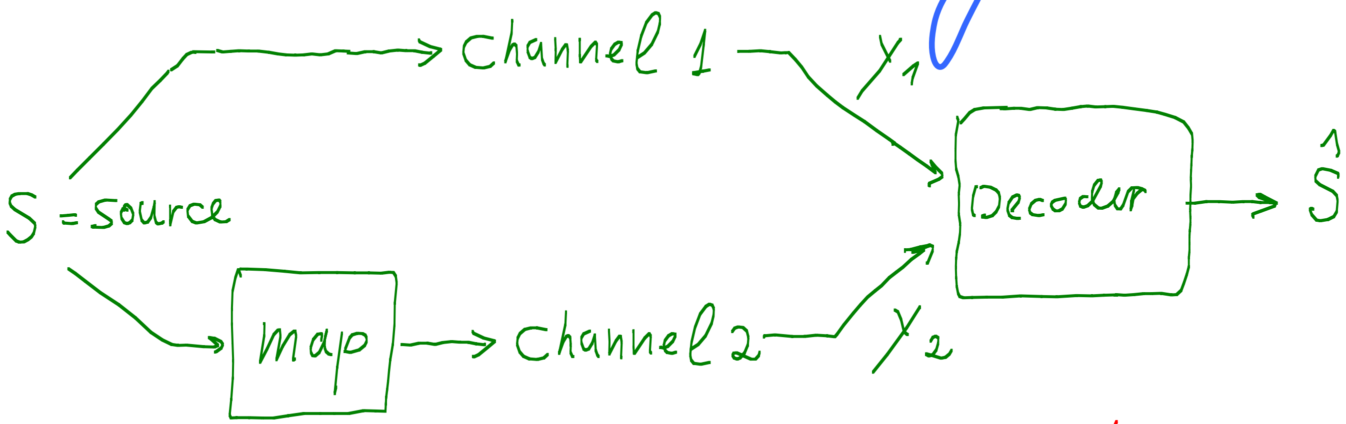
- Is structure always at least as good as random?

- The "best single letter" solution?

When Random is Better than Structure

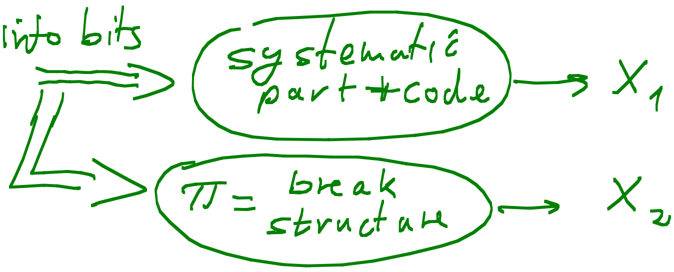
Structure sometimes creates ambiguity...

1. Information cross-checking



random mapping is better than structured!

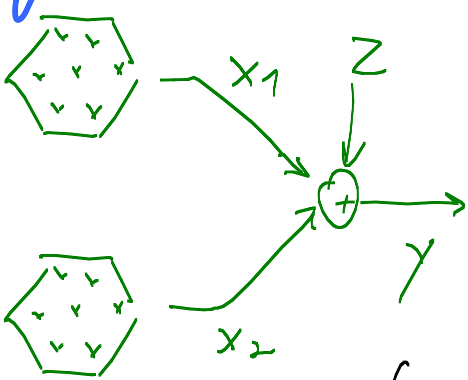
e.g. turbo coding



See also Tuncel

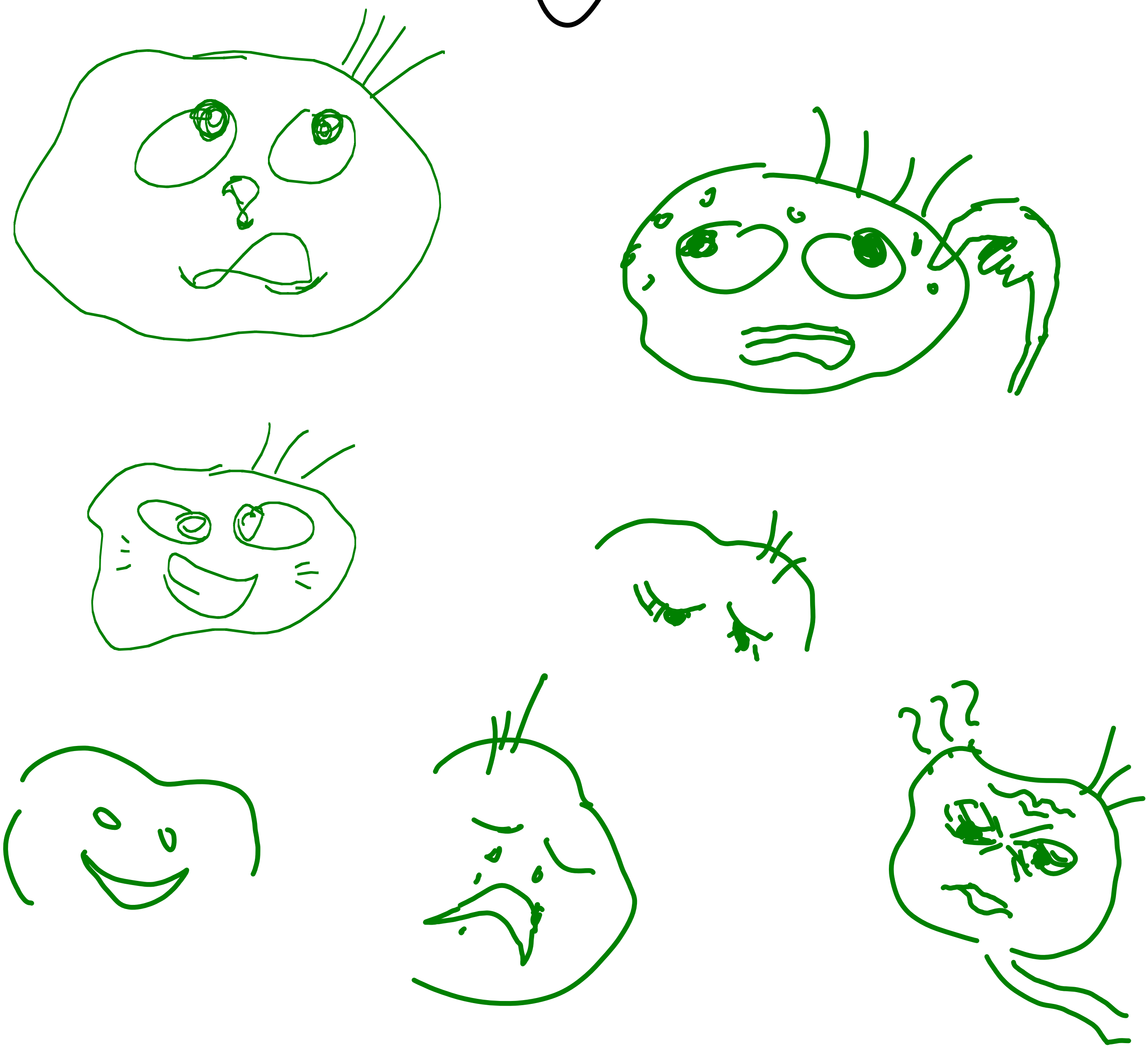
2. Decode Many-from-One [distinguishability]

Symmetric MAC



find X_1 and X_2 from $X_1 + X_2$ ($X_1, X_2 \in \Lambda \Rightarrow$ ambiguity)

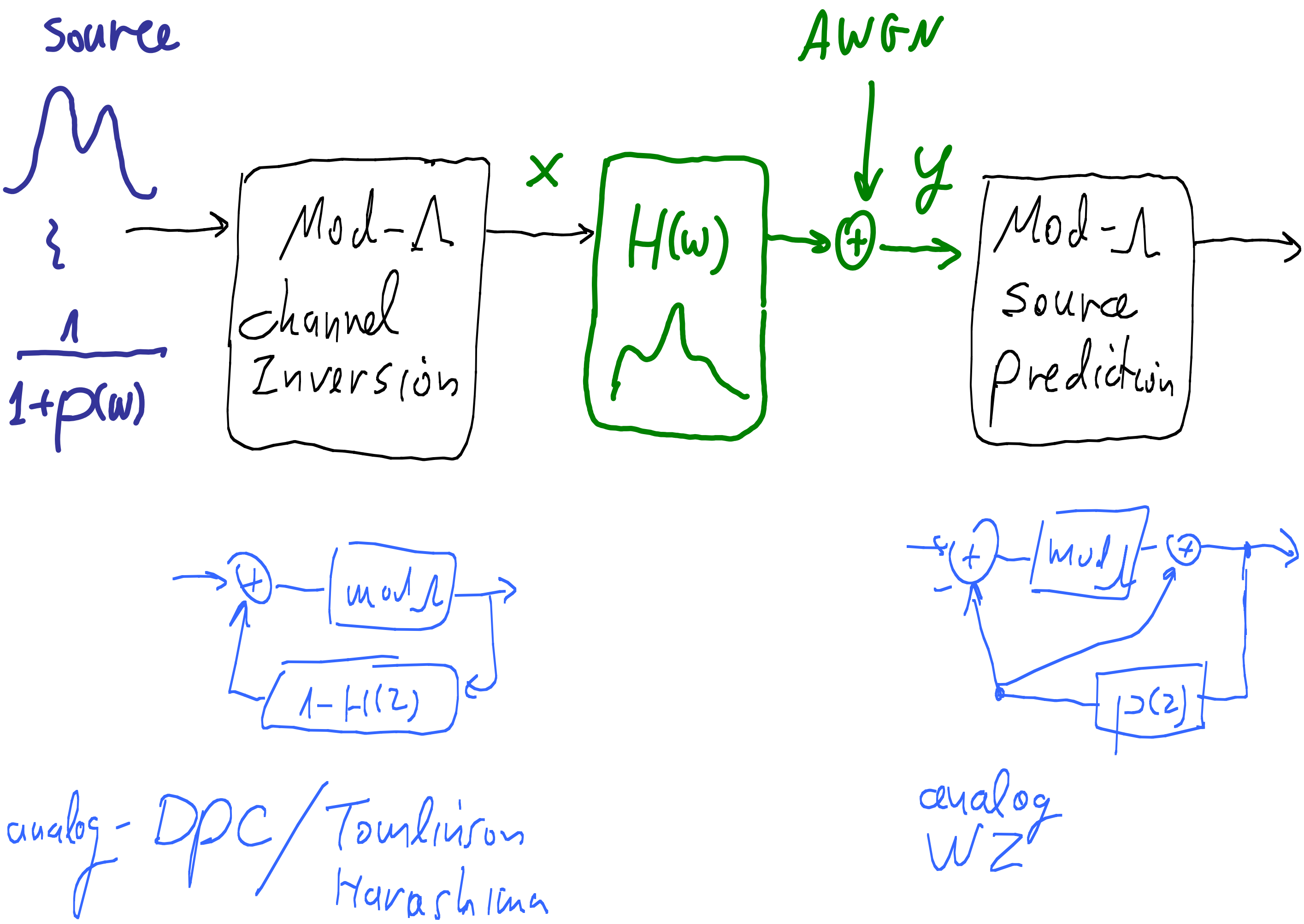
Thank You ?



Backup Slides

Analog Matching of Colored Sources to Colored Channels

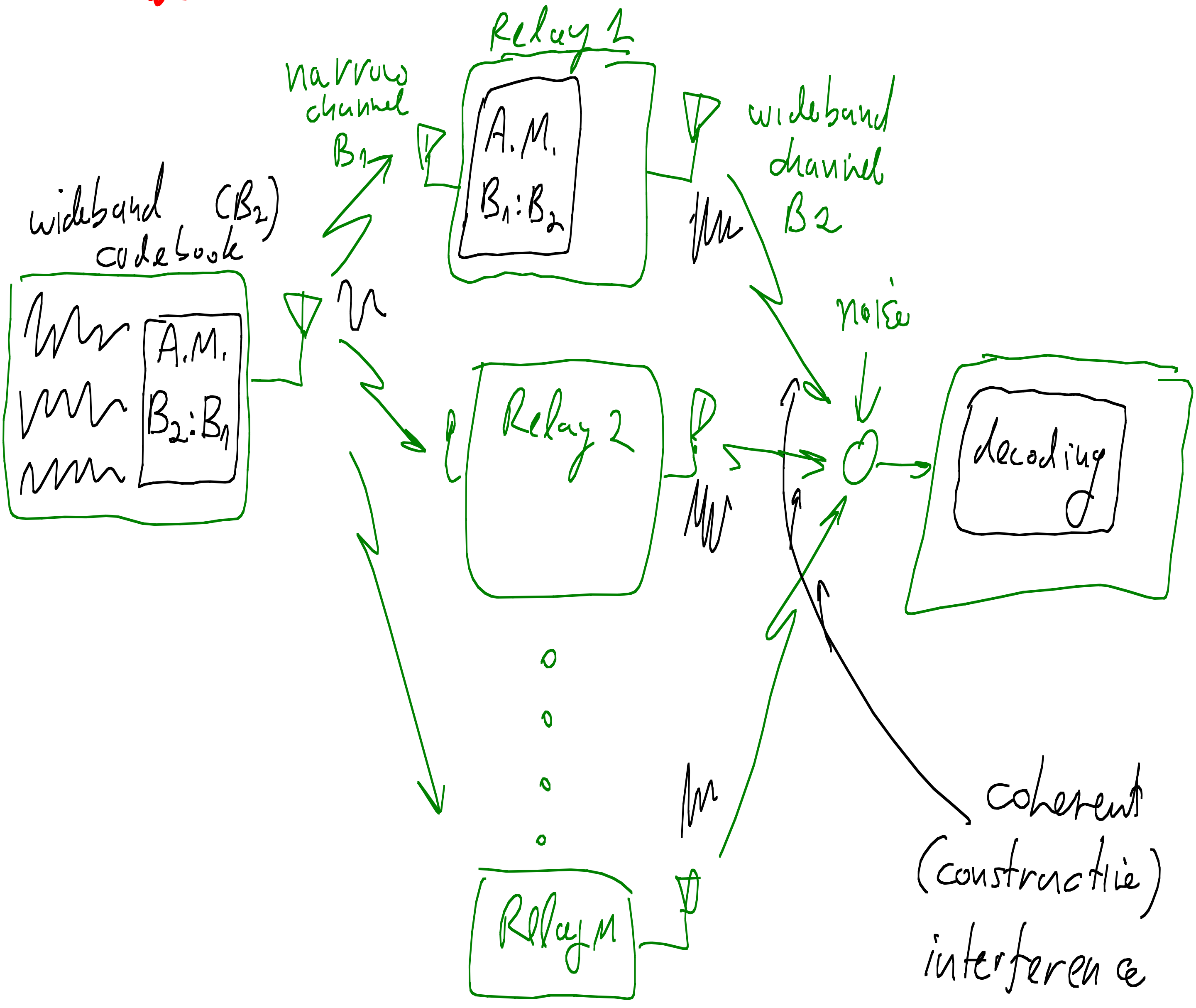
[Kochman & Z]



$\Rightarrow$ Achieves $\boxed{R(D) = C - \text{LOSS}}$

$$\text{LOSS} = \frac{1}{2}\log\left(G(\Lambda)\cdot\mu(\Lambda, P_e)\right)$$

Rematch & Forward for Parallel Relays [Kochman Khina Erez & Z]



$\Rightarrow$ extension of Amplify & Forward for $B_2 \neq B_1$